A Hard Constraint Time-Stepping Approach for Rigid Multibody Dynamics with Joints, Contact and Friction

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Goal

- Develop method for simulating rigid multibody dynamics with joints, contact, and friction
- Require the solution of only one linear complementarity problem (LCP) per step
- Progress at much larger time steps than explicit penalty methods

Introduction

Nonsmooth rigid multibody dynamics (NRMD) methods attempt to predict the position and velocity evolution of a group of rigid particles subject to certain constraints and forces.

- **non-interpenetration**
- **collision**
- adhesion
- **dry friction**
- global forces
  - electrostatic
  - gravitational

Previous Numerical Schemes for NRMD

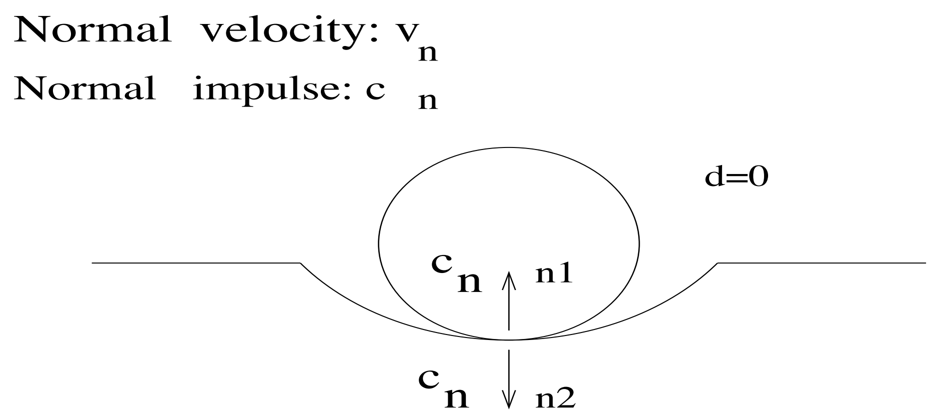
- Piecewise DAE
  - No proof of consistency
  - Collisions may produce near zero timestep
- Acceleration-force linear complementarity problem
  - Cannot handle Panleve paradoxes (no classical acceleration)
- Penalty (or regularization)
  - Stiff problem may cause near zero timestep
- Velocity-impulse LCP-based time-stepping

We Use Velocity-Impulse LCP-Based Approach

- Advantages
  - Solution exists for any choice of parameters
  - Does not suffer from artificial stiffness
  - Solves only one linear complementarity problem per step
- Disadvantages
  - Subproblem becomes harder because it has hard constraints

Model Requirements and Notations

- MBD system : generalized positions q and velocities v. Dynamic parameters: mass $M(q)$ (positive definite), external force $k(t, q, v)$.
- Non interpenetration constraints: $\Phi^{(j)}(q) \geq 0,\ 1 \leq j \leq n_{total}$ and compressive contact forces at a contact.
- Joint (bilateral) constraints: $\Theta^{(i)}(q) = 0,\ 1 \leq i \leq m$.
- Frictional Constraints: Coulomb friction, for friction coefficients $\mu^{(j)}$.
- Dynamical Constraints: Newton laws, conservation of impulse at collision.



Contact Model

- Contact configuration described by the (signed) distance function $d = \Phi(q)$, which is defined for some values of the interpenetration. Feasible set: $\Phi(q) \geq 0$.
- Contact forces are compressive, $c_n \geq 0$.
- Contact forces act only when the contact constraint is exactly satisfied, or

$$\Phi(q) \text{ is complementary to } c_n \text{ or } \Phi(q)c_n = 0, \text{ or } \Phi(q) \perp c_n.$$

Coulomb Friction Model

- Tangent space generators: $\widehat{D}(q) = \left[\widehat{d}_1(q), \widehat{d}_2(q)\right]$, tangent force multipliers: $\beta \in R^2$, tangent force $\widehat{D}(q)\beta$.
- Conic constraints: $||\beta|| \leq \mu c_n$, where μ is the friction coefficient.
- Max Dissipation Constraints: $\beta = \text{argmin}_{||\widehat{\beta}|| \leq \mu c_n} v^T \widehat{D}(q)\widehat{\beta}$.
- v_T, the tangential velocity, satisfies $|v_T| = \lambda = -v^T \widehat{D}(q)\frac{\beta}{||\beta||}$. λ is the Lagrange multiplier of the conic constraint.

