Basic Properties of Rational Expressions

A rational expression is any expression of the form

$$\frac{P}{Q}$$

where P and Q are polynomials and $Q \neq 0$. In the following properties, no denominator is allowed to be zero.

Basic Property:	Example:
$\frac{P}{Q} = \frac{R}{S}$ if and only if $P \cdot S = Q \cdot R$	$\frac{2}{3} = \frac{6}{9}$
$\frac{P}{Q} = \frac{P \cdot R}{Q \cdot R}$	$\frac{3}{5} = \frac{3 \cdot 4}{5 \cdot 4} = \frac{12}{20}$
$-\frac{P}{Q} = \frac{-P}{Q} = \frac{P}{-Q} = -\frac{-P}{-Q}$	$-\frac{7}{4} = \frac{-7}{4} = \frac{7}{-4} = -\frac{-7}{-4}$
$\frac{P}{Q} = -\frac{-P}{Q} = -\frac{P}{-Q} = \frac{-P}{-Q}$	$\frac{5}{8} = -\frac{-5}{8} = -\frac{5}{-8} = \frac{-5}{-8}$

The Domain of a Rational Function:

Unless we are told otherwise, we assume that the domain of a function is the set of all real numbers for which the function is defined.

A rational expression is undefined when the denominator is zero. Hence when you need to find the domain of a rational function, you need to determine all values (if any) which cause the denominator to be zero.

Examples: Find the domain of the following functions:

a) $f(x) = 2x + 4$

b) $g(x) = \frac{x + 5}{x - 4}$

c) $h(x) = \frac{2x + 5}{x^3 - 4x}$

d) $F(x) = \frac{x}{x^2 + 9}$

Solution:

a) $\Re$ (all real numbers)

b) $\{x \in \Re \mid x \neq 4\}$ (all real numbers except 4)

c) $\{x \in \Re \mid x \neq 0, 2, -2\}$ (all real numbers except 0, 2, –2)

d) $\Re$ (all real numbers)

Notice that

$$\frac{-P}{P} = -1.$$

Since it is true that

$$x - a = -(-x + a)$$
$$= -(a - x),$$

It must also be true that

$$\frac{x - a}{a - x} = -1.$$

WARNING:

You must reduce only factors!! If the terms are not factors, they cannot be factored out.

Nonsense like "canceling out" nonfactors will not be tolerated!!

$$\frac{2x + 8}{2} = x + 4$$

$$\frac{x^2 - 9}{x - 3} = x + 3$$

Multiplication and Division of Rational Expressions

To multiply two rational expression, multiply the numerators together and multiply the denominators together.

$$\frac{P}{Q} \cdot \frac{R}{S} = \frac{P \cdot R}{Q \cdot S}.$$

To divide two rational expressions, invert the one immediately after the division sign, and do a rational expression multiplication.

$$\frac{P}{Q} \div \frac{R}{S} = \frac{P}{Q} \cdot \frac{S}{R} = \frac{P \cdot S}{Q \cdot R}$$

Example: Perform the indicated operation and simplify:

$$\frac{x^2 + 2x - 3}{x^2 - 3x - 10} \div \frac{4x + 2}{x^2 - x} \cdot \frac{2x^2 - 9x - 5}{x^2 - 2x + 1}$$

$$= \frac{x^2 + 2x - 3}{x^2 - 3x - 10} \cdot \frac{x^2 - x}{4x + 2} \cdot \frac{2x^2 - 9x - 5}{x^2 - 2x + 1}$$

$$= \frac{(x + 3)(x - 1)}{(x - 5)(x + 2)} \cdot \frac{x(x - 1)}{2(2x + 1)} \cdot \frac{(2x + 1)(x - 5)}{(x - 1)^2}$$

$$\boxed{= \frac{x(x + 3)}{2(x + 2)}}$$

Rule for Adding or Subtracting Fractions with Equal Denominators (RASFED)

To add or subtract two rational expressions whose denominators are equal, simply add or subtract the numerators. **Make sure you use parentheses when appropriate!**

$$\frac{P}{Q} + \frac{R}{Q} = \frac{P + R}{Q} \qquad \frac{P}{Q} - \frac{R}{Q} = \frac{P - R}{Q}$$

Examples: Perform the indicated operation and simplify:

a) $\frac{x}{x^2 - 4} + \frac{2}{x^2 - 4}$

$= \frac{x + 2}{x^2 - 4}$

$= \frac{x + 2}{(x + 2)(x - 2)}$

$\boxed{= \frac{1}{x - 2}}$

b) $\frac{x^2}{x^2 - 4} - \frac{4x - x^2}{x^2 - 4}$

$= \frac{x^2 - (4x - x^2)}{x^2 - 4}$

$= \frac{x^2 - 4x + x^2}{x^2 - 4}$

$= \frac{2x^2 - 4x}{x^2 - 4}$

$= \frac{2x(x - 2)}{(x + 2)(x - 2)}$

$\boxed{= \frac{2x}{(x + 2)}}$

Rule for Adding or Subtracting Fractions with Unequal Denominators

(FLEAS)

1. **F**actor the rational expression.

