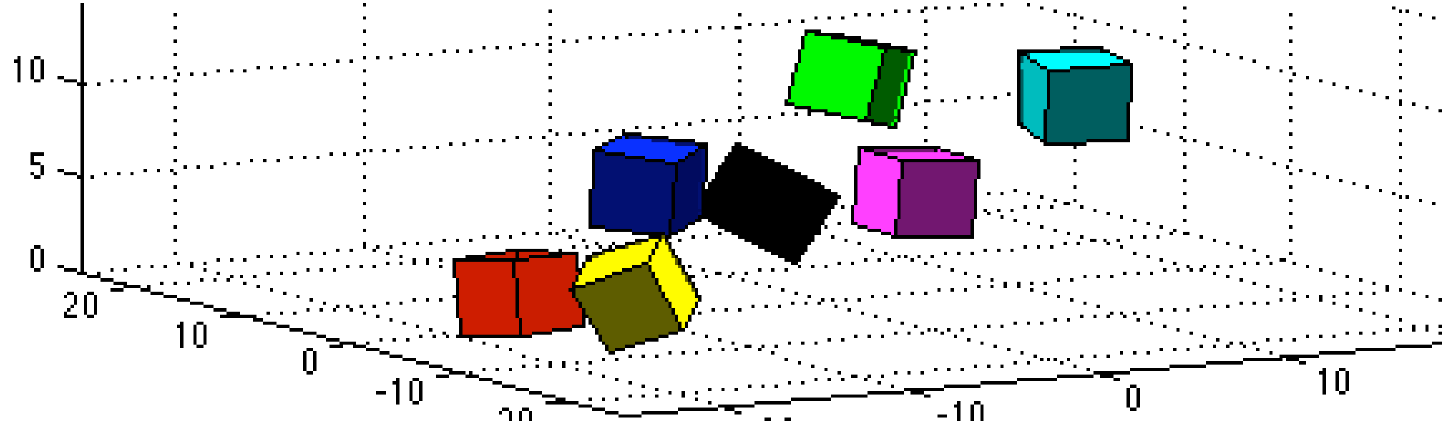
# A Linear Complementarity Time-Stepping Scheme for Rigid Multibody Dynamics with Nonsmooth Shapes.



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# Introduction

Nonsmooth rigid multibody dynamics (NRMD) methods attempt to predict the position and velocity evolution of a group of rigid particles subject to certain constraints and forces. Simulating the dynamics of a system with several rigid bodies with joint, contact, and friction constraints is important part in many areas,

- granular and rock dynamics
- masonry stability analysis
- simulation of concrete obstacle response to explosion
- tumbling mill design
- interactive virtual reality
- robot simulation and design

# Advantages of LCP Based Approaches

- We have an existing solution for any choice of parameters.
- We do not suffer from the lack of a solution that can happen in piecewise DAE and acceleration-force LCP approach.
- We can quickly and easily detect collisions and interpenetrations without suffering from artificial stiffness introduced by some methods, like penalty approach methods.
- We can implement a stable approach like current trend that can produce a linear complementarity scheme for nonsmooth shapes for large fixed time-step.

# Nonsmooth shapes problems

- Previous methods have been formulated only for differentiable shapes, and thus simulations must be stopped any time a nonsmooth point is crossed.
- If we had an easy way to compute the depth of penetration, we could proceed very easily with much the same anaylsis.
- We must reconsider our metric so that we can accommodate nonsmooth convex bodies.