Coulomb Friction Model and Complementarity

- Discretized Tangent Force: The tangent force is approximated by a polygonal convex subset.
- Tangential Generators: $\widetilde{D}(q) = [D^{(j_1)}(q), D^{(j_2)}(q), \ldots, D^{(j_s)}(q)]$
- Conic constraints: $||\beta||_1 \leq \mu c_n$, with μ friction coefficient.
- Max Dissipation Constraints: $\beta = \text{argmin}_{||\widetilde{\beta}||_1 \leq \mu c_n} v^T \widetilde{D}(q) \widetilde{\beta}$.

$$
\begin{aligned}
D^{(j)^T}(q)v + \lambda^{(j)} e^{(j)} \quad & \geq 0 \quad \perp \quad \beta^{(j)} \geq 0, \\
\mu c_n^{(j)} - e^{(j)^T} \beta^{(j)} \quad & \geq 0 \quad \perp \quad \lambda^{(j)} \geq 0.
\end{aligned}
\tag{1}
$$

Modified Linearization

- Enforce geometric constraints at the velocity level by modified linearization of the mappings $\Theta^{(i)}$ and $\Phi^{(j)}$.
- For joint constraints the modified linearization leads to

$$\nu^{(i)^T}(q^{(l)})v^{(l+1)} + \gamma \frac{\Theta^{(i)}(q^{(l)})}{h_l} = 0, \quad i = 1, 2, \ldots, m \tag{2}$$

- For a noninterpenetration constraint of index j,

$$n^{(j)^T}(q^{(l)})v^{(l+1)} + \gamma \frac{\Phi^{(j)}(q^{(l)})}{h_l} \geq 0 \perp c_n^{(j)} \geq 0. \tag{3}$$

where γ is a user-defined parameter. If $\gamma = 1$, then we would achieve proper linearization.

Equations of Motion

- Continuous : Newton's Law

$$M(q)\frac{d^2q}{dt^2} - \sum_{i=1}^{m} \nu^{(i)} c_\nu^{(i)} - \sum_{j=1}^{p} \left(n^{(j)}(q) c_n^{(j)} + D^{(j)}(q)\beta^{(j)} \right) = k(t, q, \frac{dq}{dt})$$

- Descretized : Euler discretization of Newton's Law

$$M(q^{(l)}) \left(v^{(l+1)} - v^{(l)} \right) = h_l k \left(t^{(l)}, q^{(l)}, v^{(l)} \right) + \sum_{i=1}^{m} c_\nu^{(i)} \nu^{(i)}(q^{(l)}) \quad + \sum_{j \in \mathcal{A}} \left(c_n^{(j)} n^{(j)}(q^{(l)}) + \sum_{i=1}^{m_C^{(j)}} \beta_i^{(j)} d_i^{(j)}(q^{(l)}) \right)$$

Time-stepping Equations

$$M(v^{l+1} - v^{(l)}) - \sum_{i=1}^{m} \nu^{(i)} c_\nu^{(i)} - \sum_{j\in\mathcal{A}} (n^{(j)} c_n^{(j)} + D^{(j)} \beta^{(j)}) = hk$$

$$\nu^{(i)^T} v^{l+1} = -\Upsilon^{(i)}, \qquad i = 1..m$$

$$\rho^{(j)} = n^{(j)^T} v^{l+1} \geq -\Delta^{(j)}, \quad \textbf{compl. to} \quad c_n^{(j)} \geq 0, \quad j \in \mathcal{A}$$

$$\sigma^{(j)} = \lambda^{(j)} e^{(j)} + D^{(j)T} v^{l+1} \geq 0, \quad \textbf{compl. to} \quad \beta^{(j)} \geq 0, \quad j \in \mathcal{A}$$

$$\zeta^{(j)} = \mu^{(j)} c_n^{(j)} - e^{(j)^T} \beta^{(j)} \geq 0, \quad \textbf{compl. to} \quad \lambda^{(j)} \geq 0, \quad j \in \mathcal{A}.$$

Note: The time-stepping scheme has a solution although the classical formulation doesn't!

