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Spring 2011

Algebra 1: Midterm Exam SOLUTIONS

Part 1: In class Exam

1. For p and q distinct primes, with $q \neq 1 \pmod{p}$, prove that a group of order $p^n q$ is solvable, for all integers $n \geq 0$.

Let P be a Sylow p-subgroup of group G. Since q $\neq 1$ (mod p), by Sylow's theorem we conclude that P is normal in G. Since P is a p-group, it is solvable. Group G/P is cyclic of order q, and is, therefore, solvable. It follows that G is solvable.

2. As a permutation group of symmetries of a regular $n-$gon, the Dihedral group is primitive if and only if n is a prime number.

A block of imprimitivity must be a proper subset of the vertices of the regular n-gon that are positioned in the form of a regular d-gon, with d a proper divisor of n. Such a block exists if and only if n has proper divisors.

3. Let G be a finite group, and π a set of primes that divide $|G|$. Denote by $O_\pi(G)$ the subgroup of G generated by all normal p-groups of G,with $p \in \pi$. Show that:

(a) $O_\pi(G)$ is characteristic in G.

Any automorphism of G maps normal p-groups into normal p-groups, $p\in \pi$, and thus if fixes $O_\pi(G)$.

(b) $O_\pi(G)$ is the direct product of its Sylow subgroups, and is, therefore, nilpotent.

If H and K are normal p-groups of G, then HK is also a normal p-group of G, since $|HK|$ divides $|H||K|$ (a power of p). It follows that G contains a largest normal p-group, which is a Sylow p-subgroup of $O_\pi(G)$, necessarily. For two different p in π these Sylow subgroups of $O_\pi(G)$ intersect in 1, showing that $O_\pi(G)$ is the direct product of its Sylow subgroups.

4. List all nonisomorphic abelian groups of order 100. Explain how you may construct a non-abelian, non-dihedral group of order 100. Be as specific as you can. [The more specific, the higher the grade.]

By the Basis Theorem for abelian groups, the abelian groups of order 100 are: $\mathbf{Z}_2 \times Z_2 \times Z_5 \times Z_5$, $\mathbf{Z}_4 \times Z_5 \times Z_5$, $\mathbf{Z}_2 \times Z_2 \times Z_{25}$, $\mathbf{Z}_4 \times Z_{25}$. Let $\mathbf{V} = Z_5 \times Z_5$ be a 2-dimensional vector space over $\mathbf{F}_5$. Act with $\mathbf{Z}_4 =< a >$ on V through, for example, the automorphism 2I, which has order 4 in GL(2,5). Specifically, element a conjugates $(\mathbf{x},\mathbf{y}) \in V$ into $(\mathbf{2x},\mathbf{2y}) \in V$. Since the action is nontrivial, the resulting group is non-abelian. It is not dihedral, since it has no elements of order 25, but $\mathbf{D}_{100}$ has such elements.

5. Prove that a group of order $p^n(p+1)$ cannot be simple; $n > 1$.

Assume that such a group G is simple. There are $p+1$ Sylow p-subgroups in G. As G acts transitively on its Sylow p-subgroups, and since G is simple, we obtain an embedding of G into S_{p+1}. Hence $p^n(p+1) = |G|$ divides $(p+1)! = |S_{p+1}|$, a contradiction, for $n \geq 2$.

Part 2: Due March 25 at 1 pm

1. If $H \leq G$, then $|Syl_p(H)| \leq |Syl_p(G)|$.

Write p^k for the cardinality of a Sylow p-subgroup of H. Any Sylow p-subgroup of G intersects H in a p-group of cardinality at most p^k. If the conclusion is not true, there are two Sylow p-subgroups of H that are in the same Sylow p-subgroup P of G. These two subgroups generate a group of cardinality greater than p^k in $P \cap H$, yielding a contradiction.

2. Explicitly construct all nonisomorphic groups of order 30.

By previous work we know that a group of order $2n$, with n odd, contains a normal subgroup of order n. We now have an involution a acting on $\mathbf{Z}_3 \times Z_5$. It suffices to specify its action on a generator t of $\mathbf{Z}_3$ and a generator f of $\mathbf{Z}_5$. Four possibilities arise: $a(t) = t$, $a(f) = f$; $a\{t\} = t^{-1}$, $a(f) = f$; $a(t) = t$, $a(f) = f^{-1}$; $a(t) = t^{-1}$, $a(f) = f^{-1}$. No two of these groups are isomorphic.