# Euclidean Distance: Good, Bad, & Ugly

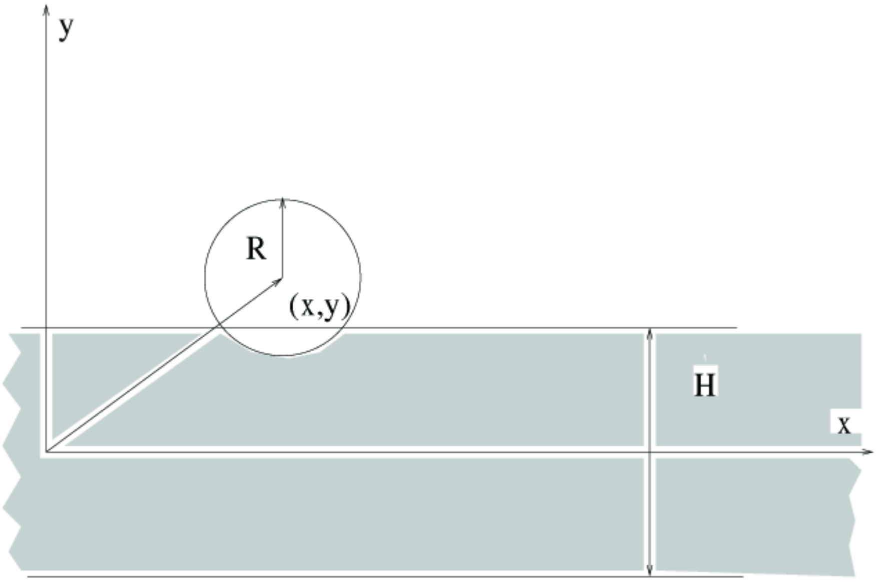


Figure 1: Euclidean Distance and Penetration

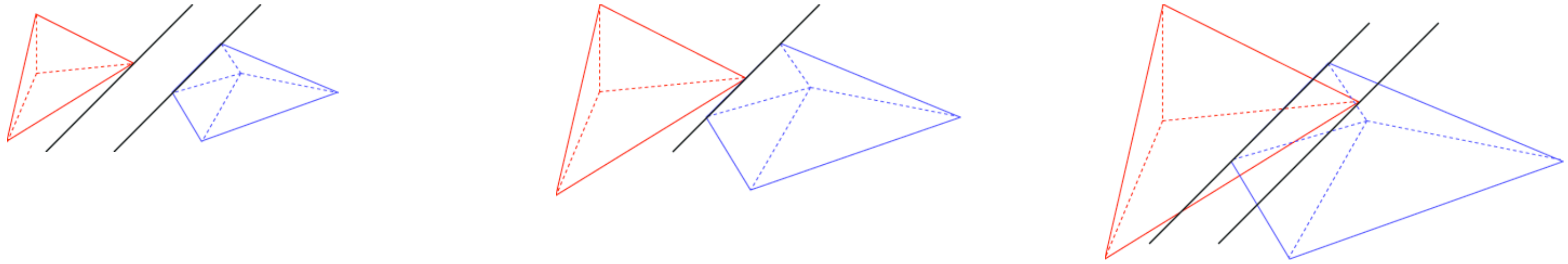
Good: Computes the distance between separated objects.

Bad: No description of extent of penetration, when it exists.

Ugly: We need to be able to determine depth of penetration.

# Signed Distance and Noninterpenetration

- Euclidean distance can detect contact, but not interpenetration
- Signed distance can detect contact and interpenetration



We need a signed distance function that satisfies these properties for convex bodies $P_1$ and $P_2$:

1. $P_1$ and $P_2$ interpenetrate if and only if $dist(P_1, P_2) < 0$,
2. $P_1$ and $P_2$ do not intersect if and only if $dist(P_1, P_2) > 0$,
3. $P_1$ and $P_2$ are in contact if and only if $dist(P_1, P_2) = 0$.

# Penetration Depth (PD)

**Definition 1** *Let $P_i$ be a convex polyhedron for $i$ = 1,2. The penetration depth between the two bodies $P_1$ and $P_2$ is defined formally as*

$$PD(P_1, P_2) = min\{||d||\,|interior(P_1 + d) \bigcap P_2 = \emptyset\}. \tag{1}$$

- *This is, more precisely, the Minkowski Penetration Depth.*
- *Note that we can slightly alter the Penetration Depth as defined to be a signed distance between two convex polyhedra.*

# Problems Computing PD

- The worst case deterministic scenario in computing the PD using Minkowski sums has complexity $O(m^2 + n^2)$
- On the other hand, Agarwal et. al. have an produced a stochastic method for approximating the PD of complexity of about $O(m^{3/4+\epsilon} n^{3/4+\epsilon})$ for any $\epsilon > 0$.