LCP Form of the Integration Step

We now rewrite the Time-Stepping Equations into the LCP form:

$$
\begin{array}{lllllll}
M^{(l)} v^{(l+1)} & -\widetilde{\nu} c_\nu & -\widetilde{n} c_n & -\widetilde{D}\beta & & = -q^{(l)} & & \\
\widetilde{\nu}^T v^{(l+1)} & & & & & = -\Upsilon & & \\
\widetilde{n}^T v^{(l+1)} & & & & & \geq -\Delta & \perp & c_n \geq 0 \\
\widetilde{D}^T v^{(l+1)} & & & & +\widetilde{E}\lambda & \geq 0 & \perp & \beta \geq 0 \\
 & & \widetilde{\mu} c_n & -\widetilde{E}^T \beta & & \geq 0 & \perp & \lambda \geq 0
\end{array}
\tag{4}
$$

where, $q^{(l)} = -Mv^{(l)} - h_l k^{(l)}$

Matrix Form of the Integration Step

$$\begin{bmatrix} M^{(l)} & -\widetilde{\nu} & -\widetilde{n} & -\widetilde{D} & 0 \\ \widetilde{\nu}^T & 0 & 0 & 0 & 0 \\ \widetilde{n}^T & 0 & 0 & 0 & -\tilde{\mu}^T \\ \widetilde{D}^T & 0 & 0 & 0 & \widetilde{E} \\ 0 & 0 & \widetilde{\mu} & -\widetilde{E}^T & 0 \end{bmatrix} \begin{bmatrix} v^{(l+1)} \\ c_\nu \\ c_n \\ \widetilde{\beta} \\ \lambda \end{bmatrix} + \begin{bmatrix} q^{(l)} \\ \Upsilon \\ \Delta \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \rho \\ \widetilde{\sigma} \\ \zeta \end{bmatrix} \quad (5)$$

$$\begin{bmatrix} c_n \\ \widetilde{\beta} \\ \lambda \end{bmatrix}^T \begin{bmatrix} \rho \\ \widetilde{\sigma} \\ \zeta \end{bmatrix} = 0, \qquad \begin{bmatrix} c_n \\ \widetilde{\beta} \\ \lambda \end{bmatrix} \geq 0, \qquad \begin{bmatrix} \rho \\ \widetilde{\sigma} \\ \zeta \end{bmatrix} \geq 0\,. \quad (6)$$

