Chapter 1. Complex Numbers

* The real numbers system:

Natural numbers

Zero and negative integers

Integer (whole) numbers

Rational numbers: n/m

Irrational numbers: solutions to $x^2 - 2 = 0$

Transcendental numbers: sin 1

Algebraic numbers (solutions of polynomial equations with integer coefficients)

Graphical representation of real numbers:
the real axis (the x-axis), the number line

Real numbers are well ordered: If $a \neq b$ then either $a > b$ or $a < b$.

Open interval: $\{x \in \mathbb{R} : a < x < b\} = (a, b)$

Closed interval: $\{x \in \mathbb{R} : a \leq x \leq b\} = [a, b]$

Absolute value: $|a| = \begin{cases} a, & a \geq 0 \\ -a, & a < 0 \end{cases}$

We need an extension of real numbers to solve the following equations:

$x^2 + 1 = 0$, $\sin x = 2$, $e^x = -5$, etc.

Denote one of the solutions of $x^2+1=0$ by i, i.e. $i^2+1=0$ or $i^2=-1$

i is called the imaginary unit.

Let a and b be real numbers. We write $z = a\cdot 1 + b\cdot i = a+bi$, where 1 is the real unit.

z is called a complex number,

$a = Re(z) = Re\, z$ is the real part of z,

$b = Im(z) = Im\, z$ is the imaginary part of z.

$$z = Re\, z\cdot 1 + Im\, z\cdot i$$

The symbol z can stand for any of complex numbers. In this case z is called a complex variable: $z = x\cdot 1 + y\cdot i = x+yi = x+iy$

If $z_1 = z_2$ where z_1 and z_2 are two complex numbers and $z_1 = a+bi$, $z_2 = c+di$ then $a=c$, $b=d$. In other words, if $z_1 = z_2$ then $Re\, z_1 = Re\, z_2$ and $Im\, z_1 = Im\, z_2$.

Ex $x+iy = 2-7i \iff x=2,\ y=-7$

[The symbol in the expression $A \iff B$ means that two statements A and B are logically equivalent.]

If $z = a + i0$ then z is a real number.
If $z = 0 + bi$ then z is a pure imaginary number.

$\mathbb{R} \subset \mathbb{C}$ (The set of real numbers is a subset of the set of complex numbers)
Indeed, for any real number a, $z = a + i0$ is a complex number.

If $z = a + bi$ then $\bar{z} = a - bi$ is called the complex conjugate to z.

- Fundamental Operations in $\mathbb{C}$

1. Addition: $(a+bi)+(c+di) = (a+c)+(b+d)i$
2. Subtraction: $(a+bi)-(c+di) = (a-c)+(b-d)i$
3. Multiplication: $(a+bi)(c+di) = (ac-bd)+(ad+bc)i$

Indeed, $(a+bi)(c+di) = ac + adi + bci + bdi^2$
$= ac + (ad+bc)i - bd = (ac-bd)+(ad+bc)i$

4. Division:
$$(a+bi) \div (c+di) = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}\, i$$
5. If $z = a + bi$ then $z\bar{z} = a^2 + b^2$

Ex Proof of 4.
$$(a+bi) \div (c+di) = \frac{a+bi}{c+di} \cdot \frac{c-di}{c-di} = \frac{(a+bi)(c-di)}{(c+di)(c-di)}$$
$$= \frac{(ac+bd)+(bc-ad)i}{c^2+d^2} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}\, i$$

$\mathbb{C}$ is not an ordered set. Complex numbers cannot be positive or negative.

Ex. 1. $(5+2i)+(11-7i)=(5+11)+(2-7)i$
$= 16-5i$

2. $(3+4i)-(7+2i)=(3-7)+(4-2)i=-4+2i$

3. $(2+3i)(1-2i)=(2+6)+(-4+3)i=8-i$

4. $(2+3i)\div(1-2i)=\frac{2-6}{1+4}+\frac{3+4}{1+4}i=-\frac{4}{5}+\frac{7}{5}i$

Check: $(-\frac{4}{5}+\frac{7}{5}i)(1-2i)=(-\frac{4}{5}+\frac{14}{5})+(\frac{8}{5}+\frac{7}{5})i=2+3i$

- Absolute value (modulus) of $z=a+bi$

$|z|=|a+bi|=\sqrt{a^2+b^2}$

Ex $z\bar{z}=|z|^2$: $\quad z\bar{z}=(a+bi)(a-bi)$
$=(a^2+b^2)+(-ab+ab)=a^2+b^2=|z|^2$