Question: Is there a faster deterministic method using some metric which is similar (at least in the limit) to the PD?

# Ultimate Goal

- We want to define a new measure that
  - C1: Easily detects collision and penetration of two convex bodies,
  - C2: Is computationally efficient,
  - C3: Is computationally fast, and
  - C4: Is metrically equivalent to the signed Euclidean distance when close to a contact.
- We want to use this measure to simulate polyhedral multibody contact problems.

# Expansion and Contraction of Convex Polyhedra

**Definition 2** *We define CP(A, b, $x_o$) to be the convex polyhedron P defined by the linear inequalities $Ax \leq b$ with an interior point $x_o$. We will often just write $P =$ CP(A, b, $x_o$).*

***Definition 3*** *Let P = CP(A, b, $x_o$). Then for any nonnegative real number t, the expansion (contraction) of P with respect to the point $x_o$ is defined to be*

$$P(x_o, t) = \{x | Ax \leq tb + (1-t)Ax_o\}$$

# Example of Expansion

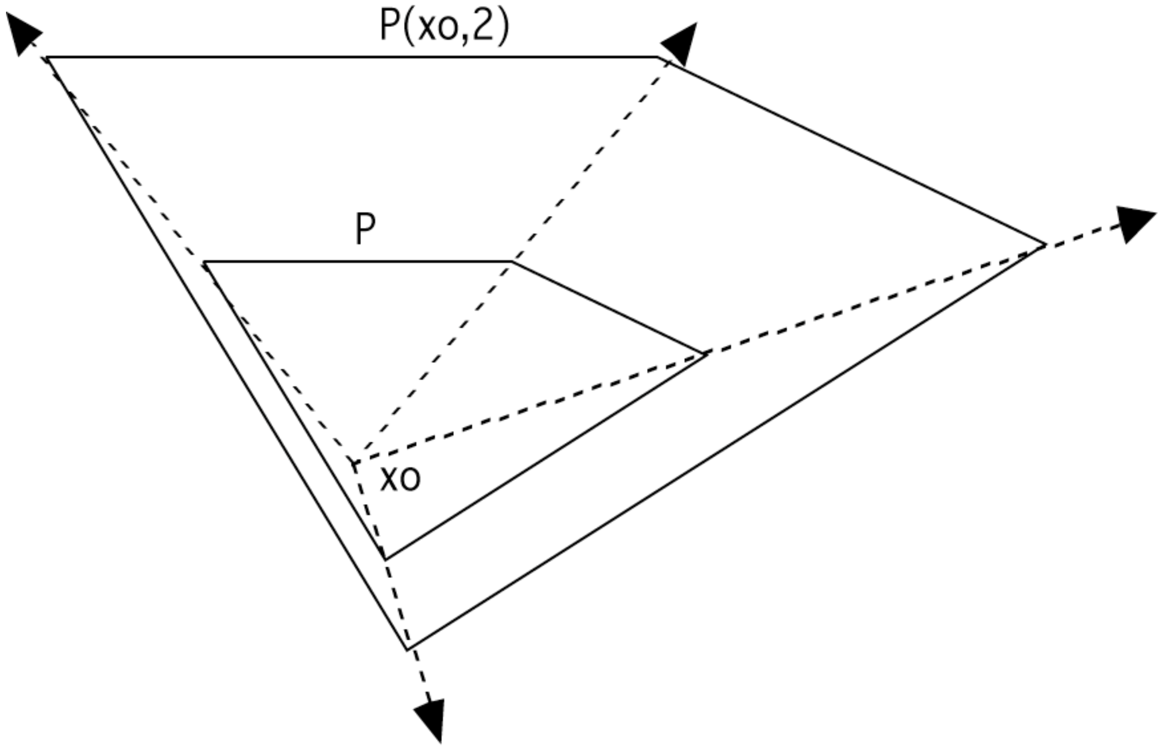


Figure 2: Demonstration of growth

# Our New Signed Distance: Ratio Metric

**Definition 4** *Let* $P_i = CP(A_i, b_i, x_i)$ *be a convex polyhedron for i = 1,2. Then the ratio measure between the two sets is given by*