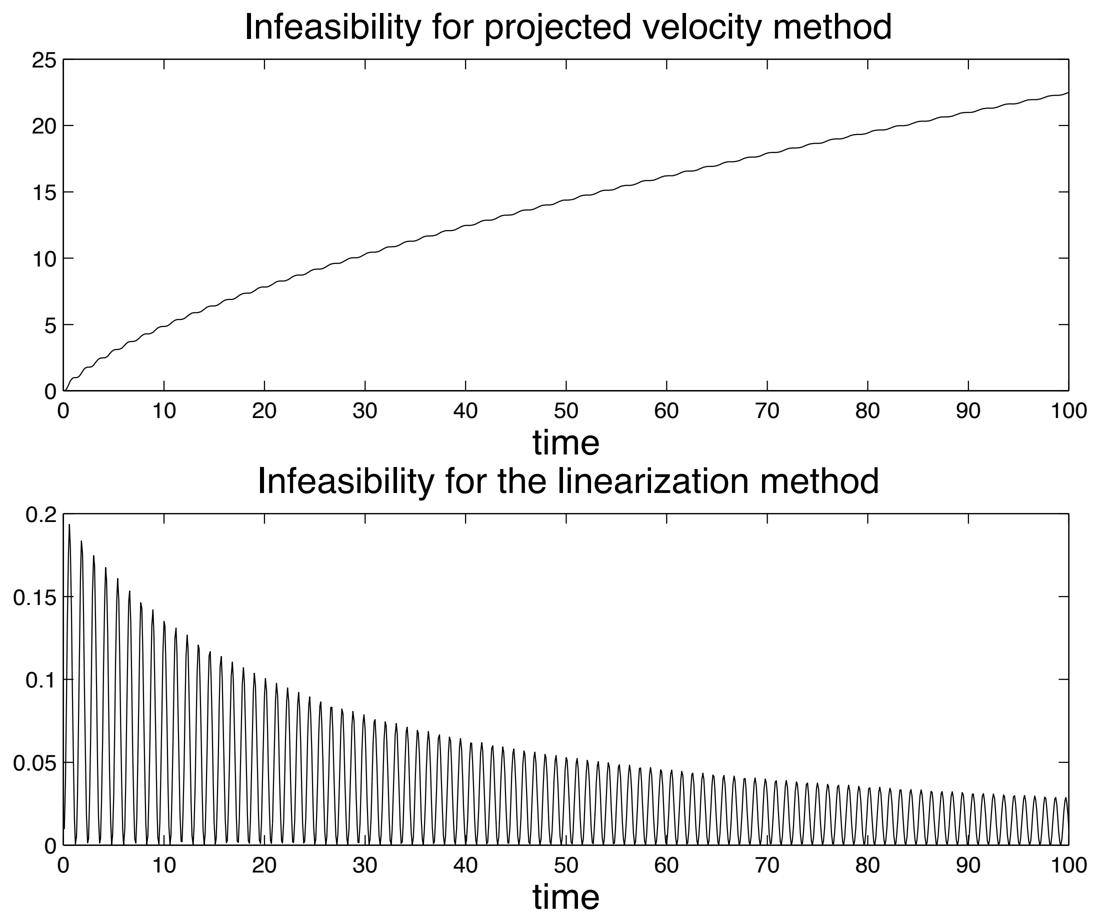
The Need for Constraint Stabilization

- The positions are updated by $q^{(l+1)} = q^{(l)} + h_l v^{(l+1)}$.
- When you do only index reduction ($\gamma = 0$), the (geometrical) joint and non interpenetration constraints, which define the feasible set

$$\mathcal{F} = \left\{ q \,\middle|\, \Theta^{(i)}(q) = 0,\ 1 \leq i \leq m, \Phi^{(j)}(q) \geq 0,\ 1 \leq j \leq n_{total} \right\}$$

  are replaced by constraints at the velocity level.
- This may create constraint drift, in which the constraint infeasibility keeps growing.
- In interactive simulation this is particulatly annoying, since geometrical inconsistency is easy to notice.

Example of Constraint Drift



Constraint error in the original method and a modified method for a pendulum example. Infeasibility ratio approaches 10^3!

Preventing Constraint Drift

- Change approach to a nonlinear (potentially nonconvex) complementarity problem.
- Perform a nonlinear projection after each LCP (and preserving the good energy properties (Anitescu and Potra 2002)).
- Perform one step of an SQP applied to the nonlinear projection problem (Cline and Pai, 2003). Extension of (Ascher, Chin, Reich 1994) from DAE. No analysis provided.
- Modify the right hand side of the LCP with an appropriate function of the infeasibility (parameter-free, (Jean, 1999,w/o analysis) and this work, (Anitescu and Hart 2002)) and (Miller and Christiansen 2002) and (Anitescu, Miller and Hart 2003).

Constraint Stabilization Analysis

- We define the feasible set:

$$\mathcal{F} = \left\{ q \,\middle|\, \Theta^{(i)}(q) = 0,\ 1 \le i \le m, \Phi^{(j)}(q) \ge 0,\ 1 \le j \le n_{total} \right\}$$

.

- Assume the geometrical data of the problem are twice continously differentiable in a neighborhood of $\mathcal{F}$.
- Measure constraint infeasibility using

$$I(q) = \max_{1 \le j \le p, 1 \le i \le m} \left\{ \Phi_{-}^{(j)}(q), \left|\Theta^{(i)}(q)\right| \right\}. \tag{7}$$

More Assumptions

- All encountered configurations have a *uniformly* pointed friction cone. (The friction cone is the set of all feasible constraint forces)
- Mass matrix $M^{(l)} = M(q^{(l)})$ is constant and positive definite
- External force satisfies

$$k(t, v, q) = k_0(t, v, q) + f_c(v, q) + k_1(v) + k_2(q), \tag{8}$$

and there exists $c_K \geq 0$ such that

$$||k_0(t, v, q)|| \leq c_K, \; ||k_1(v)|| \leq c_K \, ||v|| \, , \; ||k_2(q)|| \leq c_K \, ||q|| \tag{9}$$