- Properties of absolute value:

1. $|z_1 z_2|=|z_1||z_2|$
2. $\left|\frac{z_1}{z_2}\right|=\frac{|z_1|}{|z_2|}$
3. $|z_1+z_2|\le|z_1|+|z_2|$
4. $|z_1+z_2|\ge|z_1|-|z_2|$
$|z_1-z_2|\ge\big||z_1|-|z_2|\big|$

Ex. $z_1=3+2i$, $z_2=5-i$

(a) $|2z_1-3z_2|=|6+4i-15+3i|=|-9+7i|=\sqrt{81+49}$
$=\sqrt{130}$

(b) $\bar{z}_1^{\,4}=(3-2i)^4=((3-2i)^2)^2$

$(a+bi)^2=(a^2-b^2)+2abi$, $(a-bi)^2=(a^2-b^2)-2abi$

$\bar{z}_1^{\,4}=(5-12i)^2=(25-144)-120i=-119-120i$

$z_1^4\bar{z}_1^{\,4}=(z_1\bar{z}_1)^4=((z_1\bar{z}_1)^2)^2=(|z_1|^2)^2=13^2=169$

- Properties of conjugates

$\overline{(z_1+z_2)} = \bar{z}_1 + \bar{z}_2$, $\overline{(z_1-z_2)} = \bar{z}_1 - \bar{z}_2$

$\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$, $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$

Ex Solve the equation: $z^2 - 4z + 5 = 0$

Quadratic formula: $z = 2 \pm \sqrt{4-5} = 2 \pm i$

two solutions: $z_1 = 2+i$, $z_2 = 2-i$

Alternative solution: Let $z = a+bi$. Then

$z^2 - 4z + 5 = a^2 - b^2 + 2abi - 4a - 4bi + 5 = 0 = 0 + 0i$

Equalize real and imaginary parts:

$a^2 - b^2 - 4a + 5 = 0$, $2ab - 4b = 2b(a-2) = 0$

(1) $b=0$, a is any real number

or

(2) $a=2$, b is any real number

(1) $b=0$. Then $a^2 - 4a + 5 = 0$. No real solutions

(2) $a=2$. Then $4 - b^2 - 8 + 5 = 0$, $b^2 = 1$, $b = \pm 1$

Hence $z = 2 \pm i$

Ex $(1-2i)z = 2+3i \Rightarrow z = \frac{2+3i}{1-2i} = -\frac{4}{5} + \frac{7}{5}i$ (found before)

Ex $z^2 + (4+3i)z + 2 + 6i = 0$

Quadratic formula: $z = \frac{1}{2}\left(-4-3i \pm \sqrt{(4+3i)^2 - 4(2+6i)}\right)$

$z = \frac{1}{2}\left(-4-3i \pm \sqrt{16-9+24i-8-24i}\right) = \frac{1}{2}(-4-3i \pm i)$

$z_1 = \frac{1}{2}(-4-3i-i) = -2-2i$, $z_2 = \frac{1}{2}(-4-3i+i) = -2-i$

Ex $z^2 + (6-4i)z + 5 - 13i = 0$

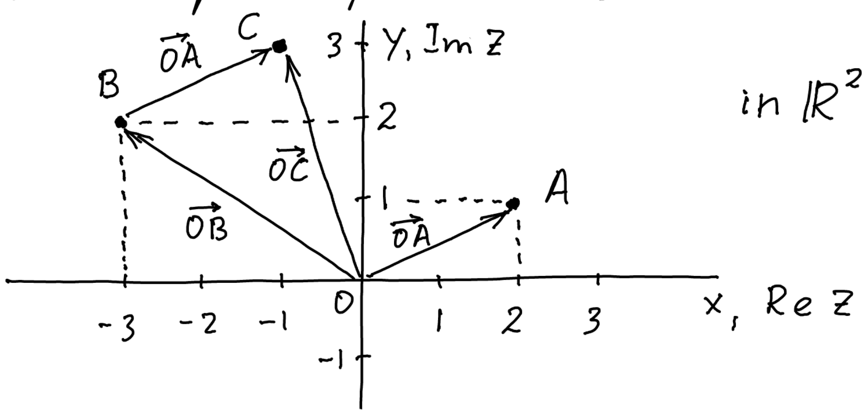
$z = \frac{1}{2}\left(-6+4i \pm \sqrt{36-16-48i-20+52i}\right) = \frac{1}{2}\left(-6+4i \pm \sqrt{4i}\right)$

$z = \frac{1}{2}(-6 + 4i \pm 2\sqrt{i})$. What is $\sqrt{i}$?