$$\hat{r}(P_1, P_2) = min\{t | P_1(x_1, t) \bigcap P_2(x_2, t) \neq \emptyset\}, \tag{2}$$

*and the corresponding ratio metric is given by*

$$\rho(P_1, P_2) = \frac{\hat{r}(P_1, P_2) - 1}{\hat{r}(P_1, P_2)}. \tag{3}$$

# Ratio Metric Equivalence to PD

**Theorem 5** *Let $P_i$ be a convex polyhedron for $i = 1,2$, and $s$ be the Penetration Depth signed distance between the two bodies. Then there exists nonnegative constants $c_1$ and $c_2$ such that the ratio metric between the two sets satisfies the relationship*

$$\frac{s}{c_1} \le \rho(P_1, P_2) \le \frac{s}{c_2} \tag{4}$$

# Alternative Formulation

Let $P_i = CP(A_i, b_i, x_i)$. The equation (2) of Definition 4 can be written as

$$\hat{r}(P_1, P_2) = min\{t | A_i x \leq t b_i + (1 - t) A_i x_i, i = 1, 2\}. \tag{5}$$

- We have an easy way to compute the Ratio Metric because this formulation is a linear program in a 3-dimensional space !!!

# Ratio Metric Measures Up

C1: We can handle convex polyhedral bodies with this elegant, yet simple way to detect collision and penetration.

C2: This metric has simplicity, in that it only involves solving a linear programming problem, which is well known.

C3: Since our resulting linear program has a primal space of dimension 3, our Ratio Metric has complexity $O(m+n)$. (N. Megiddo)

C4: It follows from Theorem 5 that the Ratio Metric is ”metrically equivalent” to the signed Euclidean distance.

# Ratio Metric Advantages

- Distances with our metric are easier to calculate than using the Penetration Distance
- Distances with our metric are faster to calculate than using the Penetration Distance.
- We can theoretically solve rigid multibody contact problems of arbitrarily fixed dimension k.
- Because the Ratio Metric is ”metrically equivalent” to the signed Euclidean distance, we will get the same solution as the time-steps asymptotically approach zero.

