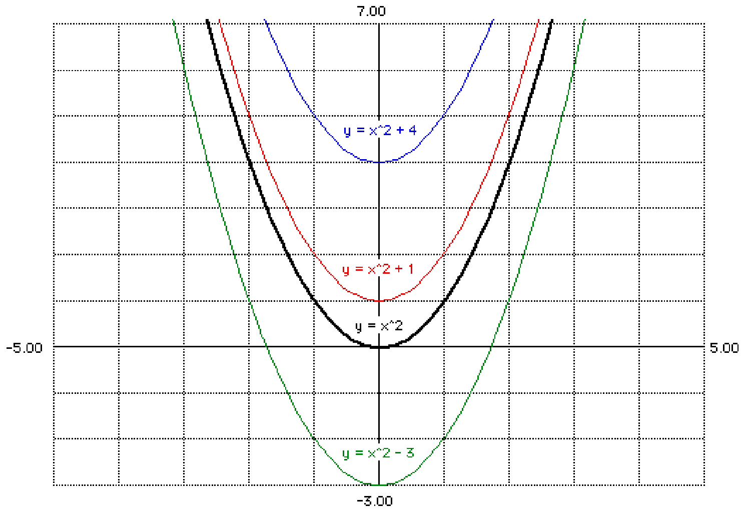
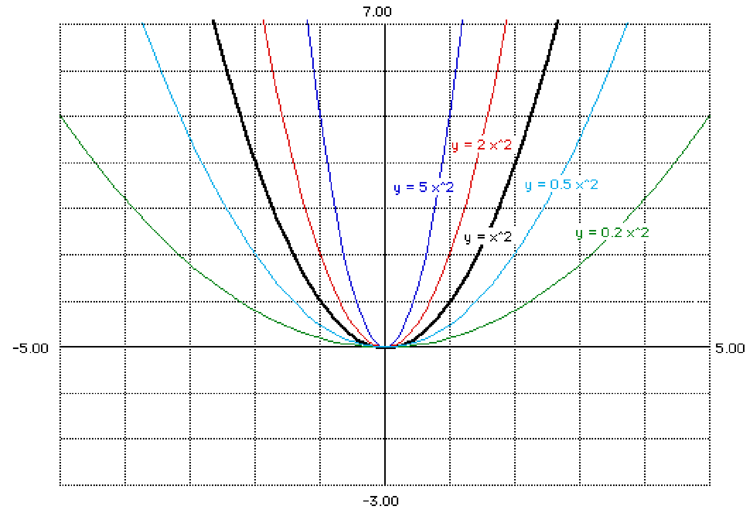
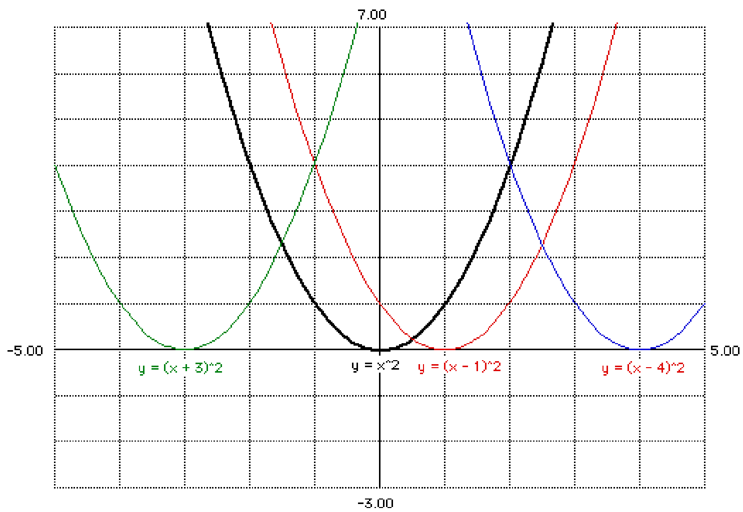
Graphs of Parabolas



