0.1 Algebras and Modules

Let \( \mathbb{F} \) be a field. An \( \mathbb{F} \)-algebra \( A \), is a ring with identity which is also a vector space over \( \mathbb{F} \) and such that the following compatibility relation holds:

\[(cx)y = c(xy) = x(cy), \forall c \in \mathbb{F}, \forall x, y \in A\]

**Note:** In what follows we identify \( c \in \mathbb{F} \) with \( c \cdot 1_A \in A \), when appropriate. Accordingly, we also identify \( \mathbb{F} \) with its copy \( \mathbb{F} \cdot 1_A \) in \( A \).

**Example 1:** If \( V \) is a vector space over \( \mathbb{F} \), then \( \text{Hom}_\mathbb{F}(V;V) \) is an \( \mathbb{F} \)-algebra, where addition and scalar multiplication of functions are pointwise, and multiplication is function composition.

**Example 2:** (the group algebra) Let \( G \) be a group and \( \mathbb{K} \) a field. Then the set \( \mathbb{K}G = \{ \sum_{g \in G} c_g \cdot g \mid c_g \in \mathbb{K} \} \) of formal sums in the elements of \( G \) with coefficients in \( \mathbb{K} \), is a \( \mathbb{K} \)-algebra under

\[
\sum_{g \in G} (c_g \cdot g) + \sum_{g \in G} (d_g \cdot g) = \sum_{g \in G} ((c_g + d_g) \cdot g)
\]

\[
\sum_{g \in G} (c_g \cdot g) \cdot \sum_{g \in G} (d_g \cdot g) = \sum_{x \in G} \left( \sum_{g, h \in G} c_g d_h \right) \cdot x
\]

\[
(c \cdot \sum_{g \in G} c_g \cdot g) = \sum_{g \in G} (cc_g \cdot g)
\]

Note also that the elements of \( G \) form a basis of \( \mathbb{K}G \) as a vector space over \( \mathbb{K} \).

Let \( A, B \) be \( \mathbb{K} \)-algebras. A map \( f : A \to B \) is an *algebra homomorphism* of \( A \) and \( B \) if:

\[f(xy) = f(x)f(y)\]

\[f(1_A) = 1_B\]

\[f(cx + y) = c \cdot f(x) + f(y)\]

\( \forall x, y \in A, c \in \mathbb{K} \)

Let \( A \) be a \( \mathbb{K} \)-algebra. An *\( A \)-module* is an abelian group \( M \), such that \( A \) acts on \( M \) (i.e. we have a map: \( A \times M \to M, (a, m) \mapsto am \)), and the following hold:

\[g(v + w) = gv + gw\]

\[(g + h)v =gv + hv\]

\[h(gv) = (hg)v\]

\[g(cv) = c(gv) = (cg)v\]

\[1_A v = v\]

\( \forall v, w \in M, g, h \in A, c \in \mathbb{K}. \)

In this case \( M \) is also a \( \mathbb{K} \)-vector space, with addition inherited from the group \( M \), and scalar multiplication defined by \( kv = (k \cdot 1_A)v \).

**Note:** Throughout this text, we will only consider modules which are finite dimensional as vector spaces.
Fix $g \in A$. Then $g_M : M \to M$, defined by $g_M(v) = gv$, is an element of $\text{End}(M)$. The map $g \mapsto g_M$ is a $\mathbb{K}$-algebra homomorphism of $A$ and $\text{End}_\mathbb{K}(M)$. The image of $A$ under this homomorphism is written as $A_M$.

**Note:** Sometimes we will identify $g$ with $g_M$, which will be clear from the context.

**Example 3:** Let $V$ be a vector space and $A$ a subalgebra of $\text{End}_\mathbb{K}(V)$, then $V$ is naturally an $A$-module (under $gv = g(v), v \in V, g \in A$).

**Example 4:** In the previous example, let $V = \mathbb{K}^n$ (the vector space of $n$-tuples over $\mathbb{K}$). If $A$ is any subalgebra of $M_n(\mathbb{K})$, then $\mathbb{K}^n$ is an $A$-module under matrix multiplication.

**Example 5:** Let $A$ be a $\mathbb{K}$-algebra. Then $A$ is a module over itself by multiplication on the left. We denote this module by $A^\circ$ (called the *left regular representation of $A$*).

**Example 6:** $\mathbb{K}$ is a $\mathbb{K}G$-module under $gk = k, \forall k \in \mathbb{K}, g \in G$, and extension by linearity (use the $\mathbb{K}G$-module relations to find $\sum_{g \in G} k_gg = \sum_{g \in G} k_gk$).

Let $V$ be an $A$-module (here $A$ is a $\mathbb{K}$-algebra). A *submodule* of $V$, is a vector subspace $W$ of $V$, which is invariant under the action of $A$ (so it is itself an $A$-module under the same action as $V$).

**Example 7:** The submodules of $A^\circ$ are precisely the left ideals of $A$.

**Example 8:** If $W$ is a submodule of the $A$-module $V$, then the quotient vector space $V/M$ is an $A$-module in a natural way.

**Exercise:** Let $I$ be a proper ideal of the $\mathbb{K}$-algebra $A$. Explain the algebraic structures $A/I$, $A^\circ/I$, and $(A/I)^\circ$. Are they all different?

### 0.2 Module Homomorphisms

Let $A$ be a $\mathbb{K}$-algebra. Let $V, W$ be $A$-modules. An *$A$-module homomorphism* is a $\mathbb{K}$-linear map $\varphi : V \to W$, such that $\varphi(gv) = g(\varphi(v)), \forall v \in V, g \in \mathbb{K}A$.

We then define

$$
\text{Hom}_A(V, W) = \{ \varphi | \varphi : V \to W \text{ is an } A\text{-module homomorphism} \} \\
\text{End}_A(V) = \text{Hom}_A(V, V)
$$

Note that $\text{End}_A(V)$ is a $\mathbb{K}$-algebra, with addition and scalar multiplication of homomorphisms defined pointwise, and multiplication defined as function composition. Moreover, $\text{End}_A(V)$ is the centralizer of $A_V$ in $\text{End}_\mathbb{K}(V)$.

**Note:** When $A$ is the group algebra $\mathbb{K}G$, we call $\mathbb{K}G$-modules just $G$-modules and $\mathbb{K}G$-module homomorphisms just $G$-homomorphisms.

A non-zero $A$-module $M$ is called *irreducible* if its only submodules are 0 and $M$.

**Theorem 1:** (*Schur’s Lemma*) Let $V, W$ be irreducible $G$-modules and assume $\varphi : V \to W$ is a $G$-homomorphism. Then:

1. Either $\varphi$ is an isomorphism, or $\varphi \equiv 0$

2. If $V = W$ and $\mathbb{K}$ is algebraically closed, then $\varphi = \lambda I_V$ for some $\lambda \in \mathbb{K}$. 


0.2. MODULE HOMOMORPHISMS

Proof: Part 1 follows from the fact that \( \ker \varphi \) and \( \text{im} \varphi \) are \( G \)-submodules of the irreducible \( G \)-modules \( V \) and \( W \), respectively. 2 \( \varphi \) is a \( \mathbb{K} \)-linear map on the vector space \( V \) and so, since \( \mathbb{K} \) is algebraically closed, it has an eigenvector \( v \neq 0 \). Thus, \( \varphi v = v \lambda \), for some \( \lambda \in \mathbb{K} \). Then \( v \in \ker(\varphi - \lambda I_V) \) and part 1 implies \( \varphi = \lambda I_V \). □

Consequences:
1. If \( V \) is an irreducible \( G \)-module, then \( \text{Hom}_G(V, W) \) is a division algebra.
2. If \( \mathbb{K} \) is algebraically closed, then \( \text{End}_G(V) = \mathbb{K} \cdot I_V \), is a field isomorphic to \( \mathbb{K} \).

Example 1: (Representations of Abelian Groups)
Let \( G \) be a group and \( V \) a \( G \)-module over an algebraically closed field \( \mathbb{K} \). If we identify the element \( g \in G \) with the map \( g_v \in \text{End}_\mathbb{K}(V) \), then by definition it follows that \( g \in G \) is a \( G \)-module homomorphism if and only if \( gh = hg \), \( \forall h \in G \iff g \in Z(G) \).

Now assume \( G \) is abelian. By above, all \( g \in G \) are \( G \)-homomorphisms. Let \( V \) be an irreducible \( G \)-module. By Schur’s lemma, we have \( g = \lambda_g I_V \) for some \( \lambda_g \in \mathbb{K} \). Hence every subspace of \( V \) is \( G \)-invariant. Since \( V \) is irreducible, this forces \( \dim V = 1 \) so that \( V \) can be identified with \( \mathbb{K} \). Then \( g \) becomes a linear functional on \( \mathbb{K} \):

\[
g \mapsto (c \in \mathbb{K} \mapsto \lambda_g c \in \mathbb{K})
\]

Since \( \lambda_g h c = (gh)(c) = g(hc) = g(\lambda_h c) = \lambda_g (\lambda_h c) = \lambda_g \lambda_h c \), \( \forall c \in \mathbb{K} \), we get \( \lambda_g h = \lambda_g \lambda_h \) for all \( g, h \in G \). Thus \( g \mapsto \lambda_g \) is a homomorphism of \( G \) and the multiplicative group \( \mathbb{K}^* \). In particular, \( g^n = 1 \Rightarrow (\lambda_g)^n = 1 \), since \( 1 \in G \) is the identity map on \( \mathbb{K} \). In other words, \( \lambda_g \) is a root of unity in \( \mathbb{K} \), of order dividing the order of \( g \).

Example 2: In the previous example let \( G = \mathbb{Z}_2 \times \mathbb{Z}_3 \) be generated by elements \( a \) of order 2 and, and \( b \) of order 3. Also let \( \mathbb{K} = \mathbb{C} \). Then all possible maps \( g \in G \mapsto \lambda_g \in \mathbb{C} \) can be arranged in the following character table of \( G \):

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>a</th>
<th>b</th>
<th>b^2</th>
<th>ab</th>
<th>ab^2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \rho_1 )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( \rho_2 )</td>
<td>1</td>
<td>1</td>
<td>w</td>
<td>w^2</td>
<td>w</td>
<td>w^2</td>
</tr>
<tr>
<td>( \rho_3 )</td>
<td>1</td>
<td>1</td>
<td>w^2</td>
<td>w</td>
<td>w^2</td>
<td>w</td>
</tr>
<tr>
<td>( \rho_4 )</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>( \rho_5 )</td>
<td>1</td>
<td>-1</td>
<td>w</td>
<td>-w</td>
<td>-w</td>
<td>-w</td>
</tr>
<tr>
<td>( \rho_6 )</td>
<td>1</td>
<td>-1</td>
<td>w^2</td>
<td>-w^2</td>
<td>-w</td>
<td>-w</td>
</tr>
</tbody>
</table>

where \( w \in \mathbb{C}, \ w^3 = 1 \)

Note that any 2 rows (or columns) are orthogonal.

Theorem 2: (Maschke) Assume char\( \mathbb{K} \nmid |G| \). Let \( V \) be a \( G \)-module and \( W \) – a submodule of \( V \). Then there exists a submodule \( W' \) of \( V \) such that \( W \bigoplus W' = V \).

Proof: Let \( U \) be a subspace of \( V \) such that \( W \bigoplus U = V \). Let \( P_0 \) be the projection of \( V \) onto \( W \) (with respect to \( U \)). Define \( P : V \mapsto W \) by \( P_v = \sum_{g \in G} g P_0(g^{-1} v) \). Note that \( P \) is \( \mathbb{K} \)-linear since \( P_0 \) and \( g \) are. For all \( v \in V \) and \( h \in G \), we have:

\[
P(hv) = \sum_{g \in G} gh^{-1} g P_0(g^{-1} hv) = \sum_{g \in G} h(h^{-1} g) P_0((h^{-1} g)^{-1} v) = \sum_{x \in G} h x^{-1} P_0(xv) = h P(v).
\]

Therefore, \( P \) is a \( G \)-module homomorphism.

Next, observe that \( P(w) = |G| w, \ \forall w \in W \). Indeed, \( P(w) = \sum_{g \in G} g P_0(g^{-1} w) = \)
\[ \sum_{g \in G} g(g^{-1}w) = |G|w, \text{ since } W \text{ is } G\text{-invariant, and on } W, P_0 \text{ is just the identity map.} \]

Take \( W' = \ker P \), which is a submodule of \( V \). We claim that this \( W' \) satisfies \( W \bigoplus W' = V \).

First, pick an arbitrary \( u \in V \). Then, since \( \text{char } \mathbb{K} \nmid |G| \), there exists \( v \in V \) such that \( u = |G|v \).

Write
\[
u = P(v) + (|G|v - P(v)) \tag{1}
\]

Note that \( P( |G|v - P(v) ) = P( |G|v) - P(P(v)) = |G|P(v) - |G|P(v) = 0 \), since \( P(v) \in W \) and on \( W \), \( P \) is multiplication by \( |G| \). Hence \( |G|v - P(v) \in W' \).

Then (1) implies that \( W + W' = V \).

Finally, if \( w \in W \cap W' \) then \( |G|w = P(w) \) on one hand, and \( P(w) = 0 \) on the other. Again since \( \text{char } \mathbb{K} \nmid |G| \), we get \( w = 0 \). Therefore, we have a direct sum: \( W \bigoplus W' = V \)

### 0.3 Basic \( G \)-Modules

Let \( G \) be any group, and \( V, W \) be \( G \)-modules over some field \( \mathbb{K} \). Then,

1. \( V \bigoplus W \) is a \( G \)-module under \( g(v, w) = (gv, gw) \), \( \forall g \in G \).

2. Recall the tensor product \( V \otimes_{\mathbb{K}} W \), which is defined as the free product of \( V \) and \( W \) modulo the "module" relations:
\[
\begin{align*}
(v_1 + v_2) \otimes w &= v_1 \otimes w + v_2 \otimes w \\
v \otimes (w_1 + w_2) &= v \otimes w_1 + v \otimes w_2 \\
c(v \otimes w) &= (cv) \otimes w = v \otimes (cw)
\end{align*}
\]

for all \( v, v_1, v_2 \in V \), \( w, w_1, w_2 \in W \) and \( c \in \mathbb{K} \).

Then \( V \otimes_{\mathbb{K}} W \) is a \( G \)-module under \( g(v \otimes w) = (gv \otimes gw) \), \( \forall g \in G \), and extension by linearity.

3. \( \text{Hom}(V, W) \) is a \( G \)-module as follows: for \( \varphi \in \text{Hom}(V, W) \), define \( g\varphi \) by \( (g\varphi)(v) = g(\varphi(g^{-1}v)) \), \( \forall g \in G \), and extend linearly.

4. Letting \( W = \mathbb{K} \) in above, we get that the dual \( V^* = \text{Hom}(V, \mathbb{K}) \) is a \( G \)-module under \( (g\varphi)(v) = g(\varphi(g^{-1}v)) = \varphi(g^{-1}v) \), since \( g \in G \) acts on \( \mathbb{K} \) as the identity map.

**Lemma 1:** \( \text{Hom}(V, W) \cong V^* \otimes W \) as \( G \)-modules.

**Proof:** Consider \( \gamma : V^* \otimes W \rightarrow \text{Hom}(V, W) \), defined by
\[
\varphi \otimes w \mapsto \gamma (v \mapsto \varphi(v) \cdot w) \tag{2}
\]

and extended linearly.

Then \( \gamma \) is injective: Indeed, let \( \xi \in \ker \gamma \). We may assume \( \xi = \sum_{i=1}^{n} \varphi_i \otimes w_i \) where \( w_i \in W \) are linearly independent (otherwise \( \xi = 0 \)). Then \( \gamma(\xi) = 0 \) implies
\[
\sum_{i=1}^{n} \varphi_i(v) \cdot w_i = 0, \forall v \in V.
\]
Hence \( \varphi_i(v) = 0, \forall v \in V, i = 1, n \), and thus \( \xi = 0 \). We conclude that \( \ker \gamma = 0 \).

Further, \( \gamma \) must also be onto, since \( \text{Hom}_G(V, W) \) and \( V^* \otimes W \) have same finite dimension \( \dim_K V \cdot \dim_K W \) over \( K \) (so they are isomorphic as vector spaces).

Finally, \( \gamma \) is a \( G \)-homomorphism: Let \( g \in G \).

We have \( \gamma g(\varphi \otimes w)(v) = (\gamma(\varphi \otimes gw))v = g\varphi(v) \cdot gw, \forall v \in V \).

We also have \( g\gamma(\varphi \otimes w)(v) = g(\varphi(v) \cdot w) = \varphi(v) \cdot gw \forall v \in V \), since \( \varphi(v) \in K \). But \( g \) acts as the identity on \( K : g\varphi(v) = \varphi(v) \). Therefore, \( g\gamma = \gamma g \), as desired. \( \blacksquare \)

### 0.4. REPRESENTATIONS AND CHARACTERS

Let \( G \) be a group and \( V \) a vector space over some field \( K \). We assume that \( \text{char} K \nmid |G| \).

A linear map (or representation) of \( G \) on \( V \), is a group homomorphism \( \rho : G \to GL(V) \).

Given a representation \( \rho \) of \( G \) on \( V \), its character \( \chi_\rho \) is a map : \( G \to K \) defined by \( \chi_\rho(g) = \text{trace}(\rho(g)) \). Thus, the following diagram commutes:

\[
\begin{array}{ccc}
G & \xrightarrow{\rho} & GL(V) \\
\downarrow{\chi_\rho} & & \downarrow{\text{trace}} \\
K & & 
\end{array}
\]

Note: If \( \rho : G \to GL(V) \) is linear, then \( V \) is a \( G \)-module under \( gv = \rho(g)(v) \).

Conversely, if we start with a \( G \)-module \( M \), the map \( g \mapsto g_M \) defined in section 0.1 is a representation of \( G \) on \( M \). We denote its character by \( \chi_M \).

**Theorem 3:** Let \( \rho \) be a linear map of \( G \) on \( M \), and \( V, W \) – any \( G \)-modules. Assume the ground field \( K \) is algebraically closed.

Then

1. The linear operator \( \rho(g) \in GL(V) \) is diagonalizable, with roots of unity as eigenvalues.

2. \( \chi_\rho(h^{-1}gh) = \chi_\rho(g), \forall g, h \in G \) (i.e. \( \chi_\rho \) is a class function on \( G \)).

3. \( \chi_{V \oplus W} = \chi_V + \chi_W \)

4. \( \chi_{V \otimes W} = \chi_V \cdot \chi_W \)

5. \( \chi_{\text{Hom}(V, W)} = \chi_{V^*} \cdot \chi_W \)

6. If \( K = \mathbb{C} \), then \( \chi_\rho(g^{-1}) = \overline{\chi_\rho(g)}, \forall g \in G \) and \( \chi_{V^*} = \overline{\chi_V} \)

**Proof:** 1 Let \( n \) be the order of \( g \), then \( (\rho g)^n = \rho(g^n) = \rho(1) = I_M \). Hence the minimal polynomial of \( \rho g \) divides \( x^n - 1 = \prod_{i=1}^n (x - \omega_i) \), where \( w_1, \ldots, w_n \in K \) are the \( n \)th roots of unity. Since \( x^n - 1 \) is separable, (because \( \text{char} K \nmid |G| \)) so is the minimal polynomial of \( \rho g \).

Therefore the Jordan form of \( \rho g \) is diagonal and all eigenvalues of \( \rho g \) are roots of unity of order dividing the order of \( g \). 2 Since \( \rho \) is a homomorphism, \( \chi_\rho(h^{-1}gh) \) and \( \chi_{\rho g} \) are traces of similar matrices, so they are equal. 3 Let \( g \in G \). Fix bases \( v_1, \ldots, v_n, w_1, \ldots, w_m \) of \( V \) and \( W \), and build the basis \( \{(v_i, 0) | i = 1, n\} \cup \{(0, w_j) | j = 1, m\} \) of \( V \oplus W \). It remains
to note that, with respect to these bases, \(g_V \oplus W\) is built of 2 diagonal blocks: \(g_V\) and \(g_W\).