# Detecting Collisions

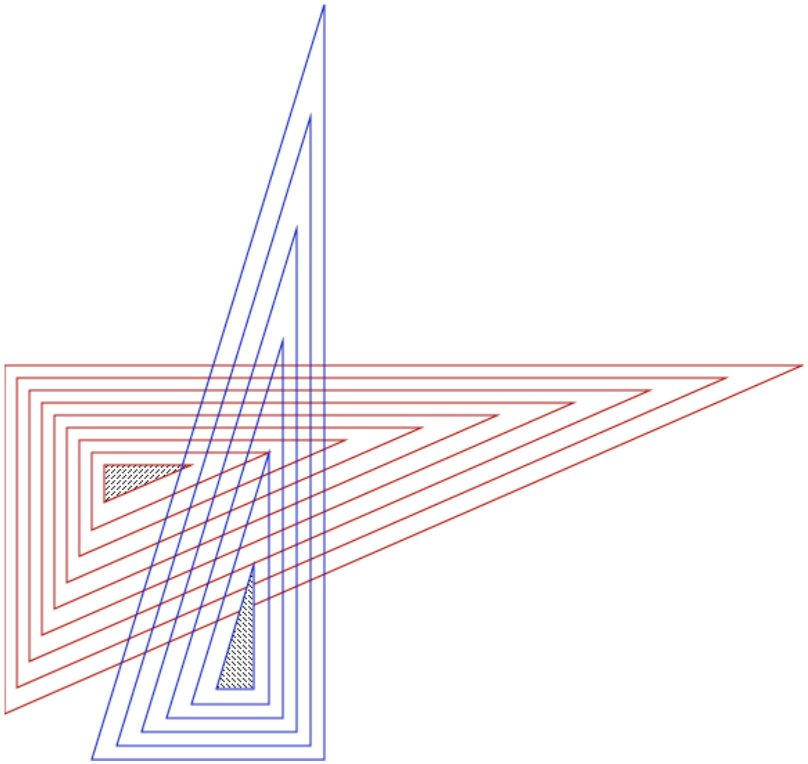
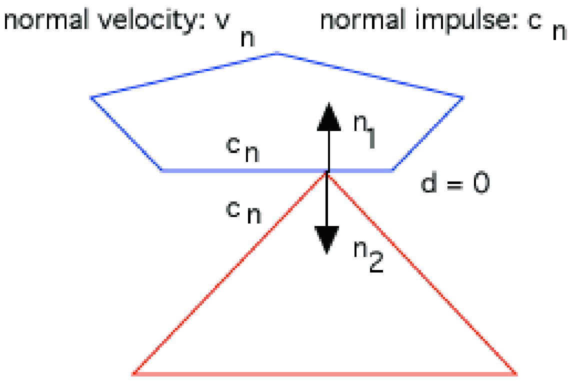


Figure 3: Visual representation of expansion or contraction

We expand or contract the two bodies until we reach contact.

# We Use Velocity-Impulse LCP-Based Approach

- Advantages
  - Solution exists for any choice of parameters
  - Does not suffer from artificial stiffness
  - Solves only one linear complementarity problem per step
- Disadvantages
  - Subproblem becomes harder because it has hard constraints



# Contact Model

- Contact configuration described by the (signed) distance function $d = \Phi(q)$, which is defined for some values of the interpenetration. Feasible set: $\Phi(q) \geq 0$.
- Contact forces are compressive, $c_n \geq 0$.
- Contact forces act only when the contact constraint is exactly satisfied, or

$$\Phi(q) \text{ is complementary to } c_n \quad \text{or } \Phi(q)c_n = 0, \quad \text{or } \Phi(q) \perp c_n.$$

# Noninterpenetration and Complementarity

The contact must be active before a nonzero compression impulse can act. Therefore, we have:

- $\Phi^{(j)}(q) \geq 0, \quad j = 1, 2, \ldots, p$
- $c_n^{(j)} \geq 0, \quad j = 1, 2, \ldots, p$
- $\Phi(q)^T c_n = 0$

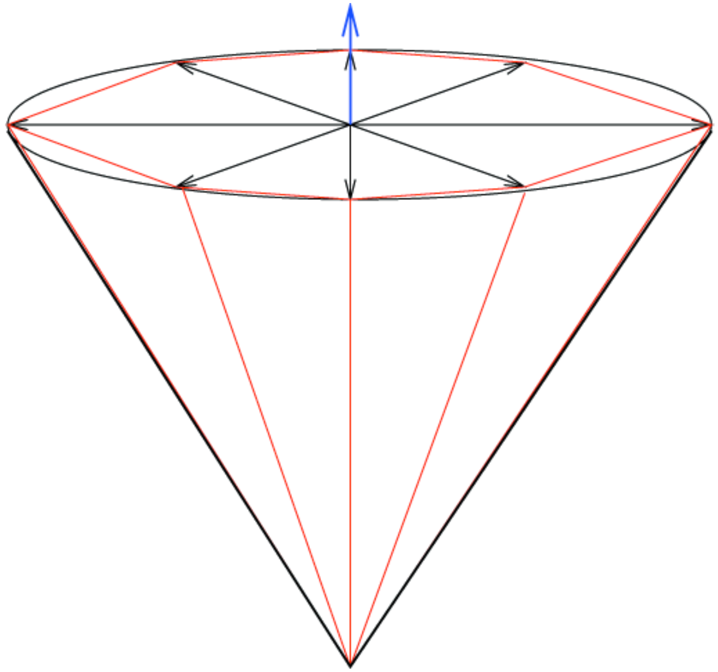
This can be expressed by the complementarity constraint

$$\Phi^{(j)}(q) \geq 0 \perp c_n^{(j)} \geq 0, \quad j = 1, 2, \ldots, p. \tag{6}$$