Main Results

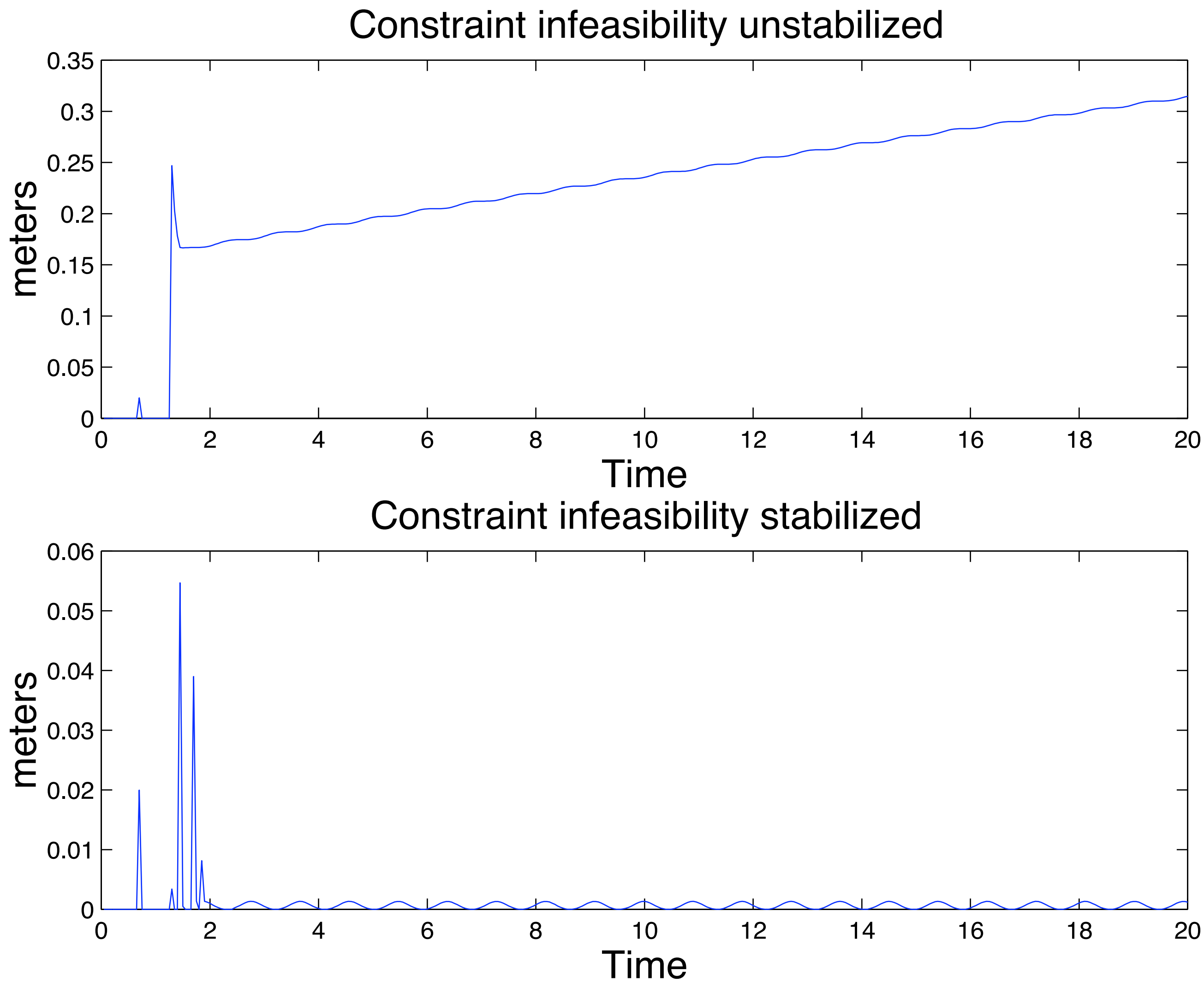
Let $I(q^{(0)}) = 0$. For fixed $\gamma \in (0,1]$ and time-steps satisfying $\frac{h_j}{h_l} \leq c_h$ for any $j \leq l$, where $c_h > 0$ is a fixed constant.