2. Find the **L**east Common Denominator (LCD).

3. **E**qualize each denominators by replacing each fraction with an equivalent one whose denominator is the LCD.

4. **A**dd or **S**ubtract using RASFED.

Example:

$$\frac{1}{2} - \frac{2}{3} + \frac{3}{4}$$

$= \frac{6}{12} - \frac{8}{12} + \frac{9}{12}$

$= \frac{6 - 8 + 9}{12}$

$\boxed{= \frac{7}{12}}$

Finding the LCD

Step 1. Factor each denominator completely, including the prime factors of any constant factor.

Step 2. Form the product of all the factors that appears in the complete factorizations.

Step 3. The number of times any factors appears in the LCD is the most number of times it appears in any one factorization.

Examples: Find the LCD for the given denominators:

a) Denominators are 24, 30, and 36

$24 = 8 \cdot 3 = 2^3 \cdot 3$

$30 = 6 \cdot 5 = 2 \cdot 3 \cdot 5$

$36 = 4 \cdot 9 = 2^2 \cdot 3^2$

LCD $= 2^3 \cdot 3^2 \cdot 5 = 360$

b) Denominators are $x^3 - x^2$ and $x^3 - x$

$x^3 - x^2 = x^2(x - 1)$

$x^3 - x = x(x^2 - 1) = x(x + 1)(x - 1)$

LCD $= x^2(x + 1)(x - 1)$

Examples: Perform the indicated operation and simplify.

a) $\frac{2}{x^2 - 1} - \frac{1}{x^2 + 2x + 1}$

$= \frac{2}{(x + 1)(x - 1)} - \frac{1}{(x + 1)^2}$

$= \frac{2(x + 1)}{(x + 1)^2(x - 1)} - \frac{x - 1}{(x + 1)^2(x - 1)}$

$= \frac{2(x + 1) - (x - 1)}{(x + 1)^2(x - 1)}$

$= \frac{2x + 2 - x + 1}{(x + 1)^2(x - 1)}$

$\boxed{= \frac{x + 3}{(x + 1)^2(x - 1)}}$

b) $\frac{x + 2}{x - 2} - \frac{x^2 + 2x}{x^2 - 4}$

$= \frac{x + 2}{x - 2} - \frac{x^2 + 2x}{(x + 2)(x - 2)}$

$= \frac{(x + 2)^2}{(x + 2)(x - 2)} - \frac{x^2 + 2x}{(x + 2)(x - 2)}$

$= \frac{(x^2 + 4x + 4) - (x^2 + 2x)}{(x + 2)(x - 2)}$

$= \frac{x^2 + 4x + 4 - x^2 - 2x}{(x + 2)(x - 2)}$

$= \frac{2x + 4}{(x + 2)(x - 2)}$

$= \frac{2(x + 2)}{(x + 2)(x - 2)}$

$\boxed{= \frac{2}{(x - 2)}}$

Complex Fractions

A simple fraction is any rational expression whose numerator and denominator contain no rational expression.

A complex fraction is any rational expression whose numerator or denominator contains a rational expression.

To simplify complex fractions:

Step 1: Identify all fractions in the numerator and denominator and find the LCD.

Step 2: Multiply the numerator and denominator by the LCD.

Examples:

a) $\dfrac{\frac{1}{2}-\frac{1}{3}}{\frac{3}{4}-\frac{1}{6}}$

$= \dfrac{12\left(\frac{1}{2}-\frac{1}{3}\right)}{12\left(\frac{3}{4}-\frac{1}{6}\right)}$

$= \dfrac{6-4}{9-2}$

$\boxed{= \dfrac{2}{7}}$

b) $\dfrac{x+y}{x^{-1}+y^{-1}}$

$= \dfrac{x+y}{\frac{1}{x}+\frac{1}{y}}$

$= \dfrac{xy\,(x+y)}{xy\left(\frac{1}{x}+\frac{1}{y}\right)}$

$= \dfrac{xy\,(x+y)}{y+x}$

$\boxed{= xy}$

Long Division of Polynomials

Monomial Denominator: When you divide a polynomial by a monomial, you use a form of the distributive property, and thus you must divide each term in the numerator by the denominator