# Discretized Friction Model

- Discretized Constraints: The set $\widehat{D}(q)\beta$ where $||\beta|| \leq \mu c_n$ is approximated by a polygonal convex subset: $D(q)\tilde{\beta}$, $\tilde{\beta} \geq 0$.
- Conic constraints: $\left|\left|\tilde{\beta}\right|\right|_1 \leq \mu c_n$.
- Frictional Constraints: $\beta = \operatorname{argmin}_{\widehat{\beta} \geq 0} v^T D(q)\widehat{\beta}$

Polygonal cone approximation to the Coulomb cone ( 3D).



# Coulomb Friction Model and Complementarity

- Tangential Generators: $\widetilde{D}(q) = [D^{(j_1)}(q), D^{(j_2)}(q), \ldots, D^{(j_s)}(q)]$
- Tangential Symmetry: $d_i^{(j)} \in \mathcal{D}^j \implies -d_i^{(j)} \in \mathcal{D}^j$
- Conic constraints: $||\beta|| \leq \mu c_n$, with $\mu$ friction coefficient.
- Max Dissipation Constraints: $\beta = \mathrm{argmin}_{||\widetilde{\beta}|| \leq \mu c_n} v^T \widetilde{D}(q) \widetilde{\beta}$.
- Retarding Constraint: Tangential velocity satisfies $|v_T| = \lambda = -v^T \widetilde{D}(q) \frac{\beta}{||\beta||}$.

$$\begin{aligned} D^{(j)^T}(q)v + \lambda^{(j)} e^{(j)} &\geq 0 \quad \perp \quad \beta^{(j)} \geq 0, \\ \mu c_n^{(j)} - e^{(j)^T} \beta^{(j)} &\geq 0 \quad \perp \quad \lambda^{(j)} \geq 0. \end{aligned} \tag{7}$$

# Notation and Definitions

**Joint Constraints** are described by the equations

$$\Theta^{(i)}(q) = 0,\ i = 1, 2, \ldots, m$$

for sufficiently smooth functions $\Theta^{(i)}(q)$, with gradients

$$\nu^{(i)}(q) = \nabla_q \Theta^{(i)}(q), \quad i = 1, 2, \ldots, m.$$

**Noninterpenetration constraints** are still written as

$$\Phi^{(j)}(q) \geq 0, \quad j = 1, 2, \ldots, p.$$

but now we will use our ratio metric.

**Infeasibility** of the constraints are measured by

$$I(q) = \max_{1 \leq j \leq p, 1 \leq i \leq m} \left\{ \Phi_{-}^{(j)}(q), |\Theta^{(i)}(q)| \right\}.$$

## Choosing the active set $\mathcal{A}$ and detecting collision

$$\mathcal{A}(q) = \left\{ j | \Phi^{(j)}(q) \leq \hat{\epsilon},\ 1 \leq j \leq p \right\}. \tag{8}$$

- No need to stop the simulation if $\hat{\epsilon}$ is appropriately chosen
- A good guideline for this choice is $\hat{\epsilon} = v_{\mathrm{max}} h$, where $h$ is of the order of the expected size of the timestep and $v_{max}$ is the expected range of the velocity

# Defining an Event

Let $\Phi^{(j)}(q) = 0$ for some position q. Then

- Two bodies $j_1$ and $j_2$ are in contact.
- Contact regions are convex combinations of extreme points.
- Extreme points are convex combinations of simple ”events”.
  - In 2D, corner-on-face intersections
  - In 3D, corner-on-face or edge-on-edge intersections
- Each ”event” produces a consistent system of equations.
  - In 2D we get a 3 x 3 system
  - In 3D we get a 4 x 4 system