4 Let \(v_1, \ldots, v_n, w_1, \ldots, w_m\) be eigenbases of \(g_V \in GL(V)\) and \(g_W \in GL(W)\) respectively, so that \(g_V(v_i) = \lambda_i v_i, g_W(w_j) = \mu_j w_j\), with \(\lambda_i, \mu_j \in \mathbb{K}\). It follows that the \((v_i \otimes w_j)\)'s form a basis of \(V \otimes W\) (they certainly span \(V \otimes W\), and there are \(n \cdot m\) of them, which is precisely \(\dim_{\mathbb{K}}(V \otimes W)\)). Note that \(g_V \otimes W(v_i \otimes w_j) = (g_V(v_i) \otimes g_W(w_j)) = \lambda_i \mu_j(v_i \otimes w_j)\), so that the above is actually a \(g_V \otimes W\)-eigenbasis. Hence \(\chi_V \otimes W(g) = \sum_{i,j} \lambda_i \mu_j = \sum_i \lambda_i \sum_j \mu_j = \chi_V(g) \cdot \chi_W(g)\). 5 follows from Lemma 1 and part 4. 6 First, if \(\lambda_1, \ldots, \lambda_n\) are the eigenvalues of \(\rho(g)\) then \(1/\lambda_1, \ldots, 1/\lambda_n\) are the eigenvalues of \(\rho(g^{-1}) = (\rho g)^{-1}\), and \(\chi_{\rho}(g^{-1}) = \chi_{\rho}(g)\) follows, since \(|\lambda_i| = 1\) by part 1. For second part, let \(g \in G\) and \(v \in V\) be an eigenvalue of \(g \in GL(V)\). Then \((g \varphi)v = \varphi(g^{-1}v) = \star\)

### 0.5 The Projection formula. Orthogonality of the irreducible characters

Throughout the section, \(G\) will be a finite group and \(\mathbb{K} - \) an algebraically closed field such that \(\text{char}\mathbb{K} \nmid |G|\).

Let \(V\) be a \(G\)-module. Define \(V^G = \{v \in V \mid gv = v, \forall g \in G\}\). Note that \(V^G\) is a submodule of \(V\). Consider

\[
P = \frac{1}{|G|} \sum_{g \in G} g
\]

as a linear map on \(V\) (we identify \(g \in G\) with \(g_V \in \text{End}(V)\)).

Then \(P\) is a \(G\)-module homomorphism: Indeed, for \(h \in G\) we have \(Ph = \frac{1}{|G|} \sum_{g \in G} gh = h \frac{1}{|G|} \sum_{g \in G} (h^{-1}gh) = h \frac{1}{|G|} \sum_{x \in G} x \) (since \(g \mapsto h^{-1}gh\) is bijective) = \(hP\).

Furthermore, \(P\) is the projection of \(V\) onto \(V^G\) (i.e. \(V = V^G \oplus \ker P\)): First, note that \(\text{im}P \subseteq V^G\), since \(hP(v) = (h \frac{1}{|G|} \sum_{g \in G} g)v = \frac{1}{|G|} \sum_{g \in G} hv = \frac{1}{|G|} \sum_{x \in G} xv = P(v)\), for all \(v \in V, h \in G\). Also, \(v \in V^G \Rightarrow P v = \frac{1}{|G|} \sum_{g \in G} g v = v\). We conclude that \(\text{im}P = V^G\) and \(P \circ P = P\), i.e. \(P\) is the projection onto \(\text{im}P = V^G\).

**Lemma 2:** Let \(V, W\) be \(G\)-modules. Consider the \(G\)-module \(H = \text{Hom}(V, W)\).

1. \(\dim_{\mathbb{K}} V^G = \text{trace} \left( \frac{1}{|G|} \sum_{g \in G} g \right)\)
2. \(H^G = \text{Hom}_G(V, W)\)
3. \(\dim_{\mathbb{K}} H^G = \frac{1}{|G|} \sum_{g \in G} \chi_{V^*}(g) \chi_W(g)\)

**Proof:** 1 Pick a basis of \(V^G\), and extend it to a basis of \(V\). From the preceding argument it follows that, in this basis, the matrix of \(P\) will be diagonal, with first \(k = \dim_{\mathbb{K}} V^G\) diagonal entries 1 and the rest 0. So \(\dim_{\mathbb{K}} V^G = \text{trace}(P)\).

2 We have \(f \in H^G \Leftrightarrow (gf)(v) = f(v), \forall g \in G, v \in V \Leftrightarrow gf(g^{-1}v) = f(v), \forall g \in G, v \in V \Leftrightarrow f(g^{-1}v) = (g^{-1}f)(v), \forall g \in G, v \in V \Leftrightarrow f = f \in \text{Hom}_G(V, W)\), as desired.

By Lemma 1, \(H \cong V^* \otimes W\), as \(G\)-modules. So, \(\chi_H = \chi_{V^*} \otimes W = \chi_V \cdot \chi_W\), by Theorem 3.
Using this and part 1, we get
\[
\dim_K H^G = \text{trace} \left( \frac{1}{|G|} \sum_{g \in G} g \right) = \frac{1}{|G|} \sum_{g \in G} \chi_H(g) = \frac{1}{|G|} \sum_{g \in G} \chi_V(g) \chi_W(g) \]

Lemma 3: Let \( V, W \) be irreducible \( G \)-modules, and \( H = \text{Hom}(V, W) \). Then
\[
\dim_K H^G = \begin{cases} 
0, & \text{if } V \ncong W \\
1, & \text{if } V \cong W 
\end{cases}
\]

Proof: If \( V, W \) are non-isomorphic \( G \)-modules, then \( \text{Hom}_G(V, W) = 0 \) by Schur’s Lemma, so Lemma 3 implies \( H^G = 0 \). Now assume \( V \cong W \), and let \( \varphi, \psi \in \text{Hom}_G(V, W) \) be non-zero. By Schur’s Lemma, \( \psi \) is invertible and hence \( \varphi \psi^{-1} \in \text{Hom}_G(W, W) \). Again, Schur’s Lemma yields \( \varphi \psi^{-1} = \lambda I_W \), for some \( \lambda \in \mathbb{K} \), i.e. \( \varphi = \lambda \psi \). This proves that \( \dim_K \text{Hom}_G(V, W) = 1 \), and then by lemma 3 we are done.

For \( \mathbb{K} = \mathbb{C} \), it follows from Theorem 3 and Lemmas 2,3 that if \( V, W \) are irreducible \( G \)-modules, then
\[
\frac{1}{|G|} \sum_{g \in G} \chi_V(g) \cdot \chi_W(g) = \begin{cases} 
0, & \text{if } V \ncong W \\
1, & \text{if } V \cong W 
\end{cases}
\]

Let \( \text{class}(G) = \{ \alpha : G \to \mathbb{C} \mid \alpha \text{ is constant on conjugacy classes of } G \} \). Then \( \text{class}(G) \) is a complex vector space under pointwise addition and scalar multiplication of functions. Its dimension is the number of conjugacy classes in \( G \), since the class indicator functions form a basis of \( \text{class}(G) \).

For any \( \alpha, \beta \in \text{class}(G) \) we define
\[
\langle \alpha, \beta \rangle = \frac{1}{|G|} \sum_{g \in G} \alpha(g) \overline{\beta(g)}
\]

It’s easy to check that \( \langle \cdot, \cdot \rangle \) is a complex inner product on \( \text{class}(G) \). Note that \( \chi_V \in \text{class}(G) \) for any \( G \)-module \( V \).

Let \( \alpha : G \to \mathbb{C} \) and let \( V \) be a \( \mathbb{C}G \)-module. Consider the map \( P_{\alpha,V} = \sum_{g \in G} \alpha(g)g : V \to V \), which is certainly in \( \text{End}_\mathbb{C}(V) \). When is \( P_{\alpha,V} \) a \( G \)-module homomorphism, regardless of \( V \)? – If \( \alpha \in \text{class}(G) \), then \( P_{\alpha,V} \) is a \( G \)-module homomorphism.

Indeed, for any \( h \in G \), we have
\[
P_{\alpha,V}h = \sum_{g \in G} \alpha(g)gh = \sum_{g \in G} h\alpha(h^{-1}gh)h^{-1}gh \quad \text{(since } \alpha \in \text{class}(G)) = \\
= h \sum_{g \in G} \alpha(h^{-1}gh)h^{-1}gh = h \sum_{x \in G} \alpha(x)x = hP_{\alpha,V}, \quad \text{as desired.}
\]

Further, if \( V \) is an irreducible \( G \)-module and \( \alpha \in \text{class}(G) \) then
\[
P_{\alpha,V} = \frac{|G|}{\dim V} \langle \alpha, \overline{\chi_V} \rangle I_V
\]
Since $\alpha \in \text{class}(G)$, $P_{\alpha,V}$ is a $G$-module homomorphism. But $V$ is irreducible, so by Schur’s Lemma $P_{\alpha,V} = \lambda I_V$ for some $\lambda \in \mathbb{C}$. The claim follows, since

$$\lambda = \frac{\text{trace}(P_{\alpha,V})}{\dim V} = \frac{1}{\dim V} \sum_{g \in G} \alpha(g)\text{trace}(g) = \frac{1}{\dim V} \sum_{g \in G} \alpha(g)\chi_V(g) = \frac{|G|}{\dim V} \langle \alpha, \overline{\chi_V} \rangle.$$

**Theorem 4:** Let $G$ be a finite group. Then the irreducible characters form an orthonormal basis of $\text{class}(G)$ as a $\mathbb{C}$-vector space.

**Proof:** First, note that (3) shows that the irreducible characters form an orthonormal set for the inner product given by (4), and hence they are linearly independent. In particular, their number cannot exceed $n$, the number of conjugacy classes of $G$. So let $\chi_1, \ldots, \chi_k$ be the irreducible characters of $G$ ($k \leq n$). To prove that $\chi_1, \ldots, \chi_k$ span $\text{class}(G)$, we show that $(\text{span}\{\chi_1, \ldots, \chi_k\})^\perp = \{0\}$. Suppose $\beta \in \text{class}(G)$ satisfies $\langle \beta, \chi_i \rangle = 0$, $\forall i$. Then $\langle \alpha, \overline{\chi}_i \rangle = 0$, $\forall i$, where $\alpha = \overline{\beta} \in \text{class}(G)$. Hence, if $U$ is any irreducible $G$-module (so $\chi_U \in \{\chi_1, \ldots, \chi_k\}$), then (5) implies that $P_{\alpha,U} \equiv 0$. In fact, it follows that $P_{\alpha,V} \equiv 0$ for all $G$-modules $V$: Indeed, by Maschke’s theorem $V = \bigoplus_{j=1}^m U_j$, where $U_1, \ldots, U_m$ are irreducible $G$-modules. Then

$$P_{\alpha,V} = \sum_{g \in G} \alpha(g)g_V = \sum_{g \in G} \alpha(g)(\bigoplus_{j=1}^m g_{U_j}) = \bigoplus_{j=1}^m P_{\alpha,U_j} = \bigoplus_{j=1}^m 0_{U_j} \equiv 0_V.$$

In particular, for the left regular representation $(\mathbb{C}G)^\circ$ we have $P_{\alpha,(\mathbb{C}G)^\circ} \equiv 0$. Hence,

$$0 = P_{\alpha,(\mathbb{C}G)^\circ}(1_G) = \sum_{g \in G} \alpha(g)(g \cdot 1_G) = \sum_{g \in G} \alpha(g)g.$$

But $\{g \mid g \in G\}$ is a basis of $\mathbb{C}G$. Therefore, $\alpha(g) = 0$, $\forall g \in G$, i.e. $\alpha \equiv 0$, as desired.

**Corollary 4’:**
1. If $V \cong \bigoplus_{i=1}^m m_iV_i$ (by $m_iV_i$ we mean the direct sum of $m_i$ copies of $V_i$), where $V_1, \ldots, V_m$ are non-isomorphic irreducible $G$-modules, then $m_i = \langle \chi_{V_i}, \chi_{V_i} \rangle$, $\forall i$. (First, from (3) it follows that 2 irreducible $G$-modules are isomorphic if and only if they have same characters. Hence $\langle \chi_{V_i}, \chi_{V_i} \rangle = \sum_{j=1}^m \langle \chi_{m_jV_j}, \chi_{V_i} \rangle = \sum_{j=1}^m m_j\langle \chi_{V_j}, \chi_{V_i} \rangle = m_i$, by the theorem). This implies
2. Let $U, V$ be $G$-modules, with $U$ irreducible. The number of times (an isomorphic copy of) $U$ occurs in a decomposition of $V$ into irreducible submodules is $\langle \chi_V, \chi_U \rangle$ (so it doesn’t depend on the decomposition).
3. For any $G$-modules $V, W$ we have $V \cong W \iff \chi_V = \chi_W$. (Clearly $V \cong W \implies \chi_V = \chi_W$. Conversely, assume $\chi_V = \chi_W$. Let $V_1, \ldots, V_k$ be a complete set of irreducible $G$-modules. Then parts 1, 2 above, along with Maschke’s theorem show that $V \cong \bigoplus_{i=1}^k a_i V_i \cong W$, where $a_i = \langle \chi_{V_i}, \chi_{V_i} \rangle = \langle \chi_W, \chi_{V_i} \rangle$).
4. Let $V_1, \ldots, V_m$ be non-isomorphic irreducible $G$-modules.
   If $V \cong \bigoplus_{i=1}^m m_iV_i$, then $\langle \chi_V, \chi_{V_i} \rangle = \sum_{i=1}^m m_i^2$.
5. A $G$-module $V$ is irreducible if and only if $\langle \chi_V, \chi_V \rangle = 1$ (follows from 4).
0.6 The left regular representation \((\mathbb{C}G)\)

Let \(G\) be a finite group and let \(\chi\) be the character of the \(\mathbb{C}G\)-module \((\mathbb{C}G)\). Consider the basis \(\mathcal{B} = \{ g \mid g \in G \}\) of \(\mathbb{C}G\). Then \(g \in G\) acts on \(\mathcal{B}\) by permuting its elements. This action is fixed-point free for any \(g \neq 1\). Hence in this basis \(\mathcal{B}\), the matrix of \(g\) (identified with \(g(\mathbb{C}G) \in GL(\mathbb{C}G)\)) has 0’s along the diagonal for \(g \neq 1\), and 1’s otherwise. This shows that

\[
\chi(g) = \begin{cases} |G| & \text{if } g = 1 \\ 0 & \text{if } g \neq 1 \end{cases}
\]  (6)

Now let \(\chi_1, \ldots, \chi_k\) be the irreducible characters of \(G\), and \(V_1, \ldots, V_k\) - (one collection of) \(G\)-modules corresponding to them. By corollary 4', we have \((\mathbb{C}G) \cong \bigoplus_{i=1}^k m_i V_i\), where

\[
m_i = \langle \chi, \chi_i \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g) \overline{\chi_i(g)} = \frac{1}{|G|} \chi(1)\text{dim}V_i = \text{dim}V_i.
\]

Thus, \(V_i\) appears \(\text{dim}V_i\) times in \((\mathbb{C}G)\). Hence

\[
(\mathbb{C}G) \cong \bigoplus_{i=1}^k (\text{dim}V_i) V_i
\]  (7)

Lemma 4: Let \(G\) be a finite group and \(V_1, \ldots, V_k\) - \(G\)-modules corresponding to all the irreducible characters \(\chi_1, \ldots, \chi_k\) of \(G\). Then

1. \(\sum_{i=1}^k (\text{dim}V_i)^2 = |G|\).

2. \(\sum_{i=1}^k (\text{dim}V_i)\bar{\chi}_i(g) = 0, \; \forall g \in G, \; g \neq 1\).

Proof: The lemma follows by taking characters in (7), and then using (6)

0.7 The Character Table

Let \(\chi_1, \ldots, \chi_r\) be the irreducible characters of a group \(G\), and let \(K_1 = \{1\}, K_2, \ldots, K_r\) be the conjugacy classes of \(G\). Pick representatives \(g_i \in K_i, \forall i\) and consider the character table \(T\) of \(G\):

<table>
<thead>
<tr>
<th></th>
<th>(K_1)</th>
<th>(K_2)</th>
<th>\ldots</th>
<th>(K_r)</th>
</tr>
</thead>
</table>
| \(\chi_1\) | 1      | \(g_2\) | \ldots | \(g_r\) |  \\
| \(\ddots\) | \(\ddots\) | \(\ddots\) | \(\ddots\) | \(\ddots\) |  \\
| \(\chi_r\) | \(\chi_i(g_j)\) | \(\chi_i(g_j)\) | \ldots | \(\chi_i(g_j)\) |  \\

Recall that

\[
\langle \chi_i, \chi_j \rangle = \frac{1}{|G|} \sum_{t=1}^r |K_t| \bar{\chi}_i(g_t) \overline{\chi_j(g_t)} = \delta_{ij}
\]
which gives orthogonality of the matrix $\sqrt{\frac{|K_i|}{|G|}}\chi_i(g_j)$. But if a matrix is orthogonal, so is its adjoint. Hence we also have column orthogonality in $T$:

$$\sum_{t=1}^{r} \chi_t(g_i)\overline{\chi_t(g_j)} = \begin{cases} \frac{|G|}{|K_i|} = |C_G(g_i)| & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

The following observations are helpful in constructing character tables of groups:

(1) **The number of (irreducible) degree 1 characters of $G$ is $|G'| = |G/[G,G]|$:*** Assume $\rho'$ is a degree 1 representation of $G' = G/[G,G]$ on $V$. Then $\rho : G \to GL(V)$ defined by $\rho(g) = \rho'(g')$ is a degree 1 representation of $G$, where $g' = g[G,G]$. Conversely, assume $\rho$ is a degree 1 representation of $G$ on $V$. Then $GL(V) \cong \mathbb{C}^\times$, so that $\rho(g)\rho(h) = \rho(h)\rho(g)$, $\forall g, h \in G$. Hence $\rho' : G' \to GL(V)$ given by $\rho'(g') = \rho(g)$, is a well-defined degree 1 representation of $G'$, because

$$\rho(xyx^{-1}y^{-1}) = \rho(x)\rho(y)\rho(x^{-1})\rho(y^{-1}) = \rho(x)\rho(x^{-1})\rho(y)\rho(y^{-1}) = I_V,$$

implies that $\rho$ acts trivially on $[G,G]$. Therefore, the number of degree 1 characters of $G$ is the number of degree 1 characters of $G'$, and the latter is $|G'|$, because $|G'|$ is abelian.

(2) **A character $\chi$ of $G$ is irreducible $\iff \langle \chi, \chi \rangle = 1$. (by Corollary 4')**

(3) **If $V,W$ are $G$-modules such that $\chi_V$ is irreducible and $\chi_W$ has degree 1, then $\chi_V \otimes W = \chi_V \cdot \chi_W$ is also irreducible:**

Since $dimW = 1$, $\chi_W(g)$ is a root of unity for any $g \in G$ and hence $\chi_W(g)\overline{\chi_W(g)} = 1$. Then we have

$$\langle \chi_V \otimes W, \chi_V \otimes W \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_V(g)\chi_W(g)\overline{\chi_V(g)}\overline{\chi_W(g)} =$$

$$= \frac{1}{|G|} \sum_{g \in G} \chi_V(g)\overline{\chi_V(g)} = \langle \chi_V, \chi_V \rangle = 1,$$

so $\chi_V \otimes W$ is irreducible.

(4) **Assume $H \triangleleft G$. Then any representation $\rho'$ of $G/H$ naturally gives a representation $\rho$ of $G$. Moreover, $\rho$ is irreducible if and only if $\rho'$ is irreducible:**

The map $\rho$ defined by $\rho(g) = \rho'(ghH)$, is a well-defined representation of $G$. Then

$$\langle \chi_{\rho'}, \chi_{\rho'} \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_{\rho'}(g)\overline{\chi_{\rho'}(g)} = \frac{1}{|G|} \sum_{g \in G} \chi_{\rho'}(ghH)\overline{\chi_{\rho'}(gh)} = \frac{|H|}{|G|} \sum_{g' \in G/H} \chi_{\rho'}(g')\overline{\chi_{\rho'}(g')},$$

$$= \langle \chi_{\rho'}, \chi_{\rho'} \rangle,$$

and the second assertion follows from 2).

It should be noted that the inner product $\langle \chi_{\rho'}, \chi_{\rho'} \rangle$ is taken in $\text{class}(G)$, whereas $\langle \chi_{\rho''}, \chi_{\rho''} \rangle$ – in $\text{class}(G/H)$.