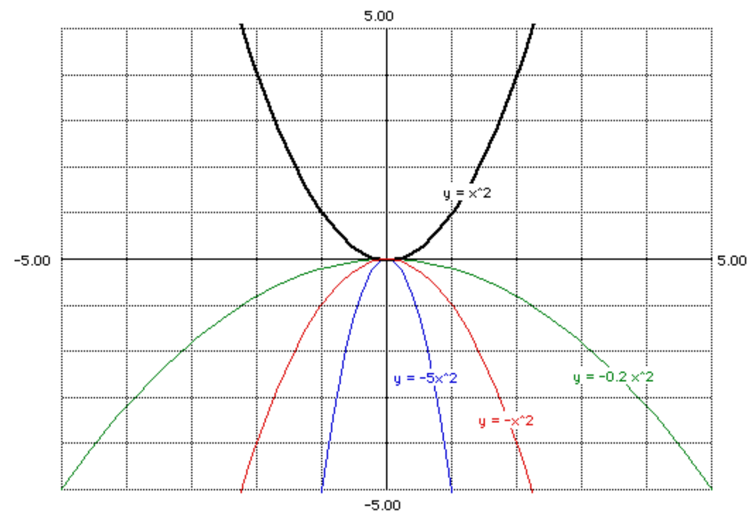
$y = x^2$	typical graph
$y = x^2 + 1$	typical graph moved up 1 unit
$y = x^2 + 4$	typical graph moved up 4 units
$y = x^2 - 3$	typical graph moved down 3 units



$y = x^2$	typical graph
$y = 2x^2$	typical graph slightly thinner
$y = 5x^2$	typical graph much thinner
$y = 0.5x^2$	typical graph slightly wider
$y = 0.2x^2$	typical graph much wider



$y = x^2$	typical graph
$y = (x - 1)^2$	typical graph moved right 1 unit
$y = (x - 4)^2$	typical graph moved right 4 units
$y = (x + 3)^2$	typical graph moved left 3 units



$y = x^2$	typical graph
$y = -2x^2$	typical graph inverted
$y = -5x^2$	typical graph much thinner, inverted
$y = -0.2x^2$	typical graph much wider, inverted

Graphing Quadratic Functions

A quadratic function is a function whose rule is a quadratic polynomial. That is, it can be written in the form

$$f(x) = ax^2 + bx + c, \qquad a \neq 0.$$

or

$$f(x) = a(x - h)^2 + k, \qquad a \neq 0.$$

The graph of such a quadratic function is a parabola with the following information:

1) The parabola opens $\begin{cases} \text{up} & \text{if } a > 0 \\ \text{down} & \text{if } a < 0 \end{cases}$

2) The parabola is $\begin{cases} \text{thinner} & \text{if } |a| > 1 \\ \text{wider} & \text{if } 0 < |a| < 1 \end{cases}$

3) The vertex of the parabola has coordinates (h, k) where

$$h = -\frac{b}{2a} \qquad k = f(h) = c - ah^2$$

4) The axis of symmetry of the parabola is the line x = h

Note that it is much easier to graph when the function has the form $f(x) = a(x - h)^2 + k$. Here are some suggestions for graphing quadratic functions.

Step 1: Write in standard form and identify a, b, and c.

Step 2: Calculate h: $h = -\frac{b}{2a}$

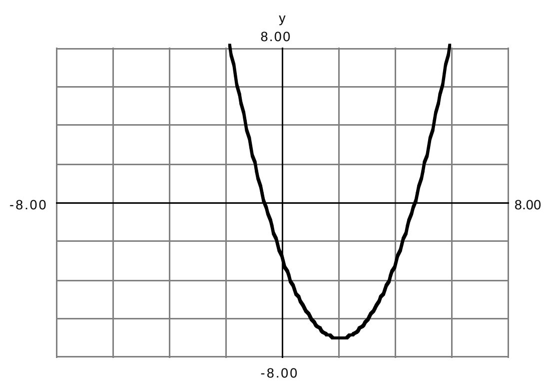
Step 3: Calculate k: $k = f(h)$ OR $k = c - ah^2$

Step 4: Find all of the intercepts.