- Geometric representation of complex numbers. Complex plane.

$z = x + iy \in \mathbb{C}$, $(x, y) \in \mathbb{R}^2 \Rightarrow \mathbb{C} \Leftrightarrow \mathbb{R}^2$

The complex plane is a 2-d plane.



in $\mathbb{R}^2$: $A(2,1)$, $B(-3,2)$

$A \leftrightarrow 2 + i \in \mathbb{C}$

$B \leftrightarrow -3 + 2i \in \mathbb{C}$

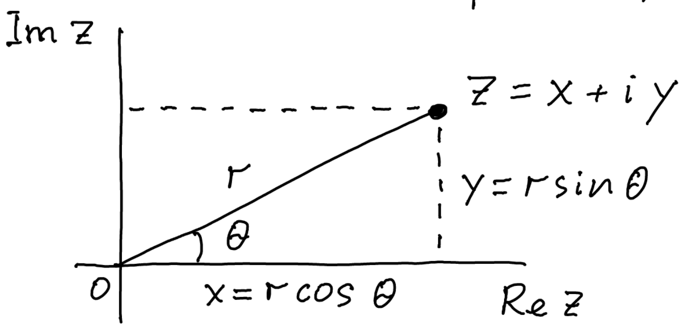
Vector interpretation of complex numbers:

$2 + i$ corresponds to the vector $\vec{OA} = \langle 2, 1 \rangle$

$-3 + 2i$ corresponds to the vector $\vec{OB} = \langle -3, 2 \rangle$

Their sum is $(2 + i) + (-3 + 2i) = -1 + 3i$ corresponds to the vector $OC = \langle -1, 3 \rangle$.

- Polar Form of Complex Numbers



$z = x + iy$

$r = \sqrt{x^2 + y^2} = |z|$

$x = r\cos\theta$, $y = r\sin\theta$

$\tan\theta = x/y$

r is modulus or absolute value of z

θ is the argument of z

$z = x + iy = r(\cos\theta + i\sin\theta) = |z| \operatorname{cis}\theta$

The last form is called a polar form of the complex number $z = x + iy$.

The argument is multivalued: $\theta + 2\pi k$ for any integer k is also an argument.

Notation: $\theta = \arg z$

A particular choice of an interval of the length of 2π for $\arg z$ is called the principal range. A value of θ inside that interval is called its principal value or principal argument.
It is common to use the interval $(-\pi, \pi]$ as the principal range. An alternative: $[0, 2\pi)$.

Ex $z = -3\sqrt{3} - 3i$, $|z| = \sqrt{27+9} = 6$

$\cos\theta = -\frac{3\sqrt{3}}{6} = -\frac{\sqrt{3}}{2}$, $\sin\theta = -\frac{3}{6} = -\frac{1}{2}$

Principal range $[0, 2\pi)$: $z = 6\operatorname{cis}\left(\frac{7}{6}\pi\right)$

Principal range $(-\pi, \pi]$: $z = 6\operatorname{cis}\left(-\frac{5}{6}\pi\right)$

- De Moivre's Theorem

Let $z_1 = x_1 + iy_1 = r_1(\cos\theta_1 + i\sin\theta_1)$
and $z_2 = x_2 + iy_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

Then $z_1 z_2 = r_1 r_2 (\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right)$$

Generalization:

$$z_1 z_2 \cdots z_n = r_1 r_2 \cdots r_n \left(\cos(\theta_1+\theta_2+\cdots+\theta_n) + i\sin(\theta_1+\theta_2+\cdots+\theta_n)\right)$$

$$z^n = r^n\left(\cos(n\theta) + i\sin(n\theta)\right) \quad \text{if } z = r(\cos\theta + i\sin\theta)$$

Ex Prove of the product.