(5) **Assume $\theta$ is a character of $G$ (over $\mathbb{C}$). Then $\overline{\theta}$, defined by $\overline{\theta}(g) = \overline{\theta(g)}$, $\forall g \in G$, is also a character of $G$. Moreover, $\overline{\theta}$ is irreducible $\iff \theta$ is irreducible:**

Let $\rho : G \to GL(V)$ be a representation such that $\theta = \chi_{\rho}$. Fix a $\mathbb{C}$-basis $B = \{v_1, \ldots, v_n\}$ of $V$. Define $\rho' : G \to GL(V)$ by $\rho'(g) = \overline{\rho(g)} \in GL(V)$ (the conjugate of the matrix of $\rho(g)$ in the basis $B$). Clearly $\rho'(gh) = \rho(g)\rho(h) = \rho(g)\rho(h) = \rho(g)\rho(h) = \rho'(g)\rho'(h)$, $\forall g, h \in$
0.7. THE CHARACTER TABLE

Let $V$ be an $n$-dimensional $G$-module over $F$, where $F$ is an algebraically closed field such that $\text{char}(F) \neq 2$. If $\{v_1, \ldots, v_n\}$ is a basis of $V$, then $\{v_i \otimes v_j \mid 1 \leq i, j \leq n\}$ is a basis of $V \otimes_F V$. Thus, there is a unique linear map $\theta$ on $V \otimes_F V$, given by $\theta(v_i \otimes v_j) = v_j \otimes v_i$. For any $a_i, b_j \in F$, we have

$$\theta(a_1 v_1 + \cdots + a_n v_n \otimes b_1 v_1 + \cdots + b_n v_n) = \sum_{i,j=1}^n a_i b_j \theta(v_i \otimes v_j)$$

$$= \sum_{i,j=1}^n b_j a_i (v_i \otimes v_j) = (b_1 v_1 + \cdots + b_n v_n) \otimes (a_1 v_1 + \cdots + a_n v_n).$$

i.e. $\theta(u \otimes v) = v \otimes u$, $\forall u, v \in V$. In particular, it follows that $\theta$ does not depend on the chosen basis $\{v_1, \ldots, v_n\}$ of $V$. Next, $\theta$ (and hence $\theta - I_{V \otimes V}$) is a $G$-module homomorphism, since $\theta(g(v_i \otimes v_j)) = \theta(gv_i \otimes gv_j) = gv_j \otimes gv_i = g(\theta(v_i \otimes v_j))$, $\forall g \in G$.

We define the symmetric square: $\text{Sym}^2(V \otimes V) = \text{ker}(\theta - I_{V \otimes V})$ (or just $\text{Sym}^2$, when $V$ is clear from the context), which is a $G$-submodule of $V \otimes V$. We have

$$u = \sum_{i,j=1}^n a_{ij} (v_i \otimes v_j) \in \text{ker}(\theta - I_{V \otimes V}) \iff \sum_{i,j=1}^n (a_{ij} - a_{ji}) v_i \otimes v_j = 0 \iff a_{ij} = a_{ji}, \forall i, j$$

$$\iff u \in \text{span}\{v_i \otimes v_j + v_j \otimes v_i \mid 1 \leq i \leq j \leq n\}$$

Clearly the $\{v_i \otimes v_j + v_j \otimes v_i\}$'s are linearly independent. Thus, they form a basis of $\text{Sym}^2$ so that $\dim_F \text{Sym}^2 = \frac{n(n+1)}{2}$. To compute $\chi_{\text{Sym}^2}$, let $g \in G$. Since $F$ is algebraically closed, it follows that $g$, regarded as a an element of $GL(V)$, is diagonalizable. Thus, we may choose $v_1, \ldots, v_n$ to be eigenvectors of $g$, with eigenvalues $\lambda_1, \ldots, \lambda_n$. Then $g(v_i \otimes v_j + v_j \otimes v_i) = \lambda_i \lambda_j (v_i \otimes v_j + v_j \otimes v_i)$ and hence $\{\lambda_i \lambda_j \mid 1 \leq i \leq j \leq n\}$ is the set of eigenvalues of $g$ as an element of $GL(\text{Sym}^2)$. Therefore,

$$\chi_{\text{Sym}^2}(g) = \sum_{1 \leq i \leq j \leq n} \lambda_i \lambda_j = \frac{1}{2}((\sum_{i=1}^n \lambda_i)^2 + \sum_{i=1}^n \lambda_i^2) = \frac{(\chi_V(g))^2 + \chi_V(g^2)}{2} \tag{8}$$

Accordingly, we also define the alternating square: $\text{Alt}^2(V \otimes V) = \text{ker}(\theta + I_{V \otimes V})$. Similarly, one shows that $\{v_i \otimes v_j - v_j \otimes v_i \mid 1 \leq i < j \leq n\}$ is a basis of $\text{Alt}^2$, so that $\dim_F \text{Alt}^2 = \frac{n(n-1)}{2}$. Likewise, we get

$$\chi_{\text{Alt}^2}(g) = \sum_{1 \leq i < j \leq n} \lambda_i \lambda_j = \frac{1}{2}((\sum_{i=1}^n \lambda_i)^2 - \sum_{i=1}^n \lambda_i^2) = \frac{(\chi_V(g))^2 - \chi_V(g^2)}{2} \tag{9}$$
0.8 Some Examples

Example 1: Let $G = S_3$ (the symmetric group on 3 symbols).

<table>
<thead>
<tr>
<th>$S_3$</th>
<th>1 2 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(12) (123)</td>
</tr>
</tbody>
</table>

Since $[S_n, S_n] = A_n$ in general, we have $S_n/[S_n, S_n] \cong \mathbb{Z}_2$ for $n \geq 2$. Hence $S_n$ has 2 irreducible characters of degree 1: the trivial one ($\chi_1$), and the sign character ($\chi_2(g) = 1$ if $g \in A_n$ and $\chi_2(g) = -1$ otherwise). The third character can be found directly from the orthogonality relations. Alternatively, for $n \geq 2$ consider the natural action of $S_n$ on a basis $B = \{e_1, e_2, \ldots, e_n\}$ of $\mathbb{C}^n$, given by $g(e_i) = e_{g(i)}$. This turns $\mathbb{C}^n$ into an $S_n$-module $V$. Clearly $W = \text{span}\{e_1 + \cdots + e_n\}$ is a submodule of $V$. The map $\phi : V \to W$

$\phi(a_1e_1 + \cdots + a_ne_n) = \frac{a_1 + \cdots + a_n}{n} (e_1 + \cdots + e_n)$

is an $S_n$-homomorphism. Indeed for all $g \in S_n$, $g(\phi(a_1e_1 + \cdots + a_ne_n)) = \frac{a_1 + \cdots + a_n}{n} (e_1 + \cdots + e_n) = \frac{a_1 + \cdots + a_n}{n} (e_{g(1)} + \cdots + e_{g(n)}) = \phi(g(a_1e_1 + \cdots + a_ne_n))$.

Since $\text{im}\phi = W$, it follows that $U = \ker\phi$ is a $n - 1$ dimensional submodule of $V$. If we represent $g \in S_n$ as a linear map on $V$ then, with respect to the basis $B$, $g$ is a permutation matrix (its $ij$ element is 1 if $g(j) = i$ and 0 otherwise). Hence $\chi_U(g)$ is the number of fixed points of $g$. Since $\chi_W(g) = 1$ and $V = W \bigoplus U$, we get $\chi_U(g) = \text{(number of fixed points of } g\text{-1}}$. It can be shown that $U$ is irreducible. We denote the $n$ dimensional module $V$ by $P_n$ (so $\chi_V = \chi_{P_n} - \chi_1$ is an irreducible character of $S_n$ of degree $n - 1$). For $n = 3$, $\chi_U = \chi_3$.

Example 2: $G = S_4$.

<table>
<thead>
<tr>
<th>$S_4$</th>
<th>1 6 8 6 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(12) (123) (1234) (12)(34)</td>
</tr>
</tbody>
</table>

$\chi_1, \chi_2$ are the trivial and sign characters; $\chi_3 = \chi_{P_3} - \chi_1$ is irreducible by above. Also $\chi_4 = \chi_3 \cdot \chi_2$ is irreducible by (3), section 0.7. The last character $\chi_5$ has degree $\sqrt{24 - 1^2 - 1^2 - 3^2 - 3^2} = 2$. Now $\chi_5 = \chi_5 \cdot \chi_2$, since both are irreducible characters of degree 2. So we get 2 zeroes in row 5. The other 2 entries are found from orthogonality
relations. Alternatively, note that $H = \{1, (12)(34), (14)(23), (13)(24)\} = \langle A_4, A_4 \rangle$ is a normal subgroup of $S_4$, and $S_4/H \cong S_3$. If we lift $\chi_3$ from $S_3 \cong S_4/H$ to $S_4$ we get $\chi_5$, which is automatically irreducible by (4) of section 0.7.

**Example 3:** $G = A_4$ (the alternating group on 4 letters).

\[
\begin{array}{c|cccc}
A_4 & 1 & 4 & 4 & 3 \\
1 & (123) & (132) & (12)(34) & \\
\chi_1 & 1 & 1 & 1 & 1 \\
\chi_2 & 1 & w & w^2 & 1 \\
\chi_3 & 1 & w^2 & w & 1 \\
\chi_4 & 3 & 0 & 0 & -1 \\
\end{array}
\]

where $w \in \mathbb{C} \setminus \mathbb{R}$, $w^3 = 1$.

We have $[A_4, A_4] = \{1, (12)(34), (13)(24), (14)(23)\} \cong V_4$, so $A_4/[A_4, A_4] \cong C_3$. Hence $A_4$ has 3 representations of degree 1, involving complex roots of unity of order dividing 3 ($\chi_1, \chi_2, \chi_3$). Recall that $\chi_{P_4} - \chi_1$ is an irreducible character of $S_4$. If we restrict to $A_4$, we obtain $\chi_4$ (which is irreducible, since $\langle \chi_4, \chi_4 \rangle = 1/12 \cdot (3^2 + 3 \cdot (-1)^2) = 1$).

**Example 4:** $G = S_5$.

\[
\begin{array}{c|ccccc}
S_5 & 1 & 10 & 20 & 30 & 24 & 15 & 20 \\
1 & (12) & (123) & (1234) & (12345) & (12)(34) & (12)(345) & \\
\chi_1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\chi_2 & 1 & -1 & 1 & -1 & 1 & -1 \\
\chi_3 & 4 & 2 & 1 & 0 & -1 & 0 & -1 \\
\chi_4 & 4 & -2 & 1 & 0 & -1 & 0 & 1 \\
\chi_5 & 6 & 0 & 0 & 0 & 1 & -2 & 0 \\
\chi_6 & 5 & 1 & -1 & -1 & 0 & 1 & 1 \\
\chi_7 & 5 & -1 & -1 & 1 & 0 & 1 & -1 \\
\end{array}
\]

$\chi_1, \chi_2$ are the trivial and sign characters; $\chi_3 = \chi_{P_5} - \chi_1$ is irreducible and hence, so is $\chi_4 = \chi_3 \cdot \chi_2$. The alternating square of $V_3$ (the representation whose character is $\chi_3$) has character $\chi_5$ (recall that $\chi_{Alt^2V}(g) = \frac{1}{2}[\chi_V(g)^2 - \chi_V(g^2)]$), and is irreducible since $\langle \chi_5, \chi_5 \rangle = 1/120 \cdot (6^2 + 24 \cdot 1^2 + 15 \cdot 2^2) = 1$. The other 2 characters are found by orthogonality relations of the character table.

**Example 5:** $G = A_5$.

\[
\begin{array}{c|cccc}
A_5 & 1 & 20 & 15 & 12 & 12 \\
1 & (123) & (12)(34) & (12345) & (21)(345) & \\
\chi_1 & 1 & 1 & 1 & 1 & 1 \\
\chi_2 & 4 & 1 & 0 & -1 & -1 \\
\chi_3 & 5 & -1 & 1 & 0 & 0 \\
\chi_4 & 3 & 0 & -1 & \frac{1 + \sqrt{5}}{2} & \frac{1 - \sqrt{5}}{2} \\
\chi_5 & 3 & 0 & -1 & \frac{1 - \sqrt{5}}{2} & \frac{1 + \sqrt{5}}{2} \\
\end{array}
\]

$\chi_2$ and $\chi_3$ are restrictions from the characters $\chi_3$ and $\chi_6$ of $S_5$ (and they are irreducible by computation). To obtain $\chi_4$ and $\chi_5$, recall that $A_5$ is isomorphic to the group $H$ of
rotations and symmetries of a regular icosahedron. This gives us a degree 3 representation $ho : A_5 \to GL(\mathbb{R}^3)$ (the origin is the center of the icosahedron). Note that $ho$ is irreducible since $A_5 \cong \mathcal{H}$ acts transitively on the set of vertices of the icosahedron. Since $|(12345)| = 5$, it’s immediate that $\rho(12345)$ is a rotation by $\alpha = \frac{2\pi}{5}$ or $2\alpha = \frac{4\pi}{5}$ (depending on the isomorphism $A_5 \cong \mathcal{H}$). In the first case, under a suitable basis of $\mathbb{R}^3$, we have

$$\rho(12345) = \begin{pmatrix} \cos \alpha & \sin \alpha & 0 \\ -\sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow \chi_\rho[(12345)] = 2 \cos \alpha + 1 = \frac{1 + \sqrt{5}}{2},$$

which reveals the character $\chi_4$. The second case yields the character $\chi_5$.

**Example 6:** $G = D_8 = \langle r, s \mid r^4 = s^2 = 1, srs = r^{-1} \rangle$ (rotations and symmetries of a square).

<table>
<thead>
<tr>
<th>$D_8$</th>
<th>1</th>
<th>$r^2$</th>
<th>$r, r^{-1}$</th>
<th>$s, r^2 s$</th>
<th>$rs, r^{-1} s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi_1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_2$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>$\chi_3$</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>$\chi_4$</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_5$</td>
<td>2</td>
<td>-2</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Since $[D_8, D_8] = \langle r^2 \rangle$ and $D'_8 = D_8/[D_8, D_8] \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, we have 4 representations of degree 1 (with characters $\chi_1, \ldots, \chi_4$). These characters involve only $\pm 1$, since every element of $D'_8$ has order at most 2 (i.e. $D'_8$ has exponent 2). Finally, $\chi_5$ is found by orthogonality relations.

**Example 7:** $G = Q_8 = \langle i, j, k \mid i^2 = j^2 = k^2 = -1, ij = k, jk = i \rangle$ (the quaternions).

<table>
<thead>
<tr>
<th>$Q_8$</th>
<th>1</th>
<th>-1</th>
<th>$\pm i$</th>
<th>$\pm j$</th>
<th>$\pm k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi_1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_2$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>$\chi_3$</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>$\chi_4$</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_5$</td>
<td>2</td>
<td>-2</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

We have $[Q_8, Q_8] = \langle -1 \rangle$ and $Q'_8 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. From these we obtain $\chi_1, \ldots, \chi_4$. Again, $\chi_5$ is determined by orthogonality.

**Remark:** The groups $D_8, Q_8$ have the ”same” character table even though they are not isomorphic. This happened since the degree 1 characters (which depend only on the derived groups $D'_8 \cong Q'_8$) were enough to determine the whole character table.

### 0.9 Properties Of The Character Table $T$

Let $G$ be a finite group and assume $\rho$ is a representation of $G$ on $V$ over some algebraically closed field, whose character is $\chi$. We define

$$K_\chi = \{ g \in G \mid g \text{ acts on } V \text{ as the identity map} \} \quad (\text{i.e. } K_\chi = \ker \rho).$$
0.10. MORE ON THE REGULAR REPRESENTATION

Note that $K_\chi = \{g \in G \mid \chi(g) = \chi(1)\}$: Indeed, $\chi(g)$ is a sum of $n = \dim V = \chi(1)$ roots of unity (namely, the eigenvalues of $\rho(g)$). Hence, $\chi(g) = n \iff$ all eigenvalues of $\rho(g)$ are 1 $\iff \rho(g) = I$ (because $\rho$ is diagonalizable).

**Theorem 5:** Let $G$ be a finite group. Let $\chi_1, \ldots, \chi_n$ be its irreducible characters.

1. $T$ (the character table of $G$) determines the lattice of normal subgroups of $G$. Specifically, $H \trianglelefteq G \iff H = \cap_{i \in I} K_{\chi_i}$, for some $I \subseteq \{1, 2, \ldots, n\}$.

2. $G$ is simple $\iff$ the only irreducible $\chi_i$ whose kernel $K_{\chi_i}$ is non-trivial, is the trivial character $\chi_1$.

3. $T$ determines whether $G$ is solvable.

4. In principle, $T$ determines whether $G$ is nilpotent.

**Proof:** 1 Let $H \trianglelefteq G$. Consider the left regular representation of $\mathcal{G} = G/H$, denoted as $(\mathcal{G}G)^0$. Lift $(\mathcal{G}G)^0$ to a representation of $G$ (which permutes $H$-cosets in $G$ by left multiplication). Let $\psi$ denote its character. Then $K_{\psi} = H$, since $g(g'H) = g'H \forall g' \in G$ if and only if $g \in H$. Hence $g \in H \iff \psi(g) = \psi(1)$. On the other hand, write $\psi = \sum_{i=1}^{n} a_i \chi_i$, where $a_1, \ldots, a_n$ are non-negative integers. Then

$$|\psi(g)| \leq \sum_{i=1}^{n} a_i |\chi_i(g)| \leq \sum_{i=1}^{n} a_i |\chi_i(1)| = \psi(1),$$

which implies $\psi(g) = \psi(1) \iff \chi_i(g) = \chi_i(1), \forall i \in I$, where $I = \{1 \leq i \leq n | a_i > 0\}$ $\iff g \in K_{\chi_i}, \forall i \in I$. Therefore, $H = \cap_{i \in I} K_{\chi_i}$.

Conversely, each $K_{\chi_i}$ is the kernel of a group homomorphism of $G$, so $K_{\chi_i}$’s (and their intersections) are normal in $G$. 2 Assume $G$ is simple. Then, since $K_{\chi_2}, \ldots, K_{\chi_n}$ are proper normal subgroups of $G$, all of them must be trivial (note that $K_\chi = G \iff \chi = \chi_1$).

Conversely, if $G$ is not simple, then $\exists N \trianglelefteq G$, $\{1\} \neq N \neq G$. By 1, $N$ is an intersection of $K_{\chi_i}$’s, so $\{1\} \neq K_{\chi_i} \neq G$ for some $i (i \neq 1)$. 3 By 1, $T$ determines the lattice of normal subgroups of $G$. In particular, we can determine if $G$ has a normal series whose successive quotients are $p$-groups. This is a criterion for solvability of $G$. 4 ■

0.10 More On The Regular Representation

**Theorem 6:** Let $G$ be a finite group and $\rho_i : G \rightarrow V_i, i = 1, \ldots, n$ be a complete set of irreducible representations of $G$. Then

$$\mathcal{C}G \cong \bigoplus_{i=1}^{n} \text{End}(V_i), \quad \text{as } \mathcal{C}\text{-algebras.}$$

(10)

**Proof:** Consider the map $\phi : \mathcal{C}G \rightarrow \bigoplus_{i=1}^{n} \text{End}(V_i),$ determined by

$$g \in G \mapsto \phi \left( \begin{array}{ccc} \rho_1(g) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \rho_n(g) \end{array} \right), \quad \text{and } \mathcal{C}\text{-linear extension.}$$

(11)
Then \( \phi \) is a \( \mathbb{C} \)-algebra homomorphism, since it is \( \mathbb{C} \)-linear, and \( \rho_i \)'s are group homomorphisms. Injectivity of \( \phi \) follows because the left regular representation is faithful. Indeed, if \( \sum_{g \in G} \alpha(g) \cdot g \in \ker \phi \), then \( \rho_i(\sum_{g \in G} \alpha(g) \cdot g) = 0_{V_i} \). Hence

\[
(\bigoplus_{i=1}^n a_i \rho_i)(\sum_{g \in G} \alpha(g) \cdot g) = 0 \bigoplus_{i=1}^n a_i V_i, \quad \forall a_i \in \mathbb{N}^*.
\]

In particular, for the regular representation \( \rho \), \( \rho(\sum_{g \in G} \alpha(g) \cdot g) = 0_{(\mathbb{C}G)^0} \).

So \( 0 = \rho(\sum_{g \in G} \alpha(g) \cdot g) [1] = \sum_{g \in G} \alpha(g) \cdot g \) in \( \mathbb{C}G \). Hence \( \alpha(g) = 0 \forall g \in G \), which proves \( \ker \phi = 0 \). Next, note that \( \dim \mathbb{C}G = |G| = \sum_{i=1}^n (\dim V_i)^2 = \dim \bigoplus_{i=1}^n \text{End}(V_i) \), so \( \phi \) is surjective too. Therefore \( \phi \) is a \( \mathbb{C} \)-algebra isomorphism \( \blacksquare \)

Identifying \( \text{End}(V_i) \) with its copy in \( W = \bigoplus_{i=1}^n \text{End}(V_i) \), note that \( \text{End}(V_1), \ldots, \text{End}(V_n) \) are (2-sided) ideals of \( W \). Hence, by the theorem, \( \mathbb{C}G \) is the direct sum of the 2-sided ideals \( \phi^{-1}[\text{End}(V_1)], \ldots, \phi^{-1}[\text{End}(V_n)] \), so we have \( 1_{(\mathbb{C}G)^0} = e_1 + \cdots + e_n \), where \( e_i \in \phi^{-1}[\text{End}(V_i)] \) are unique. But obviously

\[
1_W = \begin{pmatrix} I_{\text{End}(V_1)} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & I_{\text{End}(V_n)} \end{pmatrix} = \sum_{i=1}^n \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & I_{\text{End}(V_i)} & \vdots \\ 0 & \cdots & 0 \end{pmatrix}
\]

so we must have

\[
e_i \leftrightarrow \phi \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & I_{\text{End}(V_i)} & \vdots \\ 0 & \cdots & 0 \end{pmatrix}, \quad \forall i = 1, \ldots, n. \tag{12}\]

In particular, \( e_i e_j = 0, \forall i \neq j \) and \( e_j = e_j^2 \), so \( \phi(e_i) \) is the projection of \( \bigoplus_{i=1}^n V_i \) onto \( V_i \).