# Choosing the Normal Vectors

1. Get the active set $\mathcal{A}$.
2. For each index $j \in \mathcal{A}$, locate the proximity events
    - Proximity events are events for $\hat{\Phi}(q) = \Phi(q) - \delta$ for some $\delta < \hat{\epsilon}$ and whose corresponding points are close to being within both bodies $j_1$ and $j_2$.
    - For each index $j \in \mathcal{A}$, there are $N_j$ proximity events
3. For each proximity event, $j_k$, use the Implicit Function Theorem to find the gradient of the ratio metric, which we call the normal vector $n^{(j_k)}$ at event $j_k$.

# Constraint Linearization

- Enforce geometric constraints at the velocity level by linearization of the mappings $\Theta^{(i)}$ and $\Phi^{(j)}$.
  - Linearization of joint constraints:

$$\nu^{(i)^T}(q^{(l)})v^{(l+1)} + \frac{\Theta^{(i)}(q^{(l)})}{h_l} = 0, \quad i = 1, 2, \ldots, m.$$

  - Linearization of noninterpenetration constraints:

$$n^{(j_k)^T}(q^{(l)})v^{(l+1)} + \frac{\Phi^{(j)}(q^{(l)})}{h_l} \geq 0 \perp c_n^{(j)} \geq 0,\ 1 \leq j_k \leq N_j$$

- Linearization allows us to proceed with fixed time steps in spite of collisions.

# LCP Form of the Integration Step

We now rewrite the Time-Stepping Equations into the LCP form:

$$\begin{array}{lllllll}
M^{(l)}v^{(l+1)} & -\widetilde{\nu}c_\nu & -\widetilde{n}c_n & -\widetilde{D}\beta & & = -q^{(l)} & & \\
\widetilde{\nu}^T v^{(l+1)} & & & & & = -\Upsilon & & \\
\widetilde{n}^T v^{(l+1)} & & & & & \geq -\Delta & \perp & c_n \geq 0 \\
\widetilde{D}^T v^{(l+1)} & & & & +\widetilde{E}\lambda & \geq 0 & \perp & \beta \geq 0 \\
 & & \widetilde{\mu}c_n & -\widetilde{E}^T\beta & & \geq 0 & \perp & \lambda \geq 0
\end{array} \tag{9}$$

where, $q^{(l)} = -Mv^{(l)} - h_l k^{(l)}$

# Constraint Stabilization

Let the time-stepping algorithm be applied over a finite time interval $[0, T]$, with time steps $h_l > 0$ satisfying

$$\sum_{i=0}^{N-1} h_l = T \; and \; \frac{h_{l-1}}{h_l} \leq c_h, i = 1, 2, ..., N-1.$$

If the system is initially feasible, then, under modest conditions, there exist positive constants $H, V$, and $C_c$ such that

1. $\left|\left|v^{(l)}\right|\right| \leq V, \; for \; 1 \leq l \leq N$ and
2. $I(q^{(l)}) \leq C_c \left|\left|v^{(l)}\right|\right|^2 h_{l-1}^2, \; for \; 1 \leq l \leq N.$

Note: This result has been proven to hold for smooth bodies. We expect to show that it also holds for piecewise smooth bodies.

# Results to Date

1. We have successfully implemented our procedure into a time-stepping approach for rigid multibody dynamics with joints, contact and friction.
2. Our method has been applied with 4 orders of magnitude larger time steps than other methods.
3. We have successfully simulated a true 3-dimensional problem.