Examples: Perform the indicated operation.

a) $(x^3 - 6x^2 + 2x) \div 3x$

$= \dfrac{x^3-6x^2+2x}{3x}$

$= \dfrac{x^3}{3x} - \dfrac{6x^2}{3x} + \dfrac{2x}{3x}$

$\boxed{= \dfrac{x^2}{3} - 2x + \dfrac{2}{3}}$

b) Divide $15y^3+20y^2-5y$ by $5y$

$= \dfrac{15y^3+20y^2-5y}{5y}$

$= \dfrac{15y^3}{5y} + \dfrac{20y^2}{5y} - \dfrac{5y}{5y}$

$\boxed{= 3y^2 + 4y - 1}$

c) $(16x^3 - 8x^2 + 3x) \div 2x$

d) $(-16z^4 + 16z^3 + 8z^2 + 64z) \div 8z$

Example: Simplify the following:

a) $\dfrac{9-\frac{1}{y^2}}{3-\frac{1}{y}}$

$= \dfrac{y^2\left(9-\frac{1}{y^2}\right)}{y^2\left(3-\frac{1}{y}\right)}$

$= \dfrac{9y^2-1}{3y^2-y}$

$= \dfrac{(3y+1)(3y-1)}{y(3y-1)}$

$\boxed{= \dfrac{3y+1}{y}}$

b) $\dfrac{\frac{1}{x+h}-\frac{1}{x}}{h}$

$= \dfrac{x(x+h)\left(\frac{1}{x+h}-\frac{1}{x}\right)}{x(x+h)\,h}$

$= \dfrac{x-(x+h)}{x(x+h)\,h}$

$= \dfrac{x-x-h}{x(x+h)\,h}$

$= \dfrac{-h}{x(x+h)\,h}$

$\boxed{= \dfrac{-1}{x(x+h)}}$

Long Division of Polynomials

Polynomial Denominator: When you divide a polynomial by a polynomial, you can use the same form of long division that you used with numbers. You must remember to write polynomials in descending order and account for zero coefficients. It is convenient to remember that one can subtract a quantity by changing its sign and then adding.

Step 1. Write the division as in arithmetic. Write both polynomials in descending order and write all missing terms with a coefficient of zero. Set the current term to be the first one.

Step 2.

a. Divide the first term of the divisor into the current term of the dividend. The result is the current term of the quotient.

b. Multiply every term in the divisor by the result and write the product under the dividend (align like terms).

c. Subtract. [You may reverse the signs and then add.] Treat the difference as a new dividend.

Step 3. If the remainder is a polynomial of degree greater than or equal to the degree of the divisor, then go to step 2. Otherwise (the remainder is zero or is a polynomial of degree less than the degree of the divisor) go on to step 4.

Step 4. If there is a remainder, write it as the numerator of a fraction with the divisor as the denominator, and add this fraction to the quotient.

Note: Division by polynomials can be remembered by DMS because you have to **Divide**, **Multiply**, **Subtract**, and then repeat.

Examples: Calculate the indicated quotients by long division:

a) $\dfrac{x^3 - 2x^2 - 7x + 3}{x + 2}$

b) $\dfrac{x^4 - 8x^2 - 8}{x^2 - x + 2}$

c) $\dfrac{6x^4 + x^3 - 9x + 4}{2x - 1}$

Examples: Calculate the indicated quotients by synthetic division:

a) $\dfrac{x^3 - 2x^2 - 7x + 3}{x + 2}$

b) $\dfrac{x^4 - 8x^2 - 8}{x - 3}$

c) $\dfrac{x^4 - 81}{x + 3}$

Synthetic Division of Polynomials

When you divide a polynomial by a **first degree polynomial with linear coefficient 1**, we can perform the division by using only the necessary coefficients.

Step 1. Write the opposite of the constant term of the divisor by itself. Write all of the coefficient of the dividend (using zero when terms are missing, of course).

Step 2. Bring down the first term of the dividend. This also becomes the current term.

Step 3.

a. Multiply the current term by the divisor term.
b. Put the product under the **next** term of the dividend.
c. Add the result. The result becomes the current term.

Step 4. If there is another term of the dividend, then go to Step 3. Otherwise, go to Step 5.

Step 5. The constants of the bottom line are the coefficients of the quotient and the remainder.

Remainder Theorem

When you divide a polynomial P(x) by a the factor x – c, the remainder is P(c). Thus we sometimes evaluate a polynomial P(x) when x = c by performing the appropriate synthetic division.