Also note that \( e_i \) generates the ideal \( \phi^{-1}[\text{End}(V_i)] \). Therefore, \( e_i \mathbb{C}G \leftrightarrow \text{End}(V_i) \), and \( \mathbb{C}G = \bigoplus_{i=1}^n e_i \mathbb{C}G \).

The isomorphism given by (11) is important, as it enables us to view all the irreducible representations of \( G \) (and their characters) "simultaneously". Specifically, \( \chi_{\rho_i}(g) \) is the trace of the restriction of \( \phi(g) \) on \( V_i \), and this naturally extends to \( \mathbb{C}G \).

Thus, by (12) \( \phi(e_i a)|_{V_j} = 0_{V_j}, \forall j \neq i \) and \( \phi(e_i a)|_{V_i} = \phi(a)|_{V_i}, \forall a \in \mathbb{C}G \). Hence,

\[
\chi_{\rho_i}(e_i a) = \begin{cases} 0 & \text{for } j \neq i \\ \chi_{\rho_i}(a) & \text{for } j = i \end{cases}, \quad \forall a \in \mathbb{C}G. \tag{13}\]

**Lemma 5:** In \( \mathbb{C}G \) we have:

\[
e_i = \frac{\dim V_i}{|G|} \sum_{g \in G} \chi_{\rho_i}(g^{-1}) \cdot g. \tag{14}\]

**Proof:** Write \( e_i = \sum_{g \in G} \alpha_g g \), \( \alpha_g \in \mathbb{C} \). Fix \( g \in G \). Note that \( \alpha_g \) is the coefficient of 1 in \( e_i g^{-1} \), so \( \chi_{(\mathbb{C}G)^0}(e_i g^{-1}) = |G|\alpha_g \) (recall (6), section 0.6). But \( \chi_{(\mathbb{C}G)^0} = \sum_{i=1}^n (\dim V_i) \chi_{\rho_i} \), so

\[
\alpha_g = \frac{1}{|G|} \sum_{j=1}^n (\dim V_j) \chi_{\rho_j}(e_i g^{-1}) = \frac{\dim V_i}{|G|} \chi_{\rho_i}(g^{-1}), \quad \text{by (13). Thus (14) holds } \blacksquare
\]
0.11 Fourier Transforms And Class Functions

We will often make the identification $\alpha \in \text{class}(G) \leftrightarrow \sum_{g \in G} \alpha(g) g \in \mathbb{C}G$.

The convolution product of $\alpha, \beta \in \text{class}(G)$ is defined by $\alpha \cdot \beta \leftrightarrow (\sum_{g \in G} \alpha(g)g)(\sum_{h \in G} \beta(h)h)$. Equivalently, $(\alpha \cdot \beta)(x) = \sum_{gh=x} \alpha(g) \beta(h)$, $\forall x \in G$. This turns $\text{class}(G)$ into a $\mathbb{C}$-algebra.

Moreover,

**Lemma 6:** $Z(\mathbb{C}G) \cong \text{class}(G)$, as $\mathbb{C}$-algebras.

**Proof:** We naturally define $\phi : \text{class}(G) \to \mathbb{C}G$ by $\phi(\alpha) = \sum_{g \in G} \alpha(g)g \in \mathbb{C}G$. This is clearly an injective $\mathbb{C}$-algebra homomorphism. For any $h \in G$, $h^{-1}\phi(\alpha)h = \sum_{g \in G} \alpha(h^{-1}gh)h^{-1}gh = \sum_{g \in G} \alpha(gh^{-1}gh) = \phi(\alpha)$ (since $\alpha$ is a class function). Since $\phi(\alpha)$ is centralised by $G$, which spans $\mathbb{C}G$, we have $\phi(\alpha) \in Z(\mathbb{C}G)$. Finally, we show that $\text{im} \phi = Z(\mathbb{C}G)$: Assume $a = \sum_{g \in G} \beta_g g \in Z(\mathbb{C}G)$. Then $h^{-1}ah = a$, $\forall h \in G \iff$

$$h^{-1}g^{-1}g = \sum_{g \in G} \beta_g g \iff \sum_{g \in G} [\beta_{g^{-1}h^{-1}} - \beta_g]g = 0_{\mathbb{C}G} \iff \beta_{g^{-1}h^{-1}} = \beta_g, \forall g, h \in G \iff$$

Hence $\beta : G \to \mathbb{C}$ defined by $\beta(g) = \beta_g$ is a class function, so $a = \phi(\beta) \in \text{im} \phi$ \qedsymbol

Let $\rho : G \to GL(V)$ be a representation of $G$, and $\alpha : G \to \mathbb{C}$. The **Fourier transform of $\alpha$ at $\rho$** is defined to be

$$\hat{\alpha}(\rho) = \sum_{g \in G} \alpha(g)\rho(g) \in \text{End}(V).$$

From the definition, we see that it’s enough to consider Fourier transforms at irreducible representations. Let $\rho_i : G \to V_i$, $i = 1, n$ be a complete set of irreducible representations of $G$, so that $\rho_i \to \text{End}(V_i)$.

**Lemma 7:** $\alpha \cdot \beta = \hat{\alpha} \hat{\beta}$.

**Proof:** For any representation $\rho$ of $G$, we have: $\hat{\alpha} \hat{\beta}(\rho) = \sum_{x \in G}(\alpha \cdot \beta)(x)\rho(x) =$

$$= \sum_{x \in G} \left[ \sum_{h \in G} \alpha(h) \beta(h) \right] \rho(x) = \sum_{x \in G} \left[ \sum_{h \in G} \alpha(h) \beta(h) \rho(x) \right] = \sum_{x \in G} \left[ \sum_{h \in G} \alpha(h) \rho(g) \right] \beta(h) \rho(h) = \hat{\alpha}(\rho) \hat{\beta}(\rho), \text{ as desired} \qedsymbol$$

**Theorem 7:** *(Fourier Inversion)* Let $\alpha, \beta : G \to \mathbb{C}$. Then

$$\alpha(g) = \frac{1}{|G|} \sum_{i=1}^n (\text{dim}V_i \cdot \text{trace} [\rho_i(g^{-1}) \hat{\alpha}(\rho_i)]). \tag{15}$$

**Proof:** The right term of $(15)$ is $\frac{1}{|G|} \sum_{i=1}^n ((\text{dim}V_i \cdot \text{trace} [\rho_i(g^{-1}) (\sum_{h \in G} \alpha(h) \rho_i(h))])$

$$= \frac{1}{|G|} \sum_{h \in G} \left( \alpha(h) \cdot \sum_{i=1}^n (\text{dim}V_i) \chi_{\rho_i}(g^{-1}h) \right) = \frac{1}{|G|} \sum_{h \in G} (\alpha(h) \chi_{\mathbb{C}G}(g^{-1}h)) = \alpha(g),$$

the last equality following by Lemma 4 \qedsymbol
Corollary 7: (Plancherel’s Formula)

$$\sum_{g \in G} \alpha(g) \beta(g^{-1}) = \frac{1}{|G|} \sum_{i=1}^{n} (\dim V_i) \cdot \text{trace}[\hat{\alpha}(\rho_i) \hat{\beta}(\rho_i)].$$ \hspace{1cm} (16)

Proof: The right term of (16) is $\frac{1}{|G|} \sum_{i=1}^{n} \left( (\dim V_i) \cdot \text{trace}[\sum_{g \in G} \alpha(g) \rho_i(g) \hat{\beta}(\rho_i)] \right) =$$

$$= \frac{1}{|G|} \sum_{g \in G} \left[ \alpha(g) \cdot \sum_{i=1}^{n} (\dim V_i) \text{trace}[\rho_i(g) \hat{\beta}(\rho_i)] \right] = \sum_{g \in G} \alpha(g) \beta(g^{-1}), \text{ by (15) applied to } \beta \blacksquare$

Example: Let $G = \mathbb{Z}_n = \langle g \mid g^n = 1 \rangle$ and $\alpha : G \to \mathbb{C}$. For $j = 1, 2, \ldots, n$, let $\rho_j$ be the representation of $G$ determined by $\rho_j(g) = w^j$, where $w \in \mathbb{C}$ is a primitive $n$th root of unity. Then $\hat{\alpha}(\rho_j) = \sum_{x \in G} \alpha(x) \rho_j(x) = \sum_{k=1}^{n} \alpha(g^k) w^{jk}$. Also, Fourier inversion gives $\alpha(g^k) = \frac{1}{n} \sum_{j=1}^{n} \hat{\alpha}(\rho_j) w^{-jk}$.

0.12 Clifford’s Theorem

Let $V$ be an irreducible $G$-module over some field $\mathbb{F}$. Let $H \trianglelefteq G$. In general, $V$ is not an irreducible $H$-module. Clifford’s theorem shows exactly how $V$ breaks up as a direct sum of $H$-submodules. We establish it as a consequence of a series of observations:

(1) Let $W$ be an $H$-invariant subspace of $V$. For each $g \in G$, $gW$ is a vector subspace of $V$ and $\dim_{\mathbb{F}}(gW) = \dim_{\mathbb{F}}W$ (since $g \in GL(V)$). Also, $gW$ is an $H$-submodule of $V$: For all $g \in G, h \in H$ and $w \in W$ we have $h(gw) = g(h^{-1}hg)w \in gW$, because $g^{-1}hg \in H$ and $W$ is $H$-invariant.

(2) $W$ is an irreducible $H$-module $\iff gW$ is an irreducible $H$-module. Indeed, if $0 \neq S \neq W$ is an $H$-submodule of $W$, then by (1), $0 \neq gS \neq gW$ is an $H$-submodule of $gW$. Thus, if $W$ is $H$-reducible, then so is $gW$. Conversely, if $gW$ is $H$-reducible then so is $W = g^{-1}(gW)$.

(3) $\theta : W \to U$ is an $H$-isomorphism $\iff g\theta g^{-1} : gW \to gU$ is an $H$-isomorphism. Assume $\theta : W \to U$ is an $H$-isomorphism. Then $g\theta g^{-1}[h(gw)] = g[\theta(g^{-1}hg)]w = g[g^{-1}hg]\theta w$ (since $\theta$ is an $H$-homomorphism) $= hg\theta w = h[\theta(g^{-1})gw], \forall h \in G, w \in W$. Thus, $g\theta g^{-1} : gW \to gU$ is an $H$-homomorphism. It is an isomorphism, since $\dim_{\mathbb{F}}(gW) = \dim_{\mathbb{F}}W = \dim_{\mathbb{F}}U = \dim_{\mathbb{F}}(gU)$. Conversely, if $g\theta g^{-1}$ is an $H$-isomorphism, then so is $\theta = g(g^{-1}\theta g)^{-1}$.

(4) Consider $\sum_{g \in G} gW$. This is a $G$-submodule of $V$ (since any $g \in G$ permutes the components in the sum). Since $V$ is irreducible as a $G$-module, we have $\sum_{g \in G} gW = V$. Note that this is not a direct sum, since for example $hw = W, \forall h \in H$.

(5) From now on, assume $W$ is $H$-irreducible. If $g_1W \neq g_2W$ for some $g_1, g_2 \in G$, then $g_1W \cap g_2W = 0$, since $g_1W \cap g_2W$ is an $H$-submodule of the irreducible $H$-module $gW$. This enables us to throw away the repeating components in (4), to get a direct sum. By classifying the remaining ones according to $H$-isomorphism types, we get $V = \bigoplus_{i=1}^{k} V_i$, where $V_i = g_{i1}W \bigoplus \cdots \bigoplus g_{im_i}W$, and $g_{ij}W \cong g_{i'j}W$ as $H$-modules if and only if $i = i'$. We may assume $W = g_{11}W \subseteq V_1$. 
0.13. INDUCED REPRESENTATIONS

(6) If \( gW \subseteq V_i \) then \( gV_i = V_i \). Indeed, since \( W \) and \( g_{ij}W \) are isomorphic as \( H \)-modules, so are \( gW \) and \( g(g_{ij}W) \). Hence, \( gW \subseteq V_i \Rightarrow g(g_{ij}W) \subseteq V_i \), \( \forall j = 1, \ldots, n \), so \( gV_i = \bigoplus_{j=1}^n g_{ij}W \subseteq V_i \). Conversely, note that \( g^{-1}(gW) = W \subseteq V_i \). Since \( gW \subseteq V_i \), a similar to above argument implies that \( g^{-1}V_i \subseteq V_i \) i.e. \( V_i \subseteq gV_i \). Therefore, \( gV_i = V_i \), as claimed.

Applying the above for \( g = g_{ij} \) we get \( g_{ij}V_i = V_i \), and in particular, \( \dim \pi V_i = \dim \pi V_1 \) so \( n_i = \frac{\dim \pi V_i}{\dim \pi W} = \frac{\dim \pi V_1}{\dim \pi W} = n_1 \).

(7) By (6), \( V_i \to^{g \in G} gV_i \) defines a transitive action of \( G \) on \( \{V_1, \ldots, V_k\} \). The kernel of this action contains \( H C_G(H) \). That \( H \) is in the kernel, follows since \( gW \) is \( H \)-invariant \( \forall g \in G \). Now let \( x \in C_G(H) \). Fix \( g \in G \). Regarding \( x \) as an element of \( GL(V) \), \( x : gW \to (xg)W \) is an isomorphism of vector spaces. Since \( x^{-1}hx = h, \forall h \in H \), we have \( xhu = hxu, \forall u \in gW \), so \( x \) is actually an \( H \)-isomorphism of \( gW \) and \( (xg)W \). Thus, \( gW \subseteq V_i \Rightarrow (xg)W \subseteq V_i \), \( x \) fixes each \( V_i \).

To sum up the above, we have

**Theorem 8:** (Clifford) Let \( G \) be a group, \( H \leq G \). Assume \( V \) is an irreducible representation of \( G \), over some field \( F \), and \( W \subseteq V \) is an \( H \)-invariant subspace of \( V \). Then:

1. \( V = V_1 \bigoplus V_2 \bigoplus \cdots \bigoplus V_k \), where \( V_i = g_{i1}W \bigoplus \cdots \bigoplus g_{in}W \), \( g_{ij} \in G \), and \( g_{ij}W \cong g_{i'j'}W \) as \( H \)-modules if and only if \( i = i' \). Note also that \( n \) does not depend on \( i \).
   Also, for each \( g \in G \) there is an \( i \in \{1, \ldots, k\} \) such that \( gW \cong g_{i1}W \) as \( H \)-modules.

2. \( g \in G : V_i \to gV_i \), defines a transitive action of \( G \) on \( \{V_1, V_2, \ldots, V_k\} \), with \( HC_G(H) \) in its kernel.

**Corollary 8:**
1. \( \chi_{gW}(h) = \chi_{W}(g^{-1}hg), \quad \forall g \in G, h \in H \).
2. \( \dim \pi V = kn \cdot \dim \pi W \).

**Proof:** 1 follows, since \( g^{-1}hg = \lambda w \iff hgw = g(\lambda w) = \lambda(gw) \) (\( \lambda \in F \)). 2 follows by looking at dimensions in 1. of theorem 8 \( \blacksquare \)

0.13 Induced Representations

Let \( F \) be a field, \( G \) a finite group and \( H \leq G \). Our aim is to generate representations of \( G \), from those of \( H \). Suppose \( \rho : H \to GL(W) \) is a representation of \( H \), so that \( W \) is a left \( FH \)-module.

(1) Note that \( FG \) is naturally a right \( FH \)-module (by right multiplication in \( FG \)). Consider \( W^G = FG \bigotimes_{FH} W \) (thus, we can move elements of \( FH \) between left and right). Now \( W^G \) is a left \( G \)-module, under \( g(x \otimes w) = yx \otimes w \) (\( w \in W, x, y \in G \)), and \( F \)-linear extension. Accordingly, this gives a representation of \( G \) on \( W^G \), which we denote by \( \rho^G \).

(2) Let \( \{g_1, g_2, \ldots, g_k\} \) be a complete set of representatives of left cosets of \( H \) in \( G \). Since \( G = \bigcup_{i=1}^k g_iH \) and \( g_iH \) are \( F \)-bases of \( FG \) and \( Fg_iH \) respectively, we have \( FG = \bigoplus_{i=1}^k Fg_iH \) as \( F \)-vector spaces.

(3) For \( i = 1, \ldots, k \) we let \( W_i = g_i \bigotimes_{FH} W \). Thus, \( W_i \) is an \( F \)-vector space, isomorphic to \( W \). Also, \( W_i \cap W_j = \{0\} \). We claim that \( W^G = \bigoplus_{i=1}^k W_i \). First, note that \( W_i = \)
\[ \mathbb{F}g_iH \otimes_{\mathbb{F}H} W \] (since each \( h \in H \) acts faithfully on \( W \)). Then by above,

\[
W^G = \bigoplus_{i=1}^k \mathbb{F}g_iH \otimes_{\mathbb{F}H} W = \bigoplus_{i=1}^k \mathbb{F}g_iH \otimes_{\mathbb{F}H} W = \bigoplus_{i=1}^k W_i = \bigoplus_{i=1}^k W_i, \text{ as claimed.}
\]

In particular, we get \( \text{dim}W^G = |G : H| \cdot \text{dim}W \).

\( (4) \) Let \( g \in G \). Then \( gg_iH = g_jH \) for some \( j \). For any \( w \in W \), \( g_i \otimes w \in W_i \) and \( g(g_i \otimes w) = gg_i \otimes w = g_j^{-1}gg_i \otimes w = g_j \otimes (g_j^{-1}gg_i)w \in W_j \), since \( g_j^{-1}gg_i \in H \). Thus, \( g \in G \) acts on \( W^G = \bigoplus_{i=1}^k W_i \) by first permuting the \( W_i \)'s in the same way it permutes the left cosets of \( H \) in \( G \) (under the regular representation), and then acting within \( W_i \)'s with elements of \( H \). To be more precise, let’s combine bases of \( W_1, \ldots, W_k \) to get a basis \( \mathcal{B} \) of \( W^G \). In this basis, the matrix of \( \rho^G(g) \in GL(W^G) \) is a block-permutation matrix, where the non-zero blocks describe the action of \( H \) on \( W_i \)'s. It is convenient to set \( \rho(x) = 0_{W_i}, \forall x \in G \setminus H, i = 1, k \). Then, in this basis, \( \rho^G(g) \) has form

\[
\rho^G(g) = \begin{pmatrix}
\rho(g_1^{-1}gg_1) & \cdots & \rho(g_{k}^{-1}gg_k) \\
\vdots & \ddots & \vdots \\
\rho(g_k^{-1}gg_1) & \cdots & \rho(g_k^{-1}gg_k)
\end{pmatrix}
\]

(17)

In particular, if \( \text{dim}_\mathbb{F}W = 1 \) and \( \rho \) is the trivial representation of \( H \), then \( \rho^G(g) \) is just the permutation representation of \( G \) on the left cosets of \( H \) in \( G \).

\( (5) \) If \( W_1, W_2 \) are \( H \)-modules, then \( (W_1 \oplus W_2)^G = W_1^G \oplus W_2^G \). We have \( g \otimes (w_1 + w_2) = g \otimes w_1 + g \otimes w_2 \in W_1^G \oplus W_2^G, \forall g \in G, w_i \in W_i \), so \( W_1 \oplus W_2^G \subseteq W_1^G + W_2^G \).

By (3), \( \text{dim}_\mathbb{F}(W_1 \oplus W_2)^G = |G : H|(\text{dim}W_1 + \text{dim}W_2) = \text{dim}_\mathbb{F}(W_1^G \oplus W_2^G) \), so the two must be equal.

\( (6) \) Assume \( K \leq H \leq G \), and let \( W \) be a \( K \)-module. Then \( (W^H)^G \cong W^G \) as \( G \)-modules. Define \( \phi : (W^H)^G \to W^G \), by \( \phi[g \otimes_{\mathbb{F}H} (h \otimes_{\mathbb{F}K} w)] = gh \otimes_{\mathbb{F}K} w \), and \( \mathbb{F} \)-linear extension. Then \( \phi[x(g \otimes (h \otimes w))] = \phi[xg \otimes (h \otimes w)] = xgh \otimes w = x(\phi[g \otimes (h \otimes w)]) \), \( \forall w \in W, h \in H, x, g \in G \). Thus \( \phi \) is a \( G \)-module homomorphism. From its definition, \( \phi \) is surjective. On the other hand, (3) implies \( \text{dim}_\mathbb{F}(W^H)^G = |G : K|\text{dim}_\mathbb{F}W^H = |G : H| \cdot |H : K| \text{dim}_\mathbb{F}W = \text{dim}_\mathbb{F}W^G \). Therefore, \( \phi \) is a \( G \)-isomorphism.

