Theorem *If a finite group G, $|G| = 2^n m$, with m odd, has a cyclic Sylow 2-subgroup T, then G has a normal subgroup H of order m (and thus G is the semidirect product of H by T).*

Proof Use induction on n. For $n = 1$, let x be the involution $< x >= T$. In the left regular representation of G the involution x consists of m transpositions and is therefore an odd permutation of S_{2m}. Hence

$$C_2 = \frac{S_{2m}}{A_{2m}} = \frac{A_{2m}G}{A_{2m}} = \frac{G}{G \cap A_{2m}},$$

showing that $H = G \cap A_{2m}$ is a subgroup of index 2 in G and thus normal in G.

Assume the result true for n. Let $|G| = 2^{n+1}m$, and write $T =< x >$, with $|x| = 2^{n+1}$. The left regular representation of x is again an odd permutation and, in complete analogy to the argument used above, G has a subgroup K of index 2 that is normal. The subgroup $Q = T \cap K =< x^2 >$ is a Sylow 2-subgroup of K of order 2^n. By induction there exists H normal in K with $|H| = m$; thus $K = HQ$ is a semidirect product. But element x normalizes H. Indeed, for $h \in H$ we have $|h^x| = |h| = r$, with r a divisor of m and thus an odd number. Write $h^x = h_1 q$, with $h_1 \in H$ and $q \in Q$. Then

$$1 = (h^x)^r = (h_1 q)^r = h_2 q^r,$$

with $h_2 \in H$ and $q^r \in Q$. Since r is odd, and the order of q is even, this forces $q = 1$. Hence $N_G(H) =< K, x >= G$.