$$z_1 z_2 = r_1(\cos\theta_1 + i\sin\theta)\cdot r_2(\cos\theta_2 + i\sin\theta_2)$$

$$= r_1 r_2\left((\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \cos\theta_2\sin\theta_1)\right)$$

$$= r_1 r_2\left(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right)$$

Ex $z_1 = -2\sqrt{2} + 2\sqrt{2}\,i = 4 \text{ cis } \frac{3}{4}\pi$

$z_2 = 1 + \sqrt{3}\,i = 2 \text{ cis } \frac{\pi}{3}$

$z_1 z_2 = 8 \text{ cis } \frac{13}{12}\pi$, $\frac{z_1}{z_2} = 2 \text{ cis } \frac{5}{12}\pi$, $z_2^3 = 8 \text{ cis } \pi = -8$

$$z_1 z_2 = (-2\sqrt{2} + 2\sqrt{2}\,i)(1+\sqrt{3}\,i) = 2\sqrt{2}(-1+i)(1+\sqrt{3}\,i)$$

$$= 2\sqrt{2}\left(-1-\sqrt{3} + (1-\sqrt{3})i\right) = -2\sqrt{2} - 2\sqrt{6} + (2\sqrt{2} - 2\sqrt{6})i$$

- Roots of complex numbers

w is called an n^{th} root of z if $w^n = z$.

Notation: $w = z^{1/n} = \sqrt[n]{z}$

From De Moiver's Thm $z^{1/n} = r^{1/n} \text{ cis}\left(\frac{\theta + 2\pi k}{n}\right)$

$k = 0, 1, 2, \ldots, n-1.$

Therefore, there are n different n^{th} roots of a complex number.

Ex Find all 4^{th} roots of -16.

Sln: $w^4 = -16 = 16 \text{ cis } \pi$, $r = 16$, $\theta = \pi$

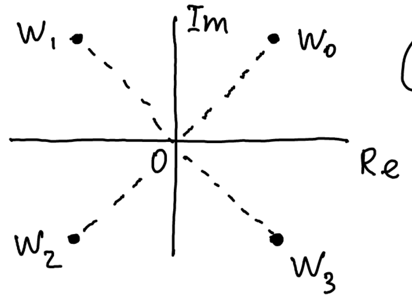
$= w = 2 \text{ cis } \frac{\pi + 2\pi k}{4}$, $k = 0, 1, 2, 3$

$k = 0$: $w_0 = 2 \text{ cis } \frac{\pi}{4} = \sqrt{2} + \sqrt{2}\,i$

$k=1: \quad w_1 = 2\,\text{cis}\,\frac{3}{4}\pi = -\sqrt{2} + \sqrt{2}\,i$

$k=2: \quad w_2 = 2\,\text{cis}\,\frac{5}{4}\pi = -\sqrt{2} - \sqrt{2}\,i$

$k=3: \quad w_3 = 2\,\text{cis}\,\frac{7}{4}\pi = \sqrt{2} - \sqrt{2}\,i$



Ex Find $\sqrt{2+i}$

Sln: $2+i = \sqrt{5}\,\text{cis}\,\theta$, where θ is such that

$$\cos\theta = \frac{2}{\sqrt{5}}, \quad \sin\theta = \frac{1}{\sqrt{5}}$$

Then $\sqrt{2+i} = (2+i)^{1/2} = 5^{1/4}\cdot \text{cis}\,\frac{\theta + 2k\pi}{2}, \quad k=0,1$

$k=0: \quad w_0 = 5^{1/4}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)$

$$\cos^2\frac{\theta}{2} = \frac{1+\cos\theta}{2} \Rightarrow \cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}} = \pm\sqrt{\frac{1+\frac{2}{\sqrt{5}}}{2}}$$

$$= \sqrt{\frac{\sqrt{5}+2}{2}}\cdot\frac{1}{5^{1/4}}$$

$$\sin^2\frac{\theta}{2} = \frac{1-\cos\theta}{2} \Rightarrow \sin\theta = \pm\sqrt{\frac{\sqrt{5}-2}{2}}\cdot\frac{1}{5^{1/4}}$$

θ is an angle in the 1st quadrant $\Rightarrow \frac{\theta}{2}$ is an angle in the 1st quadrant $\Rightarrow \cos\frac{\theta}{2} > 0$, $\sin\frac{\theta}{2} > 0$

$$\Rightarrow \cos\frac{\theta}{2} = \sqrt{\frac{\sqrt{5}+2}{2}}\cdot\frac{1}{5^{1/4}}, \quad \sin\frac{\theta}{2} = \sqrt{\frac{\sqrt{5}-2}{2}}\cdot\frac{1}{5^{1/4}}$$

Hence, $\quad w_0 = \sqrt{\frac{\sqrt{5}+2}{2}} + \sqrt{\frac{\sqrt{5}-2}{2}}\;i$