\( (7) \) Let \( \chi^G_\rho \) denote the character of \( \rho^G \). Taking traces in (17) yields \( \chi^G_\rho(g) = \sum_{i=1}^k \chi^0_\rho(g_i^{-1}gg_i) \), where \( \chi^0_\rho(x) = \begin{cases} 
\chi_\rho(x) & \text{if } x \in H \\
0 & \text{if } x \in G \setminus H 
\end{cases} \).

\[
\chi^G_\rho(g) = \frac{1}{|H|} \sum_{i=1}^k \sum_{h \in H} \chi^0_\rho(h^{-1}g_i^{-1}gg_ih) = \frac{1}{|H|} \sum_{x \in G} \chi^0_\rho(x^{-1}gx), \forall g \in G.
\]

(18)

\( (8) \) Actually (18) permits us to extend the notion of induced characters to arbitrary class functions: Let \( \theta \in \text{class}(H) \), and \( \theta^0(u) = \begin{cases} 
\theta(u) & \text{if } u \in H \\
0 & \text{if } u \in G \setminus H 
\end{cases} \).

Define \( \theta^G : G \to \mathbb{F} \), by \( \theta^G(g) = \frac{1}{|H|} \sum_{x \in G} \theta^0(x^{-1}gx) \). Then \( \theta^G \in \text{class}(G) \). Indeed, \( \theta^G(g^{-1}xg) = \frac{1}{|H|} \sum_{u \in G} \theta^0(u^{-1}g^{-1}xgu) = \theta^G(x) \), since as \( u \) runs through \( G \), so does \( gu \).
(9) (Frobenius Reciprocity) Assume $\mathbb{F} = \mathbb{C}$, $H \leq G$ as before, and let $\phi \in \text{class}(H)$, $\theta \in \text{class}(G)$. Then
\[
\langle \phi, \theta|_H \rangle = \langle \phi^G, \theta \rangle
\]
Proof: We have
\[
\frac{1}{|H|} \sum_{g \in G} \phi^G(g) \overline{\theta(g)} = \frac{1}{|G|} \sum_{g \in G} \sum_{x \in G} \phi^0(x^{-1}gx) \overline{\theta(g)} = \\
= \frac{1}{|H|} \sum_{x \in G} \sum_{g \in G} \phi^0(x^{-1}gx) \overline{\theta(x^{-1}gx)} = \\
= \frac{1}{|H|} \sum_{y \in G} \phi^0(y) \overline{\theta(y)} \quad \text{(since } \{x^{-1}gx \mid g \in G\} = G, \forall x \in G) \\
= \frac{1}{|H|} \sum_{y \in H} \phi(y) \overline{\theta(y)} = \langle \phi, \theta|_H \rangle, \quad \text{as claimed.}
\]

In particular, if $\phi$ and $\theta$ are irreducible characters of $H$ and $G$, then the number of times $\phi$ appears in $\theta|_H$ is equal to the number of times $\theta$ appears in $\phi^G$.

Note: The two inner products appearing in (19) are different. They are taken in $\text{class}(H)$ and $\text{class}(G)$, respectively.

(10) Let $\theta \in \text{class}(H)$ and $G$. From the definition, we have $\theta^G(g) = \frac{1}{|H|} \sum_{x \in \mathcal{O}_G(g)} \theta^0(x)$, where $\mathcal{O}_G(g)$ denotes the $G$-conjugacy class of $g$. Now any $H$-conjugacy class is contained in some $G$-conjugacy class, but not conversely. In general $\mathcal{O}_G(g) \cap H$ will break up into several $H$-conjugacy classes $\mathcal{O}_H(h_1), \ldots, \mathcal{O}_H(h_m)$, $h_i \in H$. Since $\theta \in \text{class}(H)$, $\theta^G(g) = \frac{1}{|H|} \sum_{i=1}^m |\{x \in G \mid x^{-1}gx \in \mathcal{O}_H(h_i)\}| \cdot \theta(h_i)$. But if $y^{-1}gy = x \in \mathcal{O}_G(g)$, then $z^{-1}gz = x \Leftrightarrow yz^{-1}gzy^{-1} = g \Leftrightarrow zy^{-1} \in \mathcal{C}_G(g)$ ($\mathcal{C}_G(g)$ is the centralizer of $g$ in $G$). Thus, $\{y \in G \mid y^{-1}gy = x\} = |\mathcal{C}_G(g)|$, $\forall x \in \mathcal{O}_G(g)$. Hence
\[
\theta^G(g) = \frac{1}{|H|} \sum_{i=1}^m |\mathcal{C}_G(g)| \cdot |\mathcal{O}_H(h_i)| \cdot \theta(h_i) = \frac{|G : H|}{|\mathcal{O}_G(g)|} \sum_{i=1}^m |\mathcal{O}_H(h_i)| \cdot \theta(h_i). \quad (20)
\]
Note also that if $\mathcal{O}_H(g) \cap H = \emptyset$ (i.e. $m = 0$), then $\theta^G(g) = 0$ by definition.

Example: $H = A_4 < G = A_5$. Let $\theta$ be the 1-dimensional character of $A_4$, given by

<table>
<thead>
<tr>
<th>$A_4$</th>
<th>1</th>
<th>4</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>(123)</td>
<td>(132)</td>
</tr>
<tr>
<td>$\theta$</td>
<td>1</td>
<td>$w$</td>
<td>$w^2$</td>
</tr>
</tbody>
</table>

where $w \in \mathbb{C} \setminus \mathbb{R}$, $w^3 = 1$.

Then $\theta^G$ is a character of $A_5$, of dimension $|A_5 : A_4| \cdot \theta(1) = 5$. By (20), we have:

1. $g = (123) \Rightarrow \mathcal{O}_G(g) \cap H = \mathcal{O}_H(123) \cup \mathcal{O}_H(132)$, so $\theta^G(123) = \frac{|G : H|}{|\mathcal{O}_G(123)|} (4\theta(123) + 4\theta(132)) = \frac{5}{2}(4w + 4w^2) = -1$.

2. $g = (12)(34) \Rightarrow \mathcal{O}_G(g) \cap H = \mathcal{O}_H[(12)(34)]$. Hence, $\theta^G[(12)(34)] = \frac{|G : H|}{|\mathcal{O}_G[(12)(34)]|} \cdot 3\theta[(12)(34)] = 1$.

3. $g = (12345)$ (or (13524)). Since $H = A_4$ doesn’t have any 5-cycles, $\mathcal{O}_G(g) \cap H = \emptyset$, so $\theta^G(g) = 0$.

<table>
<thead>
<tr>
<th>$A_5$</th>
<th>1</th>
<th>20</th>
<th>15</th>
<th>12</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta^G$</td>
<td>1</td>
<td>20</td>
<td>15</td>
<td>12</td>
<td>12</td>
</tr>
</tbody>
</table>

Note also that $\theta^G$ is irreducible.
0.14 Consequences To Permutation Groups

Consider a finite group $G$ acting transitively on a finite set $S$. For $\alpha \in S$, the stabilizer of $\alpha$, $G_\alpha = \{ g \in G \mid g\alpha = \alpha \}$ is a subgroup of $G$. The rank of $G$, $r_S$, is the number of orbits of $G_\alpha$ acting on $S$. To show that this is well-defined, consider $\alpha, \beta \in S$. There exists $g \in G$ such that $g\alpha = \beta$. Then $x\beta = \beta \iff xg\alpha = g\alpha \iff (g^{-1}xg)\alpha = \alpha$, i.e. $g^{-1}G\beta g = G_\alpha$.

If $h \in G_\alpha$, then $ha = b$ $(a, b \in S) \iff ghg^{-1}(g\alpha) = gb$ (note that $ghg^{-1} \in G_\beta$). Therefore, $a \mapsto ga$ (which is a bijection of $S$) is also a bijection between orbits of $G_\alpha$ and $G_\beta$. So the rank of $G$ is well-defined.

(1) **The rank of $G$ is 2 $\Leftrightarrow$ $G$ is doubly transitive.** By definition, the rank of $G$ is 2 $\Leftrightarrow G_\alpha$ is transitive on $S \setminus \{ \alpha \}$ $(\forall \alpha \in S)$. "$\Rightarrow$" Let $a, b, c, d \in S$, $a \neq b, c \neq d$. Since $G$ is transitive, $g(a, b) = (c, e)$ for some $g \in G, e \in S$. Since $e, d \in S \setminus \{ c \}$, there exists $h \in G_e$ such that $he = d$. So $hg(a, b) = (c, d)$, i.e. $G$ is doubly transitive. "$\Leftarrow$" Let $a \in S$. Since $G$ is doubly transitive, for any $b, c \in S \setminus \{ a \}$ there is a $g \in G$ such that $g(a, b) = (a, c)$. In particular, $g \in G_a$ and $gb = c$. Thus, $G_a$ is transitive on $S \setminus \{ a \}$.

(2) **(Burnside)** Let $\theta$ be the character of the permutation representation of $G$ on $\mathbb{C}S$. Then $r_S = \langle \theta, \theta \rangle$.

Let's take a closer look at the action of $G$ on $S$. Fix $\alpha \in S$, and let $H = G_\alpha$. Note that $g\alpha = ha\alpha (g, h \in G) \iff gh^{-1} \in H$. Thus, if $g_1H, \ldots, g_mH$ are the cosets of $H$ in $G$, then the orbit of $\alpha$, $O_G(\alpha) = \{ g\alpha \mid g \in G \} = \{ g_1\alpha, \ldots, g_m\alpha \}$. In particular, $|O_G(\alpha)| = |G : H| = |G : G_\alpha|$ (this is Burnside's Lemma, and is true regardless of the transitivity of $G$ on $S$). Since $G$ is transitive on $S$, $S = O_G(\alpha) = \{ g_1\alpha, \ldots, g_m\alpha \}$, so we have $\theta(g) = |\{ \beta \in S \mid g\beta = \beta \}| = |\{ i \mid g(i\alpha) = g_i\alpha \}| = |\{ i \mid g_i^{-1}gg_i \in H \}| = 1_H(g)$, the last equality following by (17) ($1_H$ is the trivial character of $H$). Finally, let $O_H(\alpha_1), \ldots, O_H(\alpha_k)$ be the orbits of $H$ acting on $S$. Summing up the above, $r_S = \frac{1}{|H|} \sum_{i=1}^k |O_H(\alpha_i)| \cdot |G_{\alpha_i}| = \frac{1}{|H|} \sum_{\beta \in S} |G_\beta| = \frac{1}{|H|} \sum_{h \in H} |\{ \beta \in S \mid h\beta = \beta \}| = \frac{1}{|H|} \sum_{h \in H} \theta(h) = \langle \theta, 1_H \rangle = \langle \theta, \theta \rangle$, as desired (we also used Frobenius reciprocity).

(3) **Assume $|S| \geq 2$. Then $G$ is doubly transitive on $S$ $\Leftrightarrow \theta - 1_G$ is an irreducible character of $G$.**

Let $\alpha \in S$. Since $G$ is transitive on $S$, $1 = \frac{1}{|G|} |O_G(\alpha)| \cdot |G_{\alpha}| = \frac{1}{|G|} \sum_{\beta \in S} |G_\beta| = \frac{1}{|G|} \sum_{g \in G} \theta(g) = \langle \theta, 1_G \rangle$. So $\theta - 1_G$ is a character of $G$ (of degree $|S| - 1$), and $\langle \theta - 1_G, 1_G \rangle = 0$. Hence, $G$ is doubly transitive on $S$ $\Leftrightarrow \langle \theta, \theta \rangle = 2 \Leftrightarrow \langle \theta - 1_G, \theta - 1_G \rangle + \langle 1_G, 1_G \rangle + 2 \langle \theta - 1_G, 1_G \rangle = 1 \Leftrightarrow \theta - 1_G$ is irreducible.

(4) **For all $n \geq 2$, $S_n$ and $A_n$ (except $A_3$) have irreducible characters of degree $n - 1$.** The group $S_n$ is $n$-transitive in its permutation action on $S = \{ 1, 2, \ldots, n \}$, so by (3), the claim is true for $S_n$. It is also true for $A_2$. Now let $n \geq 4$. We are done once we show that $A_n$ is doubly transitive on $S$. In fact, $A_n$ is $(n - 2)$-transitive on $S$. To see this, let $(a_1, \ldots, a_{n-2})$ and $(b_1, \ldots, b_{n-2})$ be two $(n - 2)$-tuples of distinct elements of $S$. Then $\exists \pi \in S_n$ such that $\pi(a_1, \ldots, a_{n-2}) = (b_1, \ldots, b_{n-2})$. Note that $(S_n)_{a_1} \cap \cdots (S_n)_{a_{n-2}} = \{ \tau \in S_n \mid \tau(a_i) = a_i, \forall i = 1, \ldots, n-2 \} \cong S_2$, so it has both odd and even permutations. Choose $\sigma \in (S_n)_{a_1} \cap \cdots (S_n)_{a_{n-2}}$ such that $\text{sign}\pi = \text{sign}\sigma$. Then $\sigma \pi \in A_n$, and $\sigma \pi(a_1, \ldots, a_{n-2}) = (b_1, \ldots, b_{n-2})$. 


0.15 Frobenius Groups

Theorem 9: (Frobenius) Assume a finite group $G$ acts transitively on a finite set $S$. If every non-identity element of $G$ fixes at most 1 point of $S$, then the set of fixed-point-free elements together with the identity, forms a normal subgroup of $G$.

Proof: For $i = 0, 1$, let $F_i = \{ g \in G \mid g \text{ fixes } i \text{ elements of } S \}$. Then $G = \{ 1 \} \cup F_0 \cup F_1$, and we have to show that $N = \{ 1 \} \cup F_0$ is a normal subgroup of $G$. For any $g, h \in G$, $g$ fixes $a_1, \ldots, a_k \in S \iff hgh^{-1}$ fixes $ha_1, \ldots, ha_k$. This shows that $F_0, F_1$ (and clearly $\{ 1 \}$) are normal subsets of $G$. So $N$ is normal as a subset of $G$. The strategy of the proof is to construct a representation of $G$, whose kernel is $N$. We shall do this by inducing irreducible representations of a stabilizer $G_x \ (x \in S)$, and then by considering a certain linear combination of them.

Fix $x \in S$, and let $H = G_x$. Since $gHg^{-1} = G_x$, and $G$ is transitive on $S$, all 1-point stabilizers $G_x$ are conjugate to $H$. Thus, any $g \in F_1$ is conjugate to some element of $H$. This enables us to select a full set of representatives of $G$-conjugacy classes $A = \{ 1, h_2, \ldots, h_r, y_1, \ldots, y_k \}$, with $h_i \in H$ and $y_j \in F_0 = N \setminus \{ 1 \}$. Another consequence is that $F_1 = \bigcup_{z \in S} (G_z \setminus \{ 1 \})$ (the union is disjoint since any $g \in G$ fixes at most 1 point of $S$). Thus, $G = N \bigsqcup \bigcup_{z \in S} (G_z \setminus \{ 1 \})$. Since $G$ is transitive on $S$, $|S| = |G : G_z|$, $\forall z \in S$. So the above decomposition gives $|G| = |S| \cdot (|G_z| - 1) + |N| \Rightarrow |N| = |S| = |G : H|$. Note also that $H$ is self-normalizing in $G$: indeed, if $ghg^{-1} = h' \ (h, h' \in H, g \in G)$, then $h'$ fixes $x$ and $gx$ so we must have $x = gx$, i.e. $g \in H$. This also means that $B = \{ 1, h_2, \ldots, h_r \}$ is a full set of representatives of $H$-conjugacy classes.

Let $\phi_1, \ldots, \phi_r$ be the irreducible characters of $H$. We have $\phi_k^G(1) = |G : H| \cdot \phi_k(1)$, $\phi_k^G(h_i) = \frac{1}{|H|} \sum_{g \in G} \phi^0(g^{-1}h_ig) = \phi_k(h_i)$ (since $H$ is self-normalizing in $G$) and $\phi_k^G(y_j) = 0$ (since $g^{-1}y_ig \in F_0$, which is disjoint from $H$). Let $\chi_1$ be the trivial character of $G$. For $i = 2, \ldots, r$ we define $\chi_i = \phi_i^G - \phi_i(1) \cdot \phi_i^G + \phi_i(1) \cdot \chi_1$ (which is also valid for $i = 1$). Then

1. $\chi_i(1) = \phi_i(1)$. Indeed, $\chi_i(1) = \phi_i^G(1) - \phi_i(1) \cdot \phi_i^G(1) + \phi_i(1) = |G : H| \cdot \phi_i(1) - \phi_i(1) \cdot |G : H| \cdot 1 + \phi_i(1) = \phi_i(1)$.

2. $\chi_i(h_j) = \phi_i(h_j)$ (this follows from $\phi_i^G(h_j) = \phi_i(h_j)$, $\forall i, j$).

3. $\chi_i(y_k) = \phi_i(1)$ (because $\phi_i^G(y_k) = 0$, $\forall i, k$).

These, together with the above observations, imply

$$\langle \chi_i, \chi_i \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_i(g) \overline{\chi_i(g)} = \frac{1}{|G|} \left[ \sum_{g \in N} |\chi_i(g)|^2 + \sum_{z \in S} \sum_{1 \neq g \in G_z} |\chi_i(g)|^2 \right]$$

$$= \frac{1}{|G|} \left[ |N| \cdot \phi_i(1)^2 + |S| \sum_{1 \neq h \in H} |\chi_i(h)|^2 \right]$$

$$= \frac{|G : H|}{|G|} \sum_{h \in H} |\phi_i(h)|^2 = \langle \phi_i, \phi_i \rangle = 1.$$
Now, by definition we can write \( \chi_i = \sum_{j=1}^{r+s} a_j \psi_j \), \( a_j \in \mathbb{Z} \), where \( \psi_1, \ldots, \psi_{r+s} \) are the irreducible characters of \( G \). Then 1 = \( \langle \chi_i, \chi_i \rangle = \sum_{j=1}^{r+s} a_j^2 \Rightarrow \chi_i = \pm \psi_j \) for some \( j \). But \( \chi_i(1) = \phi_i(1) > 0 \), so \( \chi_i = \psi_j \) is an irreducible character of \( G \).

Finally, let \( \chi = \sum_{i=1}^{r+s} \phi_i(1) \cdot \chi_i \), which is a character of \( G \). For any \( 1 \neq h \in H \), we have \( \chi(h) = \sum_{i=1}^{r+s} \phi_i(1) \phi_i(h) = 0 \), by lemma 4. Hence, the same holds for any \( g \in G \setminus N = F_1 \), since such a \( g \) is conjugate to some \( 1 \neq h \in H \). For any \( y \in N \) (and in particular \( y = 1 \)), \( \chi(y) = \sum_{i=1}^{r+s} \phi_i(1)^2 = |H| \). Therefore, \( N = \{ g \in G \mid \chi(g) = \chi(1) = |H| \} = K \leq G \), as desired.

This theorem motivates the following definition of an important class of groups.

A Frobenius group is a group \( G \), having a subgroup \( H \) such that \( H \cap g^{-1}Hg = \{1\} \), \( \forall g \in G \setminus H \). This implies that the distinct conjugates of \( H \) intersect at the identity (since \( x^{-1}Hx \cap y^{-1}Hy \) is conjugate to \( (yx^{-1})^{-1}H(yx^{-1}) \cap H \) via \( x \)), and there are \( |G : H| \) of them (because \( x^{-1}Hx = y^{-1}Hy \Leftrightarrow xH = yH \)). We make the following observations:

1. In the setting of the above definition, \( G \) acts transitively and faithfully on \( S = \{ gH \mid g \in G \} \), by left multiplication. (to see that the action is faithful, let \( x \in G \) be in its kernel. Then \( xgH = gH, \forall g \in G \Leftrightarrow g^{-1}xg \in H, \forall g \in G \Leftrightarrow x \in \cap_{g \in G} gH \Rightarrow x = 1 \). Furthermore, if \( 1 \neq g \in G \) fixes \( xH \) and \( yH \), then as before, \( g \in xH \cap yHy^{-1} \Rightarrow xH = yH \). Thus, each \( 1 \neq g \in G \) fixes at most one element of \( S \). Therefore, by Frobenius’ theorem, \( N = \{1\} \cup \{ g \in G \mid g \text{ acts fixed-point-free on } S \} \leq G \). By the proof of the theorem, we also have \( G = N \bigcup_{i=1}^{r} \{ g_i^{-1}Hg_i \setminus \{1\} \} \) (\( g_1, \ldots, g_r \) are the distinct representatives of cosets of \( H \)), and \( |N| = |G : H| \). Since \( N \cap H = \{1\} \), \( |N| \cdot |H| = |G| \) and \( N \leq G \), we deduce that \( G \) is the semidirect product of \( N \) and \( H \).