1. There exist V, H such that whenever $h_l \leq H$, then $v_l \leq V$, $\forall l$.
2. There exists C such that $I(q^l) \leq C\frac{1}{\gamma}\left(\max_{1\ldots l} h_l\right)^2 V^2$.
3. The velocity solution satisfies an energy bound

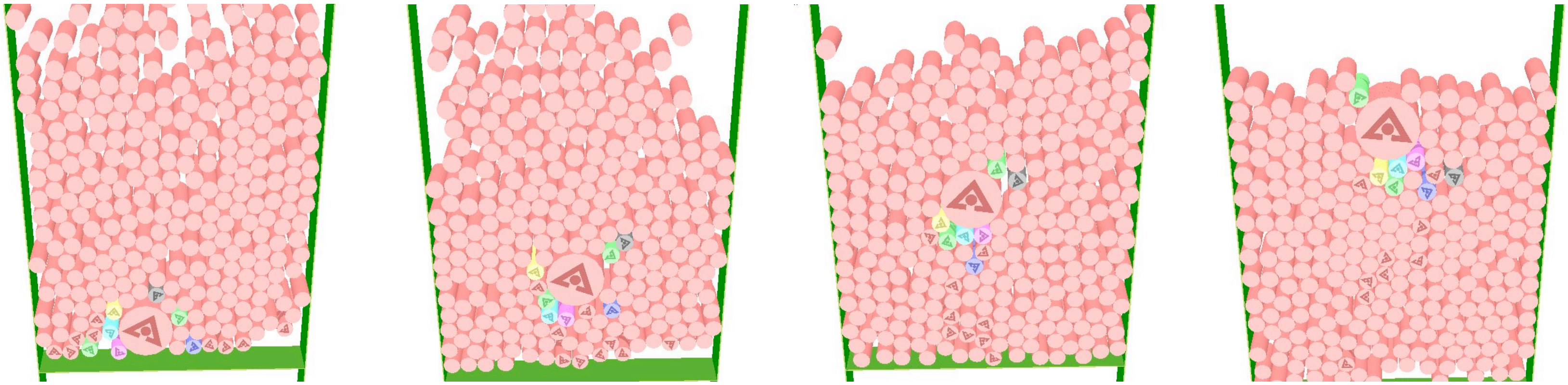
$$\begin{aligned} v^{(l+1)^T} M^{(l)} v^{(l+1)} &\leq v^{(l)} M^{(l)} v^{(l)} + h_l^2 k^{(l)} M^{(l)-1} k^{(l)} \\ &+2h_l v^{(l)^T} k^{(l)} + c(q^{(l)}, \tilde{\mu}, M^{(l)})^2 \left\| \Delta_{-}^{(l)}, \Upsilon^{(l)} \right\|_{\infty}^2 . \end{aligned} \tag{10}$$

- Results proven to hold for $\gamma = 1$.
- Results sketched for $0 < \gamma < 1$, to be proven in depth in thesis.