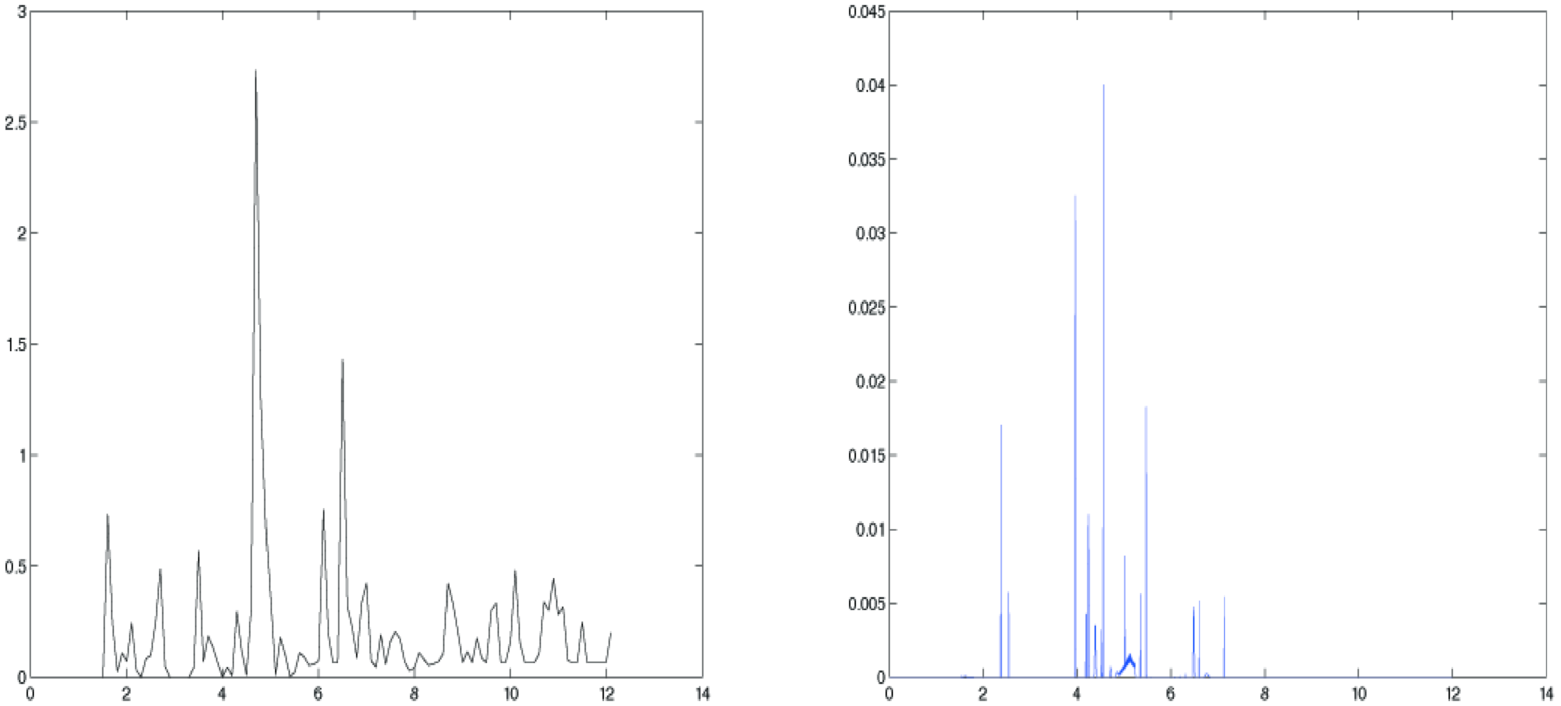


Figure 4: Feasibility Comparison - Time steps 0.1 vs 0.01

The error seems to decrease by a factor of 100, which is consistent with the $O(h^2)$ behavior in our Constraint Stabilization Theorem.

# Conclusions and Future Work

- We define a method for polyhedral multibody dynamics for contacts, joints, and friction while solving only linear complementarity problem per step.
- Our method does not need to stop and detect collisions explicitly and can advance with a constant time step and predictable amount of effort per step.
- Future work: We want to explicitly show that our method achieves constraint stabilization.

## Thanks

- **Mihai Anitescu**, Argonne National Laboratory
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