Note: \( N \) is called the Frobenius kernel of the Frobenius group \( G \), and \( H – \) the Frobenius complement of \( N \) in \( G \).

2. Consider the action of \( H \) as a group of automorphisms of \( N \). Let \( \phi : H \to \text{Aut}(N) \) be the conjugation homomorphism: \( \phi h = [n \in N \mapsto hnh^{-1} \in N] \). If \( 1 \neq h \in H \), then \( \{ n \in N \mid \phi h(n) = n \} = \{ n \in N \mid hnh^{-1} = n \} = \{ n \in N \mid h = hnh^{-1} \} \subseteq H \cap N = \{1\} \) (recall that \( H \) is self-normalizing in \( G \)). Thus, for every \( 1 \neq h \in H \), \( \phi h \) fixes only 1 in its action of \( N \). We say that \( H \) acts on \( N \) as a group of fixed-point-free automorphisms.

3. Conversely, assume now that \( G \) is the semidirect product of \( N \) by \( H \) (\( N \leq G \) and \( H \) acts on \( N \) as a group of fixed-point-free automorphisms). We show that \( G \) is a Frobenius group, with kernel \( N \) and complement \( H \). Let \( g \in G \setminus H \), so that \( g = hn \) with \( 1 \neq n \in N, h \in H \). Then \( g^{-1}Hg = n^{-1}h^{-1}Hhn = n^{-1}Hn \). Now we show \( H \cap n^{-1}Hn = \{1\} \): Indeed, \( h_1 = nh_2n^{-1}, h_1, h_2 \in H \Rightarrow \phi h_1(n) = h_1nh_1^{-1} = nh_2h_1^{-1} \in N \Rightarrow h_2h_1^{-1} \in H \cap N = \{1\} \Rightarrow h_1 = h_2 \) and \( \phi h_1(n) = n \Rightarrow h_1 = 1 \), since \( n \neq 1 \) and \( H \) acts fixed-point-free on \( N \). Therefore, \( H \cap g^{-1}Hg = \{1\}, \forall g \in G \setminus H \), and \( G \) is Frobenius with complement \( H \). To prove \( N \) is its kernel, note that \( N \leq G \) and \( N \cap H = \{1\} \) imply \( N \cap \{ g_i^{-1}Hg_i \setminus \{1\} \} = \emptyset \) (\( g_1H, \ldots, g_rH \) are the cosets of \( H \) in \( G \)). Then \( |G| = |N| \cdot |H| \) forces \( G = N \bigcup \{ g_i^{-1}Hg_i \setminus \{1\} \} \), so \( N \) is the Frobenius kernel of \( G \).

4. Let \( p, q \) be distinct primes with \( p \equiv 1 \pmod{q} \). Let \( \phi : \mathbb{Z}_q \to \text{Aut}(\mathbb{Z}_p) \) be a non-trivial homomorphism. Since \( \mathbb{Z}_q \) is generated by any non-identity element, \( \phi \) must be injective. Hence \( \phi g \) acts fixed-point-free on \( \mathbb{Z}_p \), \( \forall 1 \neq g \in \mathbb{Z}_q \) (if \( 1 \neq h \in \mathbb{Z}_p \) is fixed by an injective \( \alpha \in \text{Aut}(\mathbb{Z}_p) \) then \( \alpha \) is the identity since \( h \) generates \( \mathbb{Z}_p \)). It follows that the unique
non-abelian group of order $pq$, $\mathbb{Z}_p \times \mathbb{Z}_q$, is a Frobenius group.

(5) (Thompson) The Frobenius kernel of any Frobenius group, is a nilpotent group.

### 0.16 Algebraic Integers

Let $\alpha \in \mathbb{C}$ be an algebraic number, so that $\alpha$ satisfies some non-zero polynomial with coefficients in $\mathbb{Q}$. Denote by $m_\alpha(x) \in \mathbb{Q}[x]$ the minimal polynomial of $\alpha$, i.e. the monic rational polynomial of minimal degree that $\alpha$ satisfies.

**Theorem 10:** Let $\alpha \in \mathbb{C}$ be algebraic. The following are equivalent:

1. $m_\alpha(x)$ actually has coefficients in $\mathbb{Z}$.
2. $\mathbb{Z}[\alpha]$ is finitely generated as an (additive) abelian group.
3. $\alpha$ is a root of some monic polynomial $f \in \mathbb{Z}[x]$.

**Proof:** 1$\Rightarrow$2 Suppose $m_\alpha(x) = x^k + b_{k-1}x^{k-1} + \cdots + b_0 \in \mathbb{Z}[x]$. Then $\alpha^k = -(b_{k-1}\alpha^{k-1} + \cdots + b_0)$, so $\alpha^k$ (and $\alpha^m, \forall m \geq k$, by induction) is an integral linear combination of $1, \alpha, \ldots, \alpha^{k-1}$. Thus $\mathbb{Z}[\alpha]$ is generated as a $\mathbb{Z}$-module by $1, \alpha, \ldots, \alpha^{k-1}$.

2$\Rightarrow$3 Assume $\mathbb{Z}[\alpha] = f_1(\alpha)\mathbb{Z} + \cdots + f_m(\alpha)\mathbb{Z}$, where $f_i \in \mathbb{Z}[x]$. Let $n = 1 + \max_{1 \leq i \leq m}\{\deg f_i\}$. Since $\alpha^n \in \mathbb{Z}[\alpha]$, $\alpha^n = \sum_{i=1}^m c_if_i(\alpha)$ for some $c_1, \ldots, c_m \in \mathbb{Z}$. Then $\alpha$ is a root of the monic polynomial $f(x) = x^n - \sum_{i=1}^m c_if_i(x) \in \mathbb{Z}[x]$. Thus $f(x) = a(x)m_\alpha(x)$.

3$\Rightarrow$1 Assume $f(\alpha) = 0$, where $f \in \mathbb{Z}[x]$ is monic. Since $\mathbb{Q}[x]$ is a Euclidean domain, we may write $f(x) = a(x)m_\alpha(x) + b(x)$ in $\mathbb{Q}[x]$, with $b \equiv 0$ or $\deg b < \deg m_\alpha$. But $b(\alpha) = f(\alpha) - a(\alpha)m_\alpha(\alpha) = 0$, so $b \equiv 0$ by minimality of $m_\alpha$. Thus $f(x) = a(x)m_\alpha(x)$. Clearing out the denominators implies $am\cdot f(x) = [a\cdot a(x)][m\cdot m_\alpha(x)]$ in $\mathbb{Z}[x]$ ($a, m \in \mathbb{Z}_+)$.

### Definition

If $\alpha \in \mathbb{C}$ satisfies (any of) the conditions in theorem 10, we say that $\alpha$ is an algebraic integer. Denote by $\text{Alg}$ the set of algebraic integers in $\mathbb{C}$.

**Example 1:** $\alpha = \sqrt[n]{m} \in \text{Alg}$ ($n \in \mathbb{N}, m \in \mathbb{Z}$), since it is a root of the monic polynomial $x^n - m \in \mathbb{Z}[x]$.

**Example 2:** $\alpha \in \text{Alg} \iff \overline{\alpha} \in \text{Alg}$ (since for $f \in \mathbb{R}[x]$, $f(\alpha) = 0 \iff f(\overline{\alpha}) = 0$).

**Example 3:** If $\alpha \in \text{Q}$ is an algebraic integer, then $\alpha \in \mathbb{Z}$. Indeed, the minimal polynomial of $\alpha$ is $m_\alpha(x) = x - \alpha$, since $\alpha \in \mathbb{Q}$. But $\alpha$ - algebraic integer $\Rightarrow m_\alpha(x) \in \mathbb{Z}[x]$, so $\alpha \in \mathbb{Z}$.

**Lemma 10:** The set $\text{Alg}$ of the algebraic integers forms a subring of $\mathbb{C}$.

**Proof:** Clearly $1 \in \text{Alg}$. Assume $\alpha, \beta \in \text{Alg}$. By theorem 10, $\mathbb{Z}[\alpha] = \langle \alpha_1, \ldots, \alpha_k \rangle$ and $\mathbb{Z}[\beta] = \langle \beta_1, \ldots, \beta_m \rangle$ (as $\mathbb{Z}$-modules), with $\alpha_1 = \beta_1 = 1$. Note that $M = \langle \alpha_i \beta_j | 1 \leq i \leq k, 1 \leq j \leq m \rangle$ is a subring of $\mathbb{C}$, which contains $\mathbb{Z}[\alpha]$ and $\mathbb{Z}[\beta]$. Hence $\alpha \pm \beta, \alpha \beta \in M$, so $\mathbb{Z}[\alpha \pm \beta]$ and $\mathbb{Z}[\alpha \beta]$ are $\mathbb{Z}$-submodules of $M$. Since $M$ is a finitely generated $\mathbb{Z}$-module, so must be $\mathbb{Z}[\alpha \pm \beta]$ and $\mathbb{Z}[\alpha \beta]$. Therefore, $\alpha \pm \beta, \alpha \beta \in \text{Alg}$.

In some sense the ring $\text{Alg}$ is a generalization of $\mathbb{Z}$, but which lies in a much larger field of fractions than $\mathbb{Q}$, namely $\mathbb{C}$.
Let $G$ be a finite group. Recall that if $\rho$ is a representation of $G$ on $V$ (over $\mathbb{C}$), then the minimal polynomial of $\rho(g)$ divides $x^n - 1$, where $n$ is the order of $g \in G$. Then all the eigenvalues of $\rho$ are roots of unity, hence algebraic integers. It follows that the character table of $G$ consists of algebraic integers. Our next aim is to show that the degrees of the irreducible characters of $G$, divide $|G|$. 

(1) Let $C_1, \ldots, C_r$ be the conjugacy classes of $G$. Abuse notation and write $C_j = \sum_{g \in C_j} g \in \mathbb{C}G$. We also identify $f : G \to \mathbb{C}$ with $\sum_{g \in G} f(g)g \in \mathbb{C}G$. Under this identification, from the proof of lemma 6 we see that $\text{class}(G) = \mathbb{Z}(\mathbb{C}G)$. Also $C_1, \ldots, C_r \in \text{class}(G)$, and any $f \in \text{class}(G)$ is an integral linear combination of $C_1, \ldots, C_r$. Since $C_i C_j \in \text{class}(G)$, we have

$$C_i C_j = \sum_{k=1}^r n_{ijk} C_k, \quad n_{ijk} \in \mathbb{Z}. \quad (21)$$

(2) Let $\rho$ be an irreducible representation of $G$ on $V$. Since $C_j \in \mathbb{Z}(\mathbb{C}G)$, $\rho(C_j) : V \to V$ is a $G$-module homomorphism. By Schur’s lemma, $\rho(C_j) = w_{\rho j} \mathbb{I}$, for some $w_{\rho j} \in \mathbb{C}$. Applying this to (21), we get

$$w_{\rho i} w_{\rho j} = \sum_{k=1}^r n_{ijk} w_{\rho k} \quad \forall 1 \leq i, j \leq r.$$ 

Therefore, the ring $\mathbb{Z}[w_{\rho 1}, \ldots, w_{\rho r}]$ is finitely generated as a $\mathbb{Z}$-module (for example by $1, w_{\rho 1}, \ldots, w_{\rho r}$). It follows that $\mathbb{Z}[w_{\rho j}] \subseteq \mathbb{Z}[w_{\rho 1}, \ldots, w_{\rho r}]$ is also a finitely generated $\mathbb{Z}$-module. This implies $w_{\rho j} \in \text{Alg}$.

(3) What is $w_{\rho j}$? Recall that $\rho(C_j) = w_{\rho j} \mathbb{I}$. Taking traces $\Rightarrow |C_j| \cdot \chi_\rho(C_j) = w_{\rho j} \cdot \chi_\rho(1)$. Now since $\chi_\rho$ is irreducible, $\sum_{k=1}^r |C_k| |\chi_\rho(C_k)| \chi_\rho(C_k) = |G| \cdot |\chi_\rho(1)|$. Whence,

$$\sum_{k=1}^r w_{\rho k} \chi_\rho(C_k) = \frac{|G|}{\chi_\rho(1)}$$

The left term is an algebraic integer, whereas the right term is rational. So they are both integers and $\text{dim} \rho = \chi_\rho(1)$ divides $|G|$.

### 0.17 Burnside’s $p^a q^b$ Theorem

**Lemma 11:** If $\lambda \in \mathbb{C}$ is an average of roots of unity, and $\lambda \in \text{Alg}$, then $\lambda = 0$ or $\lambda$ is a root of unity.

**Proof:** Consider the monic irreducible polynomial $m_\lambda(x) \in \mathbb{Z}[x]$. The Galois group of the splitting field $K$ of $m_\lambda$ over $\mathbb{Q}$, acts transitively on the roots of $m_\lambda$. Let these roots be $\sigma_1 \lambda, \ldots, \sigma_m \lambda$. We have $\prod_{i=1}^m \sigma_i \lambda = (-1)^m m_\lambda(0) \in \mathbb{Z}$. We assume without loss that $\sigma_1, \ldots, \sigma_m$ are extended field automorphisms of $\mathbb{C}$. Write $\lambda = \frac{1}{n} (\lambda_1 + \cdots + \lambda_n)$, where $\lambda_i$ is a root of unity. Then $\sigma_j \lambda = \frac{1}{n} (\sigma_j \lambda_1 + \cdots + \sigma_j \lambda_n)$ and note that $\sigma_j \lambda_i$ is also a root unity (of same order as $\lambda_i$). It follows that $|\sigma_j \lambda| \leq \frac{1}{n} (|\sigma_j \lambda_1| + \cdots + |\sigma_j \lambda_n|) = 1$, with equality if and only if $\sigma_j \lambda_1 = \cdots = \sigma_j \lambda_n \Leftrightarrow \lambda_1 = \cdots = \lambda_n (= \lambda)$. Multiplying these inequalities over
1 \leq j \leq m \text{ implies } |m_\lambda(0)| = \prod_{j=1}^m |\sigma_j \lambda| \leq 1. \text{ If } |m_\lambda(0)| = 0, \text{ then } m_\lambda(x) = x \text{ and } \lambda = 0. 

Otherwise \(|m_\lambda(0)| = 1\), so we must have \(|\sigma_j \lambda| = 1, \forall j = 1, \ldots, m\), and \(\lambda = \lambda_1 = \cdots = \lambda_n\) is a root of unity. 

**Corollary 9:** Let \(\chi_\rho\) be an irreducible character of a finite group \(G\), and \(C_1, \ldots, C_r\) be the conjugacy classes of \(G\). If \(\chi_\rho(1)\) and \(|C_j|\) are relatively prime (for some \(1 \leq j \leq r\)), then either \(\chi_\rho(C_j) = 0\) or \(\frac{\chi_\rho(C_j)}{\chi_\rho(1)}\) is a root of unity.

**Proof:** Since \((\chi_\rho(1) | C_j|) = 1\), there exist \(a, b \in \mathbb{Z}\) such that \(a \cdot \chi_\rho(1) + b \cdot |C_j| = 1\). Then

\[
\frac{\chi_\rho(C_j)}{\chi_\rho(1)} = a \chi_\rho(C_j) + b \frac{|C_j| \cdot \chi_\rho(C_j)}{\chi_\rho(1)} = a \chi_\rho(C_j) + bw_\rho \in Alg.
\]

The left term is an average of roots of unity, so the conclusion follows from Lemma 11. 

**Lemma 12:** Let \(G\) be a finite group. If there exists a conjugacy class \(C\) of \(G\), such that \(|C| = p^n\) (\(p\) prime and \(n \geq 1\)), then \(G\) is not simple.

**Proof:** Let \(1 \neq g \in C\). Let \(\chi_1, \ldots, \chi_m\) be all the non-trivial irreducible characters of \(G\) which don’t vanish at \(g\). By column orthogonality in the character table of \(G\),

\[
1 + \sum_{i=1}^m \chi_i(1) \overline{\chi_i(g)} = 0. \tag{22}
\]

If \(p | \chi_i(1), \forall 1 \leq j \leq m\), then \(\frac{1}{p} = \sum_{i=1}^m \left[ \frac{\chi_i(1) \overline{\chi_i(g)}}{p} \right] \in Alg\), which is impossible, since \(\frac{1}{p} \in \mathbb{Q} \setminus \mathbb{Z}\). Therefore, \(p\) and \(\chi_j(g)\) are relatively prime for some \(j = 1, \ldots, m\). Since \(\chi_j(g) \neq 0\), Corollary 9 implies \(\chi_j(g) = \lambda \chi_j(1)\), where \(\lambda \in \mathbb{C}\) is a root of unity. Let \(\rho : G \to GL(V)\) be a representation whose character is \(\chi_j\). Since the eigenvalues of \(\rho(g)\) are roots of unity whose average is \(\lambda\), all of them have to equal \(\lambda\), so \(\rho(g) = \lambda I\). Now assume \(G\) is simple. Then \(G\) is not abelian (otherwise \(|C| = 1\), so \([G, G] = \{1\}\). It follows that \(G\) has only 1 irreducible character of degree 1 (the trivial one). Hence \(dim \rho = dim \chi_{C(V)} > 1\), which implies \(ker \rho \neq G\). Since \(G\) is simple, this forces \(ker \rho = \{1\}\), so that \(G \cong G/ker \rho \cong \rho(G)\) as groups. Since \(\rho(g) \in Z(\rho(G))\), we get \(1 \neq g \in Z(G)\), contradicting the simplicity of \(G\). 

**Theorem 11:** (Burnside) Let \(G\) be a group. If \(|G| = p^aq^b\) (\(p, q\) - primes and \(a, b \geq 1\)), then \(G\) is solvable.

**Proof:** Assume the statement is false, and let \(G\) be a counterexample such that \(|G|\) is minimal. Then \(G\) is simple. Indeed, if \(\{1\} \neq H < G\) then \(H\) and \(G/H\) are solvable. Then \(G\) is solvable (we get a normal series of \(G\) by combining the normal series of \(H\) and \(G/H\)), contradiction. Let \(P \in Syl_p(G)\). Since \(P\) is a \(p\)-group, we may choose \(1 \neq g \in Z(P)\). Then \(P \leq C_G(g)\), so \(|\mathcal{O}_G(g)| = |G : C_G(g)| = m^m\) (\(0 \leq m \leq b\)). But \(m \neq 0\) (otherwise \(1 \neq g \in Z(G)\), so the simple \(G\) must be abelian of non-prime order), so Lemma 12 implies that \(G\) cannot be simple, contradiction. 

What is striking about this proof is the way that solvability is used. Examining the proof, we see that the theorem is true if we replace solvability by any other property \(P\) such that \(\forall H \leq G, \ P(H) \land P(G/H) \Rightarrow P(G)\).
0.18 Brauer-Tate Theorems

We first introduce some preparatory concepts. Let $G$ be a finite group, and $\chi_1, \ldots, \chi_m$ be its irreducible characters (over $\mathbb{C}$).

We call $\psi \in \text{class}(G)$ a generalized character if it is the difference of 2 characters. Equivalently, $\psi$ is a generalized character if it is an integral linear combination of (irreducible) characters of $G$. Define $\text{ch}(G) = \{ \psi \in \text{class}(G) \mid \psi \text{ is a generalized character of } G \} = \{ a_1\chi_1 + \cdots + a_m\chi_m \mid a_1, \ldots, a_m \in \mathbb{Z} \}$. Since $\chi_1, \ldots, \chi_m$ are $\mathbb{C}$-linearly independent, $\text{ch}(G) \cong \mathbb{Z}^m$ as $\mathbb{Z}$-modules. By 4. of Theorem 3, the product of any 2 characters is a character, so $\text{ch}(G)$ is a commutative ring with identity (the trivial character $\chi_1$).

A finite group $E$ is elementary if $E = A \times P$, where $P$ is a $p$-group and $A$ is a cyclic $p'$-group (i.e. $(|A|, |P|) = 1$). If we wish to highlight $p$ we call $E$ $p$-elementary. Define $\mathcal{E}$ to be the set of elementary subgroups of $G$ (over all primes). The set of characters of $G$, induced from (characters of) members of $\mathcal{E}$, generates a $\mathbb{Z}$-submodule of $\text{ch}(G)$, which we denote by $v(G)$. Our primary goal is to prove

**Theorem 12:** (Brauer-Tate) $\text{ch}(G) = v(G)$.

As usual, we make a series of observations which will guide us through the proof of the theorem:

1. Let $w \in \mathbb{C}$ be a primitive root of 1, of order $n = |G|$. Let $R = \mathbb{Z}[w] \cong \mathbb{Z}[x]/m_w(x)$, where $m_w(x) = x^n - 1$. Thus, $1, w, \ldots, w^{n-2}$ is a basis for $R$ as a $\mathbb{Z}$-module. Define $\text{ch}_R(G) = \{ a_1\chi_1 + \cdots + a_m\chi_m \mid a_1, \ldots, a_m \in R \} \cong R^m$, as $R$-modules. Accordingly we define $v_R(G)$ to be the $R$-submodule of $\text{ch}_R(G)$, generated by characters of $G$, induced from members of $\mathcal{E}$. From definitions, we have the following diagram of inclusions:

$$
\begin{array}{ccc}
\text{ch}(G) & \longrightarrow & \text{ch}_R(G) \\
\downarrow & & \downarrow \\
v(G) & \longrightarrow & v_R(G)
\end{array}
$$

2. Let $H \leq G$, $\phi \in \text{class}(H)$ and $\theta \in \text{class}(G)$. Then $(\phi \cdot \theta|_H)^G = \phi^G \cdot \theta$. 