Step 5: Plot sufficiently many points and make use of the intercepts and the vertex (h, k)

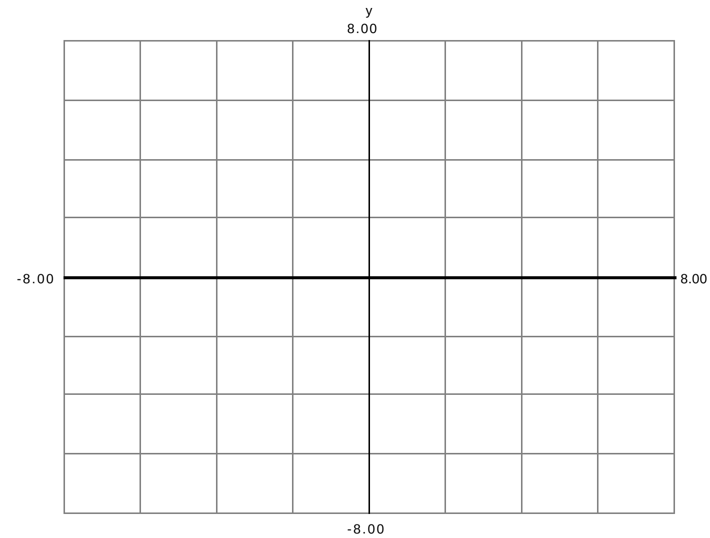
Step 6: Draw the smooth graph.

Example: Graph $f(x) = x^2 - 4x - 3$.

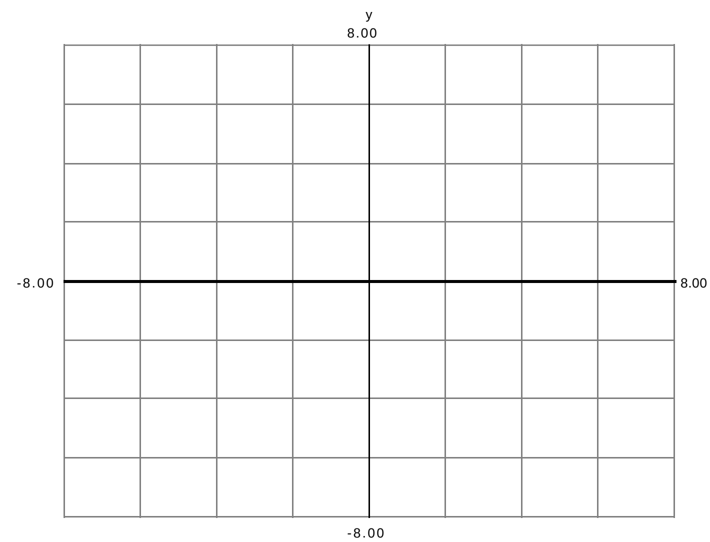


Vertex: (2, –7) Intercepts: (0,–3), (5,0), (–1,0)

Example: Graph $f(x) = x^2 + 2x - 3$. Find the vertex and any intercepts.

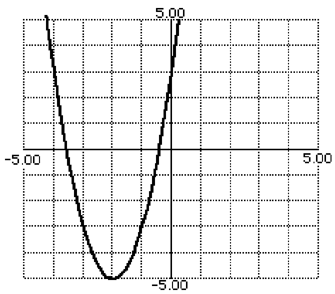


Example: Graph $f(x) = -2x^2 + 4x - 3$. Find the vertex and any intercepts.

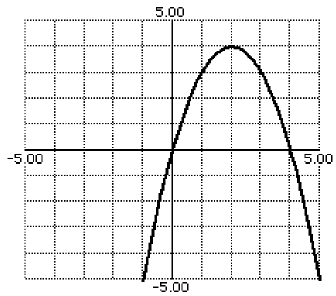


Applications of Quadratic Functions

In many applications, you need to find information about the maximum or minimum value of a function. If your model is a quadratic function, then the maximum or minimum value can be found by finding the vertex of the associated parabola. If $a > 0$, then the function has a minimum, while if $a < 0$, then the function has a maximum.