Examples 1: Let $P(x) = 2x^3 - 4x^2 + 5$.

a) By direct substitution, evaluate P(2).

b) Find the remainder when P(x) is divided by x – 2.

Examples 2: Let $P(x) = 4x^6 - 25x^5 + 35x^4 + 17x^2$. Find P(4)

Equations Involving Fractions

When we solve equations with fractions, we always assume that no denominator is zero. Therefore, we can find the LCD and multiply both sides by this nonzero factor. We must check to see that our solution does not cause any denominator to be zero.

To solve equations with (simple) fractions:

Step 1: Identify all fractions in the equation and find the LCD.

Step 2: Multiply the both sides of the equation by the LCD.

Step 3: Solve the resulting equation.

Step 4: Check the answer into the original problem. (At least check to make sure no denominator can be zero.)

WARNING:

You must know the difference between an expression and an equation. When you solve an equation, you may multiply both sides by the LCD and we get an equation without fractions. When you have an expression with fractions, you must perform the indicated operation and / or simplify. Nonsense like treating expressions like equations will not be tolerated!!

Examples: Solve the following:

a) $$\frac{2}{x-6} = \frac{3}{x-8}$$

$$(x-6)(x-8)\frac{2}{x-6} = (x-6)(x-8)\frac{3}{x-8}$$

$$2(x-8) = 3(x-6)$$

$$2x - 16 = 3x - 18$$

$$-x = -2$$

$$\boxed{x = 2}$$

b) $$\frac{z-4}{z^2-2z} = \frac{1}{z} - \frac{2}{z^2} - \frac{4}{z^3-2z^2}$$

$$\frac{z-4}{z(z-2)} = \frac{1}{z} - \frac{2}{z^2} - \frac{4}{z^2(z-2)}$$

$$z^2(z-2)\frac{z-4}{z(z-2)} = z^2(z-2)\left(\frac{1}{z} - \frac{2}{z^2} - \frac{4}{z^2(z-2)}\right)$$

$$z(z-4) = z(z-2) - 2(z-2) - 4$$

$$z^2 - 4z = z^2 - 2z - 2z + 4 - 4$$

$$z^2 - 4z = z^2 - 4z$$

$$0 = 0$$

$$\boxed{\text{all real numbers except } 0, 2}$$

c) $$\frac{y-2}{y-3} = 1 - \frac{2}{y^2-9}$$

$$\frac{y-2}{y-3} = 1 - \frac{2}{(y+3)(y-3)}$$

$$(y+3)(y-3)\frac{y-2}{y-3} = (y+3)(y-3)\left(1 - \frac{2}{(y+3)(y-3)}\right)$$

$$(y+3)(y-2) = (y+3)(y-3) - 2$$

$$y^2 + y - 6 = y^2 - 9 - 2$$

$$\boxed{y = -5}$$

d) $$\frac{2}{x^2-4} = \frac{1}{x^2} + \frac{1}{x^2-2x}$$

$$\frac{2}{(x+2)(x-2)} = \frac{1}{x^2} + \frac{1}{x(x-2)}$$

$$x^2(x+2)(x-2)\frac{2}{(x+2)(x-2)} = x^2(x+2)(x-2)\left(\frac{1}{x^2} + \frac{1}{x(x-2)}\right)$$

$$2x^2 = (x+2)(x-2) + x(x+2)$$

$$2x^2 = x^2 - 4 + x^2 + 2x$$

$$2x^2 = 2x^2 + 2x - 4$$

$$-x = -2$$

$$x = 2$$

$$\boxed{\text{no solution}}$$

e) $$2 + \frac{10}{x+2} = \frac{3}{x+3}$$

$$(x+2)(x+3)\left(2 + \frac{10}{x+2}\right) = (x+2)(x+3)\frac{3}{x+3}$$

$$2(x+2)(x+3) + 10(x+3) = 3(x+2)$$

$$2(x^2 + 5x + 6) + 10x + 30 = 3x + 6$$

$$2x^2 + 10x + 12 + 10x + 30 = 3x + 6$$

$$2x^2 + 17x + 36 = 0$$

$$(2x+9)(x+4) = 0$$

$$\boxed{x = -9/2, -4}$$

f) $$\frac{y+1}{y+3} + \frac{y+5}{y-2} = 1 + \frac{6y+23}{y^2+y-6}$$

$$(y+3)(y-2)\left(\frac{y+1}{y+3} + \frac{y+5}{y-2}\right) = (y+3)(y-2)\left(1 + \frac{6y+23}{(y+3)(y-2)}\right)$$

$$(y-2)(y+1) + (y+3)(y+5) = (y+3)(y-2) + (6y+23)$$

$$y^2 - y - 2 + y^2 + 8y + 15 = y^2 + y - 6 + 6y + 23$$

$$2y^2 + 7y + 13 = y^2 + 7y + 17$$

$$y^2 = 4$$

$$y = \pm 2$$

$$\boxed{y = -2}$$

Example: It takes Rosa, traveling at 50 mph, 45 minutes longer to go a certain distance than it takes Maria traveling at 60 mph. Find the distance traveled.