Infeasibility: Unstabilized vs Stabilized Method

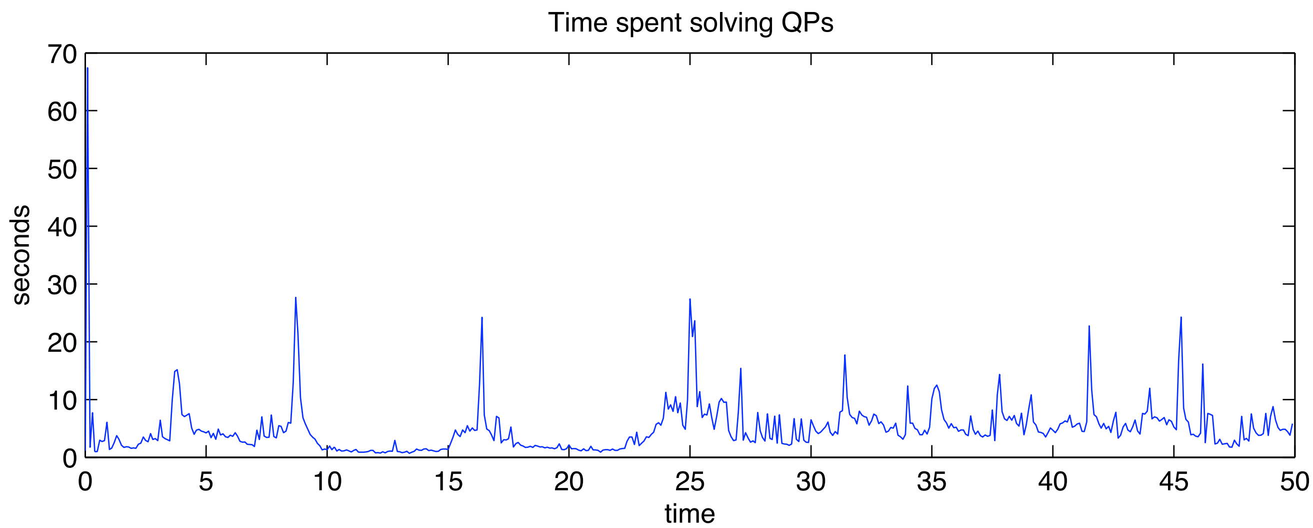


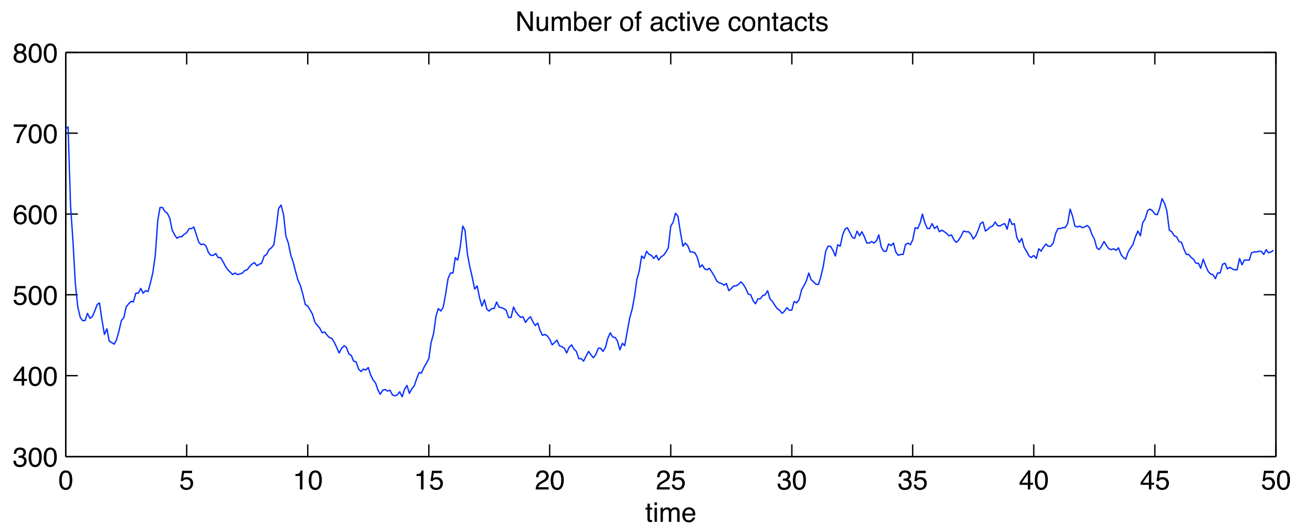
Brazil Nut Effect Simulation



- Time step of 100ms, for 50s. 270 bodies.
- Convex Relaxation Method. One QP/step. No collision backtrack.
- Friction is 0.5, restitution coefficient is 0.5.
- Large ball emerges after about 40 shakes. Results in the same order of magnitude as MD simulations (but with 4 orders of magnitude larger time step).

Brazil Nut Effect Simulations Performance





Conclusions and future work

- We define a method that achieves constraint stabilization while solving only linear complementarity problem per step.
- Our method does not need to stop and detect collisions explicitly and can advance with a constant time step and predictable amount of effort per step.
- The method has been extended to a parametric version that is used in a robotic grasp simulator (Miller and Christiansen, 2002) and (Anitescu, Miller and Hart, 2003).
- Future work: Nonsmooth particles and stabilization proof for nonzero coefficient of restitution. Convergence for relaxation scheme as $h \to 0$?

Recent Publications

- M. Anitescu and G. D. Hart. Solving nonconvex problems of multibody dynamics with contact and small friction by sequential convex relaxation. To appear in Mechanical Based Design of Machines and Structures.
- M. Anitescu and G.D.Hart. A constraint-stabilized time-stepping approach for rigid multibody dynamics with joints, contact and friction. Preprint ANL/MCS-1002-1002. Submitted to International Journal for Numerical Methods in Engineering.
- M. Anitescu and G.D. Hart. A fixed-point iteration approach for multibody dynamics with contact and small friction. Preprint ANL/MCS-P985-0802. To appear in Mathematical Programming B.
- M. Anitescu, A. Miller and G. D. Hart. Constraint stabilization for time-stepping approaches for rigid multibody dynamics with joints, contact and friction. Preprint ANL/MCS-P1023-0203. To appear in Proceeding of the ASME DETC 2003.

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