This quadratic function has a graph which is a parabola. In this case, $a > 0$. The minimum value of the function occurs at the vertex of the parabola.



This quadratic function has a graph which is a parabola. In this case, $a < 0$. The maximum value of the function occurs at the vertex of the parabola.

Example: Suppose that if the price (in dollars) of a videotape is $p(x) = 40 - \frac{x}{10}$, then x tapes will be sold.

(a) Find an expression for the total revenue from the sale of x tapes.

Solution:

$$\text{Revenue} = \text{demand} * \text{price}$$

$$R(x) = x \cdot p(x) = x\left(40 - \frac{x}{10}\right)$$

$$R(x) = 40x - \frac{x^2}{10} \qquad \text{<– answer}$$

(b) Find the number of tapes that will produce maximum revenue.

Solution: (Find the x–coordinate of the vertex!)

$$h = -\frac{b}{2a} = -\frac{40}{-\frac{2}{10}} = 200 \qquad \underline{\underline{200 \text{ tapes}}}$$

(c) Find the maximum revenue.

Solution: (Find the y–coordinate of the vertex!)

$$k = c - ah^2 = 0 - \left(-\frac{1}{10}\right)(200)^2 = 4000 \qquad \underline{\underline{\$4000}}$$

Example: Leslie Lahr owns and operates Aunt Emma's Blueberry Pies. She has hired a consultant to analyze her business operations. The consultant tells her that her profits, P(x), from the sale of x units of pies, are given by

$$P(x) = 120x - x^2.$$

How many units of pies should she sell in order to maximize profit? What is the maximum profit?

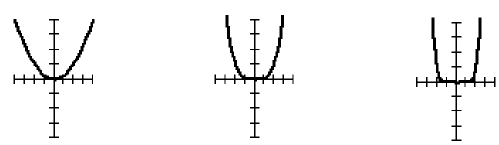
Solution: Note that the profit is given by the quadratic function $P(x) = 120x - x^2$. The graph is a downward opening parabola. Thus the function has a maximum value which we can find by getting the vertex of the parabola.

Using $a = -1$, $b = 120$, and $c = 0$, we get

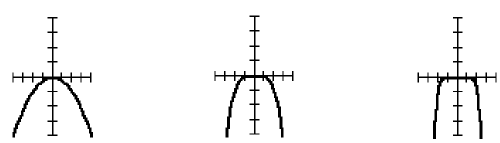
$$h = -\frac{b}{2a} = -\frac{120}{2(-1)} = 60 \qquad k = c - ah^2 = 0 - (-1)(60)^2 = 3600$$

Thus, she should sell 60 pies, and the maximum profit is $3600.

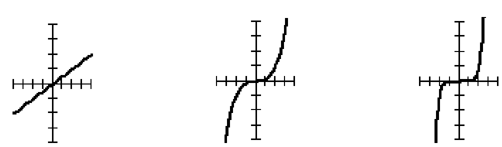
Graphing Polynomial Functions



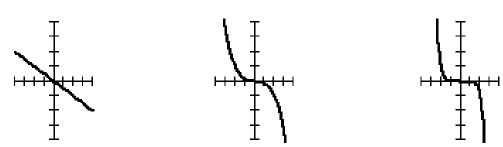
Graphs of $y = ax^n$, where n is even and $a > 0$



Graphs of $y = ax^n$, where n is even and $a < 0$



Graphs of $y = ax^n$, where n is odd and $a > 0$

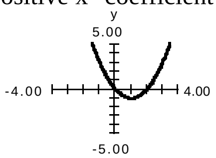


Graphs of $y = ax^n$, where n is odd and $a < 0$

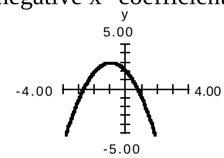
Relating Polynomial Functions and their Graphs