**Proof:** For any $y \in G$, we have $(\phi \cdot \theta|_H)^G(y) = \frac{1}{|H|} \sum_{x \in G} (\phi \cdot \theta|_H)(x^{-1}yx) = \frac{1}{|H|} \sum_{x \in G} \phi^0(x^{-1}yx)\theta(x^{-1}yx) = \theta(y) \cdot \frac{1}{|H|} \sum_{x \in G} \phi^0(x^{-1}yx) = \theta(y) \cdot \phi^G(y)$.

We used that $(\phi \cdot \theta|_H)^G = \phi^0 \cdot \theta$, which holds since both sides vanish on $G \setminus H$, and $\theta|_H = \theta$ on $H$.

3. $v(G)$ is an ideal of $\text{ch}(G)$.

Let $\chi \in \text{ch}(G)$. By definition, the set $\mathcal{A} = \{ \psi^G \mid \psi \in \text{ch}(E) \}$ for some $E \in \mathcal{E}$ generates $v(G)$ as a $\mathbb{Z}$-module, so it’s enough to check that $\chi \cdot \psi^G \in v(G)$, where $\psi \in \text{ch}(E), E \in \mathcal{E}$. By (2), $\chi \cdot \psi^G = (\psi \cdot \chi|_E)^G \in v(G)$, since $\psi$ and $\chi|_E$ (and $\psi \cdot \chi|_E$) are characters of $E$.

4. $\text{ch}(G) \cap v_R(G) = v(G)$.

First, note that $1, w, \ldots, w^{n-2}$ are linearly independent over the ring $\text{ch}(G)$. Indeed, assume $\sum_{j=1}^{n-2} (\sum_{i=1}^m c_{ij}\chi_i)w^j = 0$ in $\text{class}(G)$, with $c_{ij} \in \mathbb{C}$. Interchanging the sums we
\[ \sum_{i=1}^{m} \left( \sum_{j=1}^{n-2} c_{ij} w^j \right) \chi_i = 0 \Rightarrow \sum_{j=1}^{n-2} c_{ij} w^j = 0, \ \forall i = \overline{1,m} \Rightarrow c_{ij} = 0, \ \forall i = \overline{1,m}, j = \overline{1,n-2}. \]

In particular, it follows that \( v_R(G) = \bigoplus_{i=1}^{n-2} v(G)w^j \) as \( R \)-modules. Now suppose \( \xi = \xi_0 + \xi_1 w + \ldots + \xi_{n-2} w^{n-2} \in ch(G) \cap v_R(G), \ \xi_0, \ldots, \xi_{n-2} \in v(G) \). By the linear independence of \( 1, w, \ldots, w^{n-2} \) over \( ch(G) \), we must have \( \xi = \xi_0 \in v(G) \), which proves the claim. In particular, if \( \chi_1 \in v_R(G) \) then \( \chi_1 \in v(G) \Rightarrow ch(G) = v(G) \), since \( v(G) \) is an ideal of \( ch(G) \).

(5) Let \( p \) be prime and \( g \in G \). There exist \( g_1, g_2 \in G \) such that \( g_1 \) is a \( p \)-element (i.e. \(|g_1| = p^\alpha \) for some \( \alpha \)), \( g_2 \) is a \( p' \)-element (i.e. \(|g_2| \) and \( p \) are coprime), and \( g = g_1 g_2 = g_2 g_1 \). Moreover, this representation of \( g \) is unique. The components \( g_1, g_2 \) are called the \( p \)-part and the \( p' \)-part of \( g \), respectively. Also \( g = g_1 g_2 \) is the \( p \)-decomposition of \( g \).

Proof: Since \( g_1, g_2 \) should commute, it’s reasonable to look at powers of \( g \). Write \(|g| = p^\alpha l \), with \((p, l) = 1\). Then \( a p^\alpha + bl = 1 \) for some \( a, b \in \mathbb{Z} \). Then \( g_1 = g^b \) and \( g_2 = g^{p^\alpha} \) satisfy the conditions: indeed \(|g_1| = |g|/|g|, bl = p^\alpha \), \(|g_2| = |g|/|g, ap^\alpha| = l \) and \( g = g_1 g_2 = g_2 g_1 \). Now assume \( g = h_1 h_2 = h_2 h_1 \) were another such representation. Since \( p^\alpha l = |g| = |h_1|, |h_2| \), the hypothesis implies \(|h_1| = p^\alpha \) and \(|h_2| = l \). Then \( h_1 = h_1^{p^\alpha + bl} = h_1^b = g^b = g_1 \) (since \( h_1^b = 1 \)), and \( h_2 = g/h_1 = g/g_1 = g_2 \).

(6) Let \( g = g_1 g_2 \) be the \( p \)-decomposition of \( g \in G \). Consider \( x^{-1} g x = [x^{-1} g_1 x][x^{-1} g_2 x] \). Since \(|x^{-1} g_1 x| = |g_1|, h_1 = x^{-1} g_1 x \) is a \( p \)-element. Similarly \( h_2 = x^{-1} g_2 x \) is a \( p' \)-element. Since \( x^{-1} g x = h_1 h_2 = h_2 h_1 \), \( 5 \) implies that \( x^{-1} g x = [x^{-1} g_1 x][x^{-1} g_2 x] \) is the \( p \)-decomposition of \( x^{-1} g x \). For \( g, h \in G \), write \( g \sim h \) if the \( p' \)-parts of \( g \) and \( h \) are conjugate. Note that \( \sim \) is an equivalence relation. If \( g \) is conjugate to \( h \), then \( g \sim h \) by above. It follows that the equivalence classes of \( \sim \) are unions of conjugacy classes of \( G \).

(7) Assume \( G = G_1 \times G_2 \). Then

1. Any conjugacy class of \( G \) is the product of a conjugacy class \( G_1 \) and one of \( G_2 \).

2. Any irreducible character of \( G \) is of form \( \chi_1 \chi_2 \) (which is defined by \( \chi_1 \chi_2 (g_1 g_2) = \chi_1(g_1) \chi_2(g_2), \ \forall g_1 \in G_1, g_2 \in G_2 \)), where \( \chi_1, \chi_2 \) are irreducible characters of \( G_1 \) and \( G_2 \) respectively.

Proof: 1 Let \( g = g_1 g_2 \in G \) with \( g_i \in G_i \). Then \( \mathcal{O}_G(g) = \{(h_1 h_2)^{-1}g(h_1 h_2) \mid h_i \in G_i\} = \{h_1^{-1} g_1 h_1 [h_2^{-1} g_2 h_2] \mid h_i \in G_i\} = \mathcal{O}_{G_1}(g_1) \mathcal{O}_{G_2}(g_2) \). 2 Assume \( \chi_1, \chi_2 \) are characters of the irreducible representations \( \rho_1 : G_1 \rightarrow GL(V_1) \) and \( \rho_2 : G_2 \rightarrow V_2 \) respectively. Then \( \chi_1 \chi_2 \), is the character of the representation \( \rho : G = G_1 \times G_2 \rightarrow V_1 \otimes V_2 \) defined by \( \rho(g_1 g_2)(v_1 \otimes v_2) = (\rho_1(g_1), \rho_2(g_2)) \) [recall that if \( \{v_1, \ldots, v_k\} \) and \( \{w_1, \ldots, w_l\} \) are bases of eigenvectors of \( \rho_1(g_1) \in GL(V_1) \) and \( \rho_2(g_2) \in GL(V_2) \) respectively, then \( \{v_i \otimes w_j \mid 1 \leq i \leq k, 1 \leq j \leq l\} \) is a basis of eigenvectors of \( \rho(g_1 g_2) \in GL(V_1 \otimes V_2) \), so that the eigenvalues of \( \rho(g_1 g_2) \) are precisely the pairwise products of eigenvalues of \( \rho_1(g_1) \) and \( \rho_2(g_2) \)]. Let
\(\psi_1, \psi_2\) be irreducible characters of \(G_1, G_2\), respectively. We have

\[
\langle \chi_1 \chi_2, \psi_1 \psi_2 \rangle = \frac{1}{|G_1|} \sum_{g_1 \in G_1} \chi_1(g_1) \chi_2(g_2) \overline{\psi_1(g_1) \psi_2(g_2)} = \left( \frac{1}{|G_1|} \sum_{g_1 \in G_1} \chi_1(g_1) \overline{\psi_1(g_1)} \right) \left( \frac{1}{|G_2|} \sum_{g_2 \in G_2} \chi_2(g_2) \overline{\psi_2(g_2)} \right) = \langle \chi_1, \psi_1 \rangle \cdot \langle \chi_2, \psi_2 \rangle.
\]

For \(\psi_1 = \chi_1\) and \(\psi_2 = \chi_2\), this shows that \(\chi_1 \chi_2\) is irreducible. Otherwise, \(\langle \chi_1 \chi_2, \psi_1 \psi_2 \rangle = 0\), and in particular \(\chi_1 \chi_2 \neq \psi_1 \psi_2\). Thus, the number of such characters is the number of irreducible characters of \(G_1\) times that of \(G_2\), which by 1 is equal to the number of irreducible characters of \(G\). Hence these are all irreducible characters of \(G\).

(8) If \(E = A \times B \in \mathcal{E}\), with \(A = \langle a \rangle\), then \(\exists \xi \in \text{ch}_R(E)\) such that \(\xi(ab) = |a|, \xi(a^jb) = 0, \forall j = 2, 3, \ldots, |a| - 1, b \in B\).

**Proof:** Define \(\xi : E \to \mathbb{C}\) as above. Then \(\xi \in \text{class}(E)\), because \(\Theta_E(a^jb) = a^j \Theta_B(b), \forall j = 1, |a|, b \in B\). Let \(\chi_1, \ldots, \chi_k\) and \(\psi_1, \ldots, \psi_l\) be the irreducible characters of \(A\) and \(B\) respectively. We may assume that \(\chi_j(a) = w^{j-1}\) for \(1 \leq j \leq k\), where \(w \in \mathbb{C}\) is a primitive root of unity of order \(k = |a|\). By 7, we may write

\[
\xi = \sum_{1 \leq i \leq k, 1 \leq j \leq l} a_{ij} \chi_i \psi_j, \text{ with } a_{ij} \in \mathbb{C}.
\]

Let \(\Theta_1, \ldots, \Theta_l\) be the conjugacy classes of \(B\). From 7 and the definition of \(\xi\), we get

\[
a_{ij} = \langle \xi, \chi_i \psi_j \rangle = \frac{1}{|E|} \sum_{s=1}^l \left| \Theta_s \right| \cdot \xi(a \Theta_s) \chi_i(a) \overline{\psi_j(\Theta_s)} = \frac{1}{|a| \cdot |B|} \sum_{s=1}^l \left| \Theta_s \right| \cdot \left| a \right| w^{i-1} \cdot \psi_j(\Theta_s) = \frac{w^{i-1}}{|B|} \sum_{s=1}^l \left| \Theta_s \right| \overline{\psi_j(\Theta_s)} = w^{i-1} \cdot \langle \psi_1, \psi_j \rangle \in \mathbb{Z}[w] \subseteq R.
\]

Thus \(a_{ij} \in R, \forall i, j\), implying \(\xi \in \text{ch}_R(E)\).

(9) Let \(p\) be a prime and \(a\) a \(p'\)-element of \(G\). There exists an integer valued function \(\theta \in v_R(G)\) such that, for \(g \in G\), \(\theta(g) \equiv 1 \pmod{p}\) if the \(p'\) part of \(g\) is conjugate to \(a\), and \(\theta(g) = 0\) otherwise.

**Proof:** Consider the centralizer \(C_G(a)\) of \(a\) in \(G\). Define \(E = \langle a \rangle \times B\), where \(B \in Syl_p(C_G(a))\). By 8, \(\exists \xi \in \text{ch}_R(E)\) such that \(\xi(ab) = |a|, \xi(a^jb) = 0, \forall j = 2, 3, \ldots, |a| - 1, b \in B\). Now \(E\) is an elementary subgroup of \(G\), so \(\xi^G \in v_R(G)\). Since all values of \(\xi\) are multiples of \(|a|\), the same is true for \(\xi^G\), because \(\xi^G(g) = \sum_{i=1}^s \xi^0(x_i^{-1}gx_i)\), with \(x_1, \ldots, x_s\) distinct coset representatives of \(E\) in \(G\). If \(g\) is not conjugate to some \(ab\), with \(b \in B\), we have \(\xi^G(g) = 0\) (in this case \(x_i^{-1}gx_i \notin aB\), so \(\xi^0(x_i^{-1}gx_i) = 0, \forall 1 \leq i \leq s\)). Assume
now that $x^{-1}gx = ab$, where $b \in B$. We have

$$\xi^G(g) = \xi^G(ab) = \sum_{1 \leq i \leq s, x_i^{-1}abx_i \in E} \xi(x_i^{-1}abx_i) = |a| \cdot |\{i \mid 1 \leq i \leq s, \ x_i^{-1}abx_i \in aB\}|.$$  

Since $b \in B \subseteq Syl_p(C(a))$, $b$ is a $p$-element and $ab = ba$. But $a$ is a $p'$-element, so $b$ and $a$ are the $p$- and $p'$-parts of $ab$, respectively. Therefore, by 5 and 6 we have:

$$x_i^{-1}abx_i = ab', \ b' \in B \iff x_i^{-1}ax_i = a, x_i^{-1}bx_i = b' \iff x_i \in C_G(a) \cap N_G(B).$$

Hence $\xi^G(g) = |a| \cdot |\{i \mid 1 \leq i \leq s, \ x_i \in C_G(a) \cap N_G(B)\}| = |a| \cdot |C_G(a) \cap N_G(B) : E|$, the latter being true since $E$ is a subgroup of $C_G(a) \cap N_G(B)$. Further $\xi^G(g) = |C_G(a) \cap N_G(B) : B| \neq 0 (\text{mod } p)$, because $|C_G(a) \cap N_G(B) : B|$ divides $|C_G(a) : B|$, which is not divisible by $p$. Finally, define $\theta = \alpha \xi^G$ where $\alpha \in \mathbb{Z}$ is such that $\alpha \cdot |C_G(a) \cap N_G(B) : B| \equiv 1 \pmod{p}$. Then $\theta \in v_R(G)$ satisfies the conditions in the hypothesis. Note that $\theta$ takes only 2 distinct values by construction (0 and $\alpha \cdot |C_G(a) \cap N_G(B) : B|$).

(10) For a fixed prime $p$, let $L_1, \ldots, L_t$ be the equivalence classes of $\sim$. By 9, there exist $\theta_1, \ldots, \theta_t \in v_R(G)$ such that, for each $i$, $\theta_i$ is congruent to 1 (mod $p$) on $L_i$, and equal to 0 otherwise. Then $\psi = \theta_1 + \cdots + \theta_t \in v_R(G)$ is integer valued, and satisfies $\psi(g) \equiv 1 \pmod{p}$, $\forall g \in G$.

(11) If $\eta$ is an integer valued class function on $G$, such that $\eta(g)$ is divisible by $|G|, \forall g \in G$, then $\eta \in v_R(G)$.

Proof: Choose a prime $p$ such that $(p, |G|) = 1$. Then $g \sim h \iff g$ is conjugate to $h, \forall g, h \in G$, because all elements of $G$ are their own $p'$-parts. Let $\mathcal{O}_{G}(a_1), \ldots, \mathcal{O}_{G}(a_t)$ be the conjugacy classes of $G$. By 9, there exist $\theta_1, \ldots, \theta_t \in v_R(G)$ such that, for each $i$, $\theta_i$ is $|a_i|$ on $\mathcal{O}_{G}(a_i)$, and 0 otherwise. Since $|a_i|$ divides $G$, we get that $v_R(G)$ contains the functions which are $|G|$ on a conjugacy class of $G$, and 0 outside it. But $\eta$ is an integral linear combination of these, so $\eta \in v_R(G)$.

(12) If $|G| = mp^\alpha$ with $p$ prime and $(p, m) = 1$, then $m\chi_1 \in v_R(G)$.

Proof: By 10, $\exists \psi \in v_R(G)$ such that $\psi(g) \equiv 1 \pmod{p}, \forall g \in G$. Define $\theta = \psi \circ \phi \in v_R(G)$ (recall that $v_R(G)$ is an ideal of $ch_R(G)$). We claim that $\theta(g) \equiv 1 \pmod{p}$, $\forall g \in G$: Indeed, fix $g \in G$. We have $\psi(g) = 1 + pl, \ l \in \mathbb{Z} \Rightarrow \theta(g) = (1 + pl)^{p^\alpha} = 1 + \sum_{k=1}^{p^\alpha} \binom{p^\alpha}{k} p^k l^k$. For $1 \leq k \leq p^\alpha$, the exponent of $p$ in the numerator of $\binom{p^\alpha}{k} p^k = p^k p^{\alpha-\alpha-k} \binom{p^\alpha}{k} \in \mathbb{Z}$ is at least $k + \alpha$, while the exponent of $p$ in its denominator is at most $k$. So each term in the above sum is divisible by $p^\alpha$, which yields $\theta(g) \equiv 1 \pmod{p^\alpha}$. It follows that all values of $\chi_1 - \theta$ are divisible by $p^\alpha$. Hence all values of $m(\chi_1 - \theta)$ are multiples of $|G|$. By 11, we have $m(\chi_1 - \theta) \in v_R(G)$. Since $m\theta \in v_R(G)$, we conclude that $m\chi_1 = m(\chi_1 - \theta) + m\theta \in v_R(G)$, as desired.

Finally, write $|G| = p_1^\alpha_1 \cdots p_k^\alpha_k$, where $p_1, \ldots, p_k$ are the distinct primes dividing $|G|$. For $1 \leq i \leq k$, let $m_i = |G|/p_i^\alpha$. Then $m_1, \ldots, m_k$ are relatively prime, so $1 = c_1 m_1 + \cdots + c_k m_k$, for some $c_1, \ldots, c_k \in \mathbb{Z}$. By 12, $m_i \chi_1 \in v_R(G)$ for each $1 \leq i \leq k$. Consequently, $\chi_1 = c_1 (m_1 \chi_1) + \cdots + c_k (m_k \chi_1) \in v_R(G)$, which completes the proof of Brauer-Tate’s theorem. ■
0.19 Consequences of Brauer-Tate’s Theorem

In this section, we give some important corollaries and applications of Brauer-Tate’s result. As usual, $G$ denotes a finite group.

**Lemma 13:** Let $\theta \in \mathrm{class}(G)$. Then $\theta \in \mathrm{ch}(G) \iff \theta|_E \in \mathrm{ch}(E)$, $\forall E \in \mathcal{E}$.

**Proof:** Let $u(G) = \{ \theta \in \mathrm{class}(G) \mid \theta|_E \in \mathrm{ch}(E), \forall E \in \mathcal{E} \}$. First note that $\mathrm{ch}(G) \subseteq u(G)$. The reason for this is that, if $H \leq G$ and $\chi$ is a character of $G$, then $\chi|_H$ is a character of $H$. Next, $u(G)$ is a ring with identity (the trivial character of $G$). Indeed, if $\phi, \psi \in u(G)$ then $\phi|_E, \psi|_E \in \mathrm{ch}(E)$, $\forall E \in \mathcal{E}$, so $\phi \pm \psi \in u(G)(\phi \cdot \psi)|_E = \phi|_E \cdot \psi|_E \in \mathrm{ch}(E)$, $\forall E \in \mathcal{E}$ (we used the fact that $\mathrm{ch}(E)$ is a ring). By Brauer-Tate’s theorem, $\chi_1 \in u(G)$. Hence, if we show that $\phi \in u(G)$ is an ideal of $u(G)$, then $v(G) = u(G) = \mathrm{ch}(G)$, as claimed. Let $\phi \in v(G)$, so that $\phi = \theta^G$ for some $\theta \in \mathrm{ch}(E)$ and $E \in \mathcal{E}$. By 2 of section 18, for any $\psi \in u(G)$ we have $\phi \cdot \psi = \theta^G \cdot \psi = (\theta \cdot \psi)|_E \in v(G)$. Since $\theta, \phi|_E \in \mathrm{ch}(E)$.