$k=1: \quad w_1 = 5^{\frac{1}{4}}\,\text{cis}\left(\frac{\theta}{2}+\pi\right) = -5^{\frac{1}{4}}\,\text{cis}\,\frac{\theta}{2} = -w_0$

$$w_1 = -\sqrt{\frac{\sqrt{5}+2}{2}} - \sqrt{\frac{\sqrt{5}-2}{2}}\;i$$

Alternative solution: Need to find z such that $z^2 = 2+i$. Let $z = x+iy$. Then

$(x+iy)^2 = 2+i \iff x^2 - y^2 = 2, \quad 2xy = 1$

$y = \frac{1}{2x}$, $x^2 - \frac{1}{4x^2} = 2$, $4x^4 - 1 = 8x^2$

$4x^4 - 8x^2 - 1 = 0$, $x^2 = \frac{4 \pm \sqrt{16+4}}{4} = \frac{4 \pm 2\sqrt{5}}{4}$

$x^2 = 1 \pm \frac{\sqrt{5}}{2}$, $x^2 \geq 0 \Rightarrow x^2 = 1 + \frac{\sqrt{5}}{2}$

$x = \pm\sqrt{1 + \frac{\sqrt{5}}{2}} = \pm\sqrt{\frac{\sqrt{5}+2}{2}}$, $y = \frac{1}{2x} = \pm\frac{1}{2}\sqrt{\frac{2}{\sqrt{5}+2}}$

$y = \pm\frac{1}{2}\sqrt{\frac{2}{\sqrt{5}+2}} \cdot \frac{\sqrt{\sqrt{5}-2}}{\sqrt{\sqrt{5}-2}} = \pm\frac{1}{2}\sqrt{\frac{2(\sqrt{5}-2)}{5-4}} = \pm\sqrt{\frac{\sqrt{5}-2}{2}}$

- If $z = r \operatorname{cis}\theta$ then $\bar{z} = r \operatorname{cis}(-\theta)$

Ex $\left(\frac{\sqrt{3}-i}{\sqrt{3}+i}\right)^{12} = \left(\frac{2\operatorname{cis}(-\frac{\pi}{6})}{2\operatorname{cis}(\frac{\pi}{6})}\right)^{12} = \left(\operatorname{cis}\left(-\frac{\pi}{3}\right)\right)^{12} = \operatorname{cis}(-4\pi) = 1$

- The n^{th} root of unity

Solutions of $z^n = 1$, $n \in \mathbb{N}$, are called the n^{th} root of unity

$1 = \operatorname{cis} 0 \Rightarrow z = \operatorname{cis}\frac{2\pi k}{n}$, $k = 0, 1, 2, \ldots, n-1$

Set $\omega = \operatorname{cis}\frac{2\pi}{n}$. Then ω is an n^{th} root of unity.

The complete set of n^{th} roots of unity is
$1, \omega, \omega^2, \ldots, \omega^{n-1}$
Geometrically they represent n vertices of a regular polygon of n sides inscribed into the unit circle $|z| = 1$ with one of the vertices at the point $z = 1$.

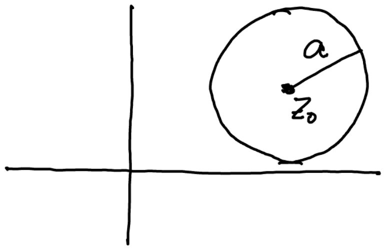
Ex $z^6 = 1 \Rightarrow \omega = \operatorname{cis}\frac{\pi}{3}$
All roots are $1, \operatorname{cis}\frac{\pi}{3}, \operatorname{cis}\frac{2\pi}{3}, \operatorname{cis}\pi, \operatorname{cis}\frac{4\pi}{3}, \operatorname{cis}\frac{5\pi}{3}$

- Sets in the Complex Plane

A complex number z represents a point in the complex z-plane ($\mathbb{C}$-plane)

- Circle : Let z_0 be a fixed complex number and a be a fixed positive real number.

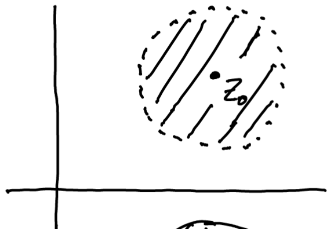
Then $|z - z_0| = a$ is the equation of a circle of radius a centered at z_0.



- Disks

$|z - z_0| < a$ is the equation of an open disk.