Degree 2:

positive x^2 coefficient

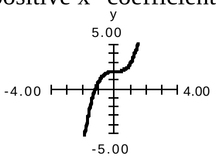


negative x^2 coefficient

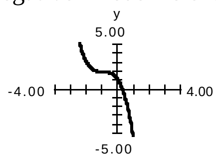


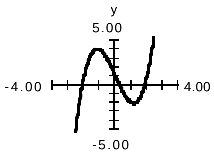
Degree 3:

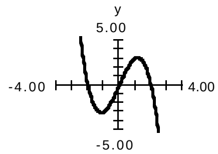
positive x^3 coefficient



negative x^3 coefficient



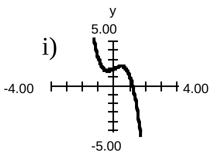




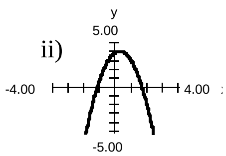
Example: Determine which of the following graphs is the graph of

a) $f(x) = -2x^2 + x - 4$ b) $g(x) = 4x^3 - 5x + 3$

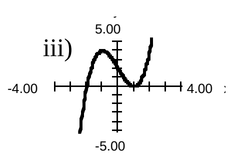
i)



ii)



iii)



Examples: Sketch the graph of the following polynomial functions:

a) $y = f(x) = (2x + 3)(x - 1)(x + 2)$

b) $y = f(x) = 3x^4 + x^3 - 2x^2$

c) $y = f(x) = x^4 - 4x^2$

d) $y = f(x) = x^3 - 4x^2$

Things to notice about graphs of polynomials

1. The graph of a polynomial function is a smooth, unbroken curve whose domain is the set of real numbers. Thus, the graph extends forever to the left and right.

2. When | x | is large, then the graph behaves similar to the graph of the associated monic polynomial of highest degree. The graph moves sharply to or from the x–axis.

3. A polynomial function of degree n has **at most n** x–intercepts.

4. A polynomial function of degree n has **less than n** total peaks and valleys.

Suggestion for graphing a polynomial function:

1. Factor the function.
2. Find all intercepts.
3. Create a sign chart for the function.
4. Use the sign chart and information about polynomial functions to produce the graph.

Horizontal asymptotes of rational functions

Case 1: If the degree of the numerator is **less than** the degree of the denominator, then the line $y = 0$ is a horizontal asymptote

Case 2: If the degree of the numerator is **equal to** the degree of the denominator, then the horizontal asymptote is the line $y = \frac{a}{c}$, where a and c are the highest coefficients of the numerator and denominator, respectively.

Case 3: If the degree of the numerator is **greater than** the degree of the denominator, then there is no horizontal asymptote.

Example: Find the horizontal asymptotes of the following, if possible.

a) $y = \frac{3}{x + 4}$

b) $y = \frac{4 - x}{3 + x}$

c) $y = \frac{2x^2}{3x - 4}$

d) $y = \frac{2x^2 - x + 6}{x^2 + x - 6}$

Solution: a) $y = 0$ b) $y = -1$ c) none d) $y = 2$

If a number c makes the denominator zero, but not the numerator, then the line x = c is a **vertical asymptote**. The graph will never touch a vertical asymptote!

Example: Find the vertical asymptotes of the following, if possible.

a) $y = \frac{3}{x + 4}$

b) $y = \frac{4 - x}{3 + x}$

c) $y = \frac{2x^2}{3x - 4}$

d) $y = \frac{2x^2 - x + 6}{x^2 + x - 6}$

Solution:

a) $x = -4$ b) $x = -3$ c) $x = \frac{4}{3}$ d) $x = -3$, $x = 2$

Suggestion for graphing a polynomial function: FIASCo

1. **F**actor the function.
2. Find all **I**ntercepts.
3. Find all **A**symptotes.
4. Create a **S**ign **C**hart for the function.
5. Use the sign chart and information about rational functions t**o** produce the graph.

Examples: Sketch the graph of the following:

a) $y = f(x) = \frac{x + 3}{x - 1}$

b) $y = f(x) = \frac{2x - 3}{2x + 1}$

c) $y = f(x) = \frac{4}{x - 2}$

d) $y = f(x) = \frac{2x^2}{x^2 - 4}$