Homework 4

1. Prove that, except for the quarternion group Q_8, any non-Abelian group of order 14 or less is a (nontrivial) semidirect product.

2. Show that Z_{p^n} is not a split extension. The same is true of Q_8.

3. Prove that every group of order pq is a split extension of Z_p by Z_q; here $q < p$ are primes.

4. Prove that if $A \leq G$ acts on group G and centralizes a normal subgroup H of G, then $[G, A] \leq C_G(H)$.

5. Classify all groups of order 18 and 20 up to isomorphism.

6. Show that a maximal subgroup of a solvable group is of prime power index in the group.

7. A minimal normal subgroup of a nilpotent group is isomorphic to Z_p, with p a prime.

8. Find all composition series of $S_3 \times Z_2$. Verify the Jordan-Hölder theorem directly in this case.

9.(a) Every Abelian group is generated by its elements of highest order.

(b) Give an example of a group which is not generated by its elements of highest order.

10. Find all the characteristic subgroups of an Abelian group of order p^6 and type (3,2,1).