	distance	= rate	• time
Rosa	x	50	
Maria	x	60	

$$\frac{x}{50} - \frac{x}{60} = \frac{3}{4}$$

$$300\left(\frac{x}{50} - \frac{x}{60}\right) = 300\left(\frac{3}{4}\right)$$

$$6x - 5x = 180$$

$$x = 180$$

180 miles

Example: Toni needs 4 hours to complete the yard work. Her husband, Sonny, needs 6 hours to do the work. How long will the job take if they work together?

Toni	4 hours
Sonny	6 hours
together	x hours

$$\frac{1}{4} + \frac{1}{6} = \frac{1}{x}$$

$$12x\left(\frac{1}{4} + \frac{1}{6}\right) = 12x\left(\frac{1}{x}\right)$$

$$3x + 2x = 12$$

$$5x = 12$$

$$x = \frac{12}{5}$$

2 hours 24 minutes

Example: Beth can travel 208 miles in the same length of time it takes Anna to travel 192 miles. If Beth's speed is 4 mph greater than Anna's, find both rates.

	distance	= rate	• time
Beth	208	x + 4	
Anna	192	x	

$$\frac{208}{x + 4} = \frac{192}{x}$$

$$x(x + 4)\left(\frac{208}{x + 4}\right) = x(x + 4)\left(\frac{208}{x + 4}\right)$$

$$208x = 192(x + 4)$$

$$208x = 192x + 768$$

$$16x = 768$$

$$x = 48$$

Beth 52 mph
Anna 48 mph

Example: Working together, Rick and Rod can clean the snow from the driveway in 20 minutes. It would have taken Rick, working alone, 36 minutes. How long would it have taken Rod alone?

Rick	36 minutes
Rod	x minutes
together	20 minutes

$$\frac{1}{36} + \frac{1}{x} = \frac{1}{20}$$

$$180x\left(\frac{1}{36} + \frac{1}{x}\right) = 180x\left(\frac{1}{20}\right)$$

$$5x + 180 = 9x$$

$$180 = 4x$$

$$45 = x$$

45 minutes

Example: John, Ralph, and Denny, working together, can clean a store in 6 hours. Working alone, Ralph takes twice as long to clean the store as does John. Denny needs three times as long as does John. How long would it take each man working alone?

John	x hours
Ralph	2x hours
Denny	3x hours
together	6 hours

$$\frac{1}{x} + \frac{1}{2x} + \frac{1}{3x} = \frac{1}{6}$$

$$6x\left(\frac{1}{x} + \frac{1}{2x} + \frac{1}{3x}\right) = 6x\left(\frac{1}{6}\right)$$

$$6 + 3 + 2 = x$$

$$11 = x$$

John 11 minutes
Ralph 22 minutes
Denny 33 minutes

Example: You can row, row, row your boat on a lake 5 miles per hour. On a river, it takes you the same time to row 5 miles downstream as it does to row 3 miles upstream. What is the speed of the river current in miles per hour?

	distance	= rate	• time
downstream	5	5 + x	
upstream	3	5 – x	

$$\frac{5}{5 + x} = \frac{3}{5 - x}$$

$$(5 + x)(5 - x)\left(\frac{5}{5 + x}\right) = (5 + x)(5 - x)\left(\frac{3}{5 - x}\right)$$

$$5(5 - x) = 3(5 + x)$$

$$25 - 5x = 15 + 3x$$

$$10 = 8x$$

$$\frac{5}{4} = x$$

$\frac{5}{4}$ mph

Example: An inlet pipe on a swimming pool can be used to fill the pool in 12 hours. The drain pipe can be used to empty the pool in 20 hours. If the pool is empty and the drain pipe is accidentally opened, how long will it take to fill the pool?

inlet pipe	12 hours
drain pipe	20 hours
together	x hours

$$\frac{1}{12} - \frac{1}{20} = \frac{1}{x}$$

$$60x\left(\frac{1}{12} - \frac{1}{20}\right) = 60x\left(\frac{1}{x}\right)$$

$$5x - 3x = 60$$

$$2x = 60$$

$$x = 30$$

30 hours