Homework 2

1. In a letter to C. Hermite [Paris *Comptes Rendus,* vol. 49 (1859), p. 115], E. Betti states that the following substitutions generate a group of order 12:

$$w = 4z,\ w = \frac{1}{z},\ w = 3\frac{(z+1)}{(z-1)} \pmod 5.$$

Verify this statement. Prove that the first two substitutions generate the noncyclic group of order 4. Prove that the third and either of the the first two substitutions generate the group of order 12 mentioned above. Represent this group as a regular permutation group using Cayley's method.

2. Prove that the transformations of the form $w = \frac{az+b}{cz+d} \pmod 3$, with a, b, c, d being integers such that $ad - bc = 1 \pmod 3$, constitute a group of order 12, which has the same number of elements of each order as the group in Exercise 1. [F. Klein, *Mathematische Annalen,* vol. 14 (1879), p. 418]. Are the two groups isomorphic?

3. By means of Sylow's theorem prove that every group of order 20 contains only one subgroup of order 5, and either one or five subgroups of order 4.

4. Prove that a group of order 15 is cyclic.

5. A group of order pq, with p and q being distinct primes and $p > q$, is cyclic, unless q divides $p-1$, in which case there are exactly two groups of order pq. Describe these two groups.

6. Show that there are $(p-2)!$ subgroups of order p in the symmetric group on p letters, p being a prime. Conclude that $(p-2)! = 1 \pmod p$.

6. The symmetric group of degree n does not contain any subgroup of index m, if m is greater that 2 but less than the largest prime factor on n. *(Cauchy, 1815)*

7. Prove that the group on letters $x_1, \ldots, x_6$ which transforms the function $x_1x_2 + x_3x_4 + x_5x_6$ into itself has order 48, and contains a normal subgroup of order 8.

8.∗ Prove that the permutations *(125)(346)* and *(17)(26)* generate a group of order 168 and that this group is simple.

9. If a group contains a subgroup of index 2 this subgroup is normal.

10. If the number of elements whose order divides d is equal to d, for all divisors d of the order of the group, then the group is cyclic.

11.∗ If d divides the order of the group, and there are exactly d elements whose orders divide d, then these elements must form a subgroup.

12. Determine the number of elements of each order in each of the four Abelian groups of order 100.

13. A group of order p^2q must be Abelian when q is a prime number which is less than the prime number p and does not divide $p^2 - 1$.

14. Show that a group in which all elements besides 1 are of order 2 is Abelian.

15. If p is an odd prime, then the number of subgroups of order p in any noncyclic $p-$group is of the form $1 + p + kp^2$.

16. Up to isomorphism, find all groups of order 15 or less.