$|z - z_0| \leq a$ is the equation of a closed disk.



open disk

closed disk

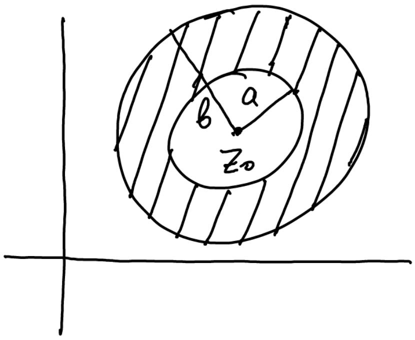
- Annulus

$a \leq |z - z_0| \leq b$

$a < |z - z_0| \leq b$

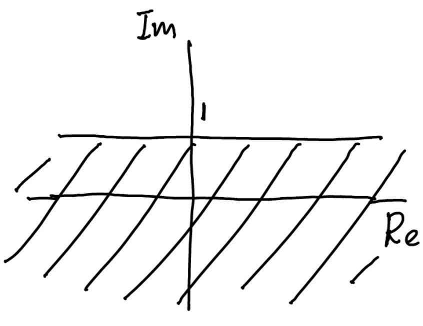
$a \leq |z - z_0| < b$

$a < |z - z_0| < b$

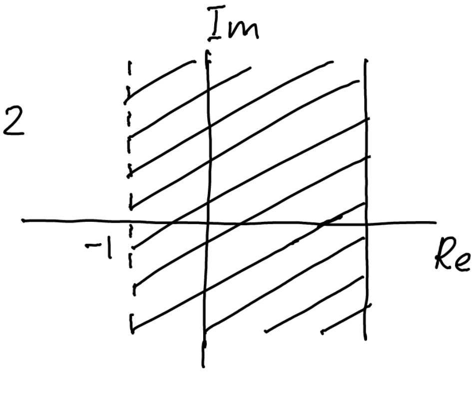


- Locus

$\operatorname{Im} z \leq 1$



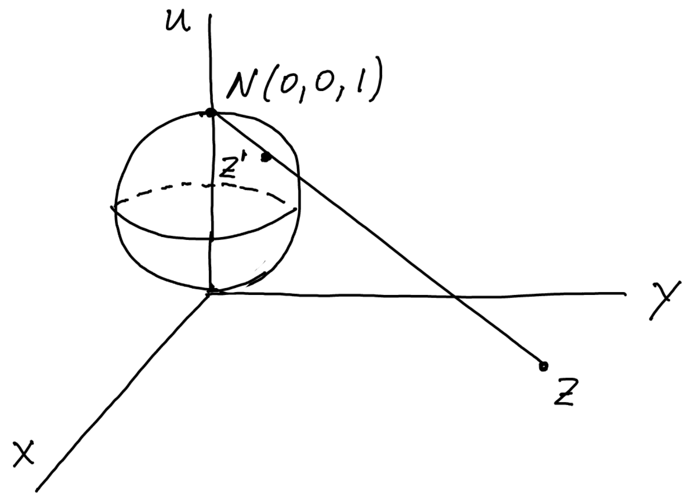
$-1 < \operatorname{Re} z \leq 2$



- A neighborhood of radius r of a point z_0 is the set of all points inside the circle $|z-z_0|=r$. These points satisfy the inequality $|z-z_0|<r$.
- A deleted neighborhood of z_0 of radius r is the set of all points z satisfying $0<|z-z_0|<r$
- An open set (or domain) is one every point of which has a neighborhood that entirely lies inside the set.
- A boundary of a set is a collection of points whose every neighborhood contains at least one point of the set and one point not belonging to the set.
- A closed set is a set that contains all its boundary points
- An empty set has no points (aka null set).
- A connected set is one in which any two points can be joined by a path that entirely lies in the set.
- A domain is an open conneted set.
- A region is a domain that may contain some of the boundary points.
- A set is called bounded if it can be enclosed by a circle of a finite radius. Otherwise the set is called unbounded. As an example : $\operatorname{Re} z > 3$

- Infinity. Riemann number sphere. Stereographic Projection.

The Riemann number sphere is a sphere in the xyu-space, where xy-plane is z-plane. It has radius $\frac{1}{2}$ and center at the point $x=0, y=0, u=\frac{1}{2}$. Its pole N has coordinates $x=0$, $y=0$, $u=1$



There is a one-to-one correspondence between all points on the z-plane and point on the sphere except N. To make this correspondens for any complex number z in the z-plane we draw a line that connects it with N. Then the line intersect the sphere at exactly one point, that we label z'. z' is the projection of z on the sphere. Then N is a projection of infinity ($z=\infty$) in the complex plane.