What follows is a nice application of the previous results, which yields a result of practical value about the character table of a group.

**Theorem 13:** Let $p$ be a prime number, and $n \in \mathbb{Z}$ such that $p^n$ is the highest power of $p$ dividing $|G|$. If $\chi$ is an irreducible character of $G$ such that $p^n|\chi(1)$, then $\chi(y) = 0$ for any $y \in G$ whose order is a multiple of $p$.

**Proof:** Define $\theta : G \to \mathbb{C}$ by $\theta(y) = \begin{cases} \chi(y) & \text{if } (p, |y|) = 1 \\ 0 & \text{if } p \text{ divides } |y| \end{cases}$. Then $\theta \in \mathrm{class}(G)$. We have to show that $\theta = \chi$. We will achieve this by showing that $\theta \in \mathrm{ch}(G)$. Indeed, if so, then $\theta = \sum a_i \chi_i + \cdots + a_m \chi_m$ where $a_1, \ldots, a_m \in \mathbb{Z}$ and $\chi_1, \ldots, \chi_m$ are irreducible characters of $G$. Since $\theta(1) = \chi(1) > 0$, we get $0 < a_1^2 + \cdots + a_m^2 = \langle \theta, \theta \rangle = \frac{1}{|G|} \sum_{\theta \in \mathcal{E}} |\theta(g)|^2 \leq \sum_{\theta \in \mathcal{E}} |\chi(g)|^2 = \langle \chi, \chi \rangle = 1$, so all of $a_1, \ldots, a_m$ are 0, except one which is 1 (it cannot be −1, since $\theta(1) > 0$). Hence $\theta$ is an irreducible character of $G$. But $\langle \theta, \chi \rangle = \langle \theta, \theta \rangle > 0$, which forces $\theta = \chi$. Next, by lemma 13 it suffices to show that $\theta|_E \in \mathrm{ch}(E)$, $\forall E \in \mathcal{E}$. When $E$ is a $p'$-group this is trivial, because by definition it follows that $\theta|_E = \chi|_E \in \mathrm{ch}(E)$. Assume now that $|p||E|$. We have $E = A \times B$ where $A$ is cyclic, $B$ is of prime power order, and $(|A|, |B|) = 1$. If $p||A|$, then $A = P \times Q_1$ where $P$ is the Sylow-$p$ subgroup of $A$, and $Q_1$ is a $p'$ group (the product of the other Sylow subgroups of $A$). So $E = (P \times Q_1) \times B = P \times (Q_1 \times B)$. If $p||B|$ (so $A$ is a $p'$-group), then $E = A \times B = B \times A$. In any case, we see that $E = P \times Q$, where $P$ is a $p$-group and $Q$ a $p'$-group. To establish $\theta|_E \in \mathrm{ch}(E)$, we show that $\langle \theta|_E, \xi \rangle$ (which is the coefficient of $\xi$ in $\theta|_E$ written as a linear combination of the irreducible characters) is an integer for any irreducible character $\xi$ of $E$. Let $\xi$ be any irreducible character of $E$. Since $E$ is the direct product $P \times Q$, elements of $P$ commute with those of $Q$. Hence the order of $gh \in E \ (g \in P, h \in Q)$, $|gh|, |h|$, is a multiple of $p$ if and only if $g \neq 1$. Therefore $\theta|_E = \chi$ on $Q$, and $\theta|_E = 0$ otherwise. So we have

$$\langle \theta|_E, \xi \rangle = \frac{1}{|E|} \sum_{y \in Q} \chi(y)\overline{\xi(y)} = \frac{|Q|}{|E|} \langle \chi|_Q, \xi|_Q \rangle \in \mathbb{Q}.$$ 

It follows that $\langle \theta|_E, \xi \rangle = \frac{1}{|P|} \langle \chi|_Q, \xi|_Q \rangle = \frac{a}{b}$, for some $a, b \in \mathbb{Z}$, with $(a, b) = 1$ and $b||P|$. 


In particular, we see that $b$ is a power of $p$. On the other hand,

\[
\langle \theta_E, \xi \rangle = \frac{1}{|E|} \sum_{y \in Q} \chi(y) \overline{\xi(y)} = \frac{\chi(1)}{|G| \cdot |E|} \sum_{y \in Q} |C_G(y) : P| \cdot \frac{|G(y)| \chi(y)}{\chi(1)} \overline{\xi(y)} = \frac{\chi(1)}{|G| \cdot |Q|} \cdot c, \text{ for some } c \in \text{Alg}, \text{ cf. 3 of section 16.}
\]

Therefore, $\frac{|G|}{\chi(1)} \langle \theta_E, \xi \rangle \in \mathbb{Q} \cap \text{Alg} = \mathbb{Z}$. But $\frac{|G|}{\chi(1)}$ (which is an integer by section 16) is coprime to $p$ by hypothesis. Also $(|Q|, p) = 1$. Since $b$ is a power of $p$, and it divides $\frac{|G|}{\chi(1)}$, we must have $b = 1$. Therefore $\langle \theta_E, \xi \rangle = a \in \mathbb{Z}$, as desired. □

**Lemma 14:** (Brauer) Let $E$ be a $p$-elementary group, and $\theta$ an irreducible character of $E$. Then $\theta = \theta^E$, for some linear (i.e. degree 1) character $\xi$ of some subgroup of $E$.

**Proof:** Let $E = A \times P$, with $A$ a cyclic $p'$-group and $P$ a $p$-group. By 7 of section 18, $\theta = \theta_1 \theta_2$, where $\theta_1$ and $\theta_2$ are irreducibles of $A$ and $P$, respectively. Since $\theta_1$ is linear (because $A$ is abelian), and the degree of $\theta_2$ is a power of $p$ (since it divides $|P|$), we get $\deg \theta = \deg(1) = p^n$, for some $n \geq 0$. We proceed by induction on $n$. The theorem statement is true for $n = 0$ ($\theta = \theta^E$ is linear), so we assume $n > 0$. Let $\lambda$ be a linear character of $E$. We have $\langle \overline{\theta}, \lambda \rangle = \frac{1}{|E|} \sum_{g \in E} \overline{\theta(g) \theta(g) \cdot \lambda(g)} = \langle \theta, \lambda \theta \rangle = \begin{cases} 1 & \text{if } \lambda \theta = \theta \\ 0 & \text{otherwise} \end{cases}$. Note that $\lambda \theta$ and $\overline{\theta}$ are irreducible characters of $E$, since $\theta$ is (cf. section 8). Let $\Lambda = \{ \lambda | \lambda$ is a linear character of $E$, such that $\lambda \theta = \theta \}$. Note that $\Lambda$ is a multiplicative group: indeed, let $\lambda_1, \lambda_2 \in \Lambda$. Since all values of $\lambda$ are roots of unity, $\theta_1^{-1} = \overline{\theta_1}$, so $\lambda_1^{-1} \lambda_2 = \overline{\lambda_1 \lambda_2}$ is a linear character of $E$ (cf. section 8). Then $\lambda_1^{-1} \lambda_2 \theta = \lambda_1^{-1} \theta = \lambda_1^{-1} \lambda_1 \theta = \theta$, which proves that $\lambda_1^{-1} \lambda_2 \in \Lambda$. By definition of $\Lambda$, we may write

$$\theta \overline{\theta} = \sum_{\lambda \in \Lambda} \lambda + \phi,$$

where $\phi$ is a non-linear character of $E$. Since $p | \deg \theta \overline{\theta}$ and $p | \deg \phi$, we get $p | |\Lambda|$. By Cauchy's theorem, there exists $\eta \in \Lambda$ such that $|\eta| = p$. Now consider $K = k \eta \eta$. By the 1st Isomorphism Theorem, $E/K \cong \text{im} \eta = \{ \mu \in \mathbb{C} | \mu^p = 1 \} \cong \mathbb{Z}/p\mathbb{Z}$. This forces $K = A \times Q$, where $Q$ is a subgroup of $P$, of index $p$. (To see this, write $K = M \times Q$ with $M \leq A$, $Q \leq P$. Since $A$ is a $p'$-group, $P$ is a $p$-group, and $|K| = |E|/p$, we must have $M = A$). Restricting (23) to $K$, and noting that $|1|_E|_K = \eta|K = 1|_K$, we find that $1|_K$ occurs at least twice in $|\theta \overline{\theta}|_K$, i.e. $\langle 1|_K, (\theta \overline{\theta})|_K \rangle \geq 2$. Next, $2 \leq \langle 1|_K, \theta|_K \overline{\theta|_K} \rangle = \langle \theta|_K, \theta|_K \rangle$, so $\theta|_K$ is a reducible character of $K$. Let then $\psi$ be an irreducible character of $K$, which occurs in $\theta|_K$. We have $\deg \psi < \deg \theta|_K = \deg \theta = p^n$. But $\deg \psi$ is a power of $p$ (because $K$ is $p$-elementary), so $\deg \psi \leq p^{n-1}$. On the other hand, applying Frobenius reciprocity yields $1 \leq \langle \theta|_K, \psi \rangle = \langle \theta, \psi^E \rangle$, which implies that $\theta$ occurs in $\psi^E$ (since $\theta$ is irreducible). In particular, $p^n = \deg \theta \leq \deg \psi^E = |E : K| \cdot \deg \psi = p \cdot \deg \psi \Rightarrow p^{n-1} \leq \deg \psi$. The two inequalities force $\deg \psi = p^{n-1}$ and $\deg \psi^E = p^n = \deg \theta$. Again, since $\theta$ occurs $\psi^E$, we actually have $\theta = \psi^E$. Now $\psi$ is an irreducible character of the $p$-elementary group $K$. 

0.19. CONSEQUENCES OF BRAUER-TATE'S THEOREM

33
of degree $p^{n-1}$, so by induction $\psi = \xi^K$ for some linear character $\xi$ of some subgroup of $K$. Finally, $\theta = \psi^E = (\xi^K)^E = \xi^E$ (by 6 of section 13, character induction is transitive), which completes the proof.$\blacksquare$

**Theorem 14:** (Brauer-Tate) Every character of $G$ is an integral linear combination of characters of $G$, induced from linear characters of elementary subgroups of $G$.

**Proof:** Let $\theta$ be a character of $G$. By theorem 12 (Brauer-Tate), $\theta = \sum_{i=1}^{k} a_i \psi_i^G$ where $a_1, \ldots, a_k \in \mathbb{Z}$ and for each $i$, $\psi_i$ is an irreducible character of some $E_i \in \mathcal{E}$. By lemma 14, $\psi_i = \xi_i^E$, where $\xi_i$ is a linear character of some subgroup $H_i \leq E_i$. Hence $\theta = \sum_{i=1}^{k} a_i (\xi_i^E)^G = \sum_{i=1}^{k} a_i \xi_i^G$, and we are done since $H_i \in \mathcal{E}$, $\forall 1 \leq i \leq k$. (subgroups of elementary groups are elementary)$\blacksquare$

## 0.20 Mackey’s Theorems

We consider representations over $\mathbb{C}$. Let $G$ be a finite group. If we induce a representation of $G$ from an irreducible character of a subgroup, it is generally not irreducible anymore. Mackey’s theorems help us understand how it decomposes into irreducibles.

1. Let $H \leq G$ and $\rho : H \to GL(V)$ be a representation of $H$. For $x \in G$, we define $\rho^x : xHx^{-1} \to GL(V)$ by $\rho^x(z) = \rho(x^{-1}zx)$, $\forall z \in xHx^{-1}$ (i.e. $\rho^x(xy^{-1}) = \rho(y)$, $\forall y \in H$).

Since conjugation by $x$ is a group isomorphism of $xHx^{-1}$ and $H$, $\rho^x$ is a representation of $xHx^{-1}$ on $V$. In some sense, $\rho^x$ is really just $\rho$, which is precisely stated in the following commuting diagram ($\sigma$ is conjugation by $x^{-1}$):

$$
\begin{array}{ccc}
H & \xrightarrow{\rho} & GL(V) \\
\sigma \downarrow & & \downarrow{id.} \\
xHx^{-1} & \xrightarrow{\rho^x} & GL(V)
\end{array}
$$

In particular, $M$ is an $H$-submodule of $V$ if and only if $M$ is an $xHx^{-1}$-submodule of $V$.

We clearly have $\chi_{\rho^x}(xy^{-1}) = \chi_{\rho}(y)$, $\forall y \in H$.

More generally, if $\phi \in \text{class}(H)$, we define $\phi^x : xHx^{-1} \to \mathbb{C}$ by $\phi^x(xy^{-1}) = \phi(y)$, $\forall y \in H$.

Then $\phi^x \in \text{class}(xHx^{-1})$ (again since conjugation by $x$ is a homomorphism). Moreover, if $\phi, \psi \in \text{class}(H)$ then $\langle \phi^x, \psi^x \rangle = \langle \phi, \psi \rangle$.

2. (Mackey) Let $K, H \leq G$ and let $\{x_1 = 1, x_2, \ldots, x_s\}$ be a set of $K - H$ double coset representatives in $G$ (recall that this means $G = \bigcup_{i=1}^{s} Kx_iH$). For $1 \leq i \leq s$, let $K_i = K \cap x_iHx_i^{-1}$. If $\psi \in \text{class}(H)$, then

$$
(\psi^G)|_K = \sum_{i=1}^{s} (\psi^{|K_i}|_{K_i})^K.
$$

(24)

**Proof:** $K$ acts on the left cosets $\{xH \mid x \in G\}$ by left multiplication. The double coset $KxH$ is a union of such cosets, namely the ones in the orbit of $xH$. The stabilizer of $xH$ under this action is $K \cap xHx^{-1}$ (indeed $k \in K$, $kxH = xH$ $\iff$ $k \in K$, $x^{-1}kx \in H$ $\iff$ $k \in K \cap xHx^{-1}$). Now, for $1 \leq i \leq s$ let $k_{i1}, \ldots, k_{is_i}$ be left coset representatives of $K_i$ in $K$. Thus, $Kx_iH = k_{i1}x_{i1}H \uplus \cdots \uplus k_{is_i}x_{si}H$ (since the action of $K$ on $x_iH$ can be identified
with the action of $K$ on the left cosets of the stabilizer $K_i$ of $x_iH$ in $K$). It follows that
\{k_{ij}x_i | 1 \leq i \leq s, 1 \leq j \leq s_i\} is a complete set of left coset representatives of $H$ in $G$. Next, since $K_i = K \cap x_iHx_i^{-1}$, we have $(\psi^{x_i}|_{K_i})^0(z) = (\psi^{x_i})^0(z), \forall z \in K, 1 \leq i \leq s$. Hence, for $y \in K$ we get
\[
\psi^G(y) = \sum_{1 \leq i \leq s} \sum_{j=1}^{s_i} \psi^0(x_i^{-1}k_{ij}^{-1}y^{-1}k_{ij}x_i) = \sum_{i=1}^{s} \sum_{j=1}^{s_i} (\psi^{x_i})^0(k_{ij}^{-1}y^{-1}k_{ij}) = \sum_{i=1}^{s} \sum_{j=1}^{s_i} (\psi^{x_i}|_{K_i})^0(k_{ij}^{-1}y^{-1}k_{ij}) = \sum_{i=1}^{s} (\psi^{x_i}|_{K_i})^K, as desired \]

(3) (Mackey) If $\phi \in \text{class}(K)$ and $\psi \in \text{class}(H)$ then
\[
\langle \phi^G, \psi^G \rangle = \sum_{i=1}^{s} \langle \phi|_{K_i}, \psi^{x_i}|_{K_i} \rangle.
\]

Proof: Using Frobenius reciprocity repeatedly, and (24), we have
\[
\langle \phi^G, \psi^G \rangle = \langle \phi, \psi^G|_{K} \rangle = \sum_{i=1}^{s} \langle \phi, (\psi^{x_i}|_{K_i})^K \rangle = \sum_{i=1}^{s} \langle \phi|_{K_i}, \psi^{x_i}|_{K_i} \rangle \]

The next result, which is a consequence of the above, in some way fulfils the goal stated in the beginning of the section.

(4) Let $H \leq G$ and $\psi$ be an irreducible character of $H$.

1. $\psi^G$ is irreducible $\iff$ for each $x \in G \setminus H$, there is no irreducible character (of $H \cap xHx^{-1}$) occurring in both $\psi|_{H \cap x_iHx_i^{-1}}$ and $\psi^{x_i}|_{H \cap x_iHx_i^{-1}}$.

2. If $H \trianglelefteq G$, then $\psi^G$ is irreducible $\iff$ $\psi^G \neq \psi$ for all $x \in G \setminus H$.

Proof: 1 Letting $K = H$ and $\phi = \psi$ in (25), we get
\[
\langle \psi^G, \psi^G \rangle = \sum_{i=1}^{s} \langle \psi|_{H \cap x_iHx_i^{-1}}, \psi^{x_i}|_{H \cap x_iHx_i^{-1}} \rangle = 1 + \sum_{i=2}^{s} \langle \psi|_{H \cap x_iHx_i^{-1}}, \psi^{x_i}|_{H \cap x_iHx_i^{-1}} \rangle.
\]
Hence $\psi^G$ is irreducible $\iff \sum_{i=1}^{s} \langle \psi|_{H \cap x_iHx_i^{-1}}, \psi^{x_i}|_{H \cap x_iHx_i^{-1}} \rangle = 0 \iff$ for $2 \leq i \leq s$ there is no common irreducible character occurring in both $\psi|_{H \cap x_iHx_i^{-1}}$ and $\psi^{x_i}|_{H \cap x_iHx_i^{-1}}$. This is equivalent to what we want, since any $x \in G \setminus H$ can be taken to be one of the double coset representatives $x_2, \ldots, x_s$. 2 Since $xHx^{-1} = H, \forall x \in G$, part 1 above implies: $\psi^G$ is irreducible $\iff$ $\psi$ and $\psi^x$ don’t have common irreducible constituents for all $x \in G \setminus H \iff \psi \neq \psi^x, \forall x \in G \setminus H$, since $\psi$ and $\psi^x$ are both irreducible characters of $H$. 

0.21 An Application of Mackey’s Results

In this section, our purpose is to describe irreducible characters of a semidirect product, where the normal component is abelian. For a group $G$, let $\text{Irred}(G)$ denote the set of irreducible characters of $G$.

(1) Let $G = A \rtimes B$ where $A$ is an abelian group. Thus, $A \trianglelefteq G$, $B \trianglelefteq G$, $AB = G$ and $A \cap B = 1$. Also $(a_1b_1)(a_2b_2) = a_1(b_1a_2b_1^{-1})b_1b_2$, $\forall a_1, a_2 \in A$, $b_1, b_2 \in B$. Let $\phi \in \text{Irred}(A)$ and $b \in B$. By previous section, we know that $\phi^b$, defined by $\phi^b(a) = \phi(bab^{-1})$, $\forall a \in A$, is an irreducible character of $A = bab^{-1}$. This gives a well-defined action of $B$ on $\text{Irred}(A)$ (note that $(\phi^b)^{b_2}(a) = \phi(b_2b_1ab_1^{-1}b_2^{-1}) = \phi^{b_1b_2}(a)$, $\forall a \in A, b_1, b_2 \in B$). Let $B_\phi$ be the centralizer in $B$ of $\phi$, under this action.

(2) Note that $\phi$ is linear, since $A$ is abelian, so that we may identify $\phi$ with its character. Now let $\rho : B_\phi \to GL(V)$ be a representation of $B_\phi$. The map $\phi \times \rho : A \rtimes B_\phi \to GL(V)$, given by $\phi \times \rho(ab) = \phi(a)\rho(b)$ ($a \in A, b \in B$) is thus well-defined. Moreover, $\phi \times \rho$ is a representation of $A \rtimes B_\phi$. Indeed, for all $a_1, a_2 \in A, b_1, b_2 \in B$ we have

$$
\begin{align*}
\phi \times \rho([a_1b_1](a_2b_2)) &= \phi \times \rho[a_1(b_1a_2b_1^{-1})b_1b_2] = \phi(a_1)\phi(b_1a_2b_1^{-1})\rho(b_1)\rho(b_2) \\
&= \phi(a_1)\phi(a_2)\rho(b_1)\rho(b_2) \quad \text{(since $b_1$ centralizes $\phi$)} \\
&= \phi(a_1)\rho(b_1)\phi(a_2)\rho(b_2) \quad \text{(since $\phi(a_2) \in \mathbb{C}$)} \\
&= \phi \times \rho(a_1b_1)\phi \times \rho(a_2b_2).
\end{align*}
$$

Denote the character of $\phi \times \rho$ by $\phi \times \chi_\rho$. We have $\phi \times \chi_\rho(ab) = \phi(a)\chi_\rho(b)$, $\forall a \in A, b \in B$.

(3) Let $\phi \in \text{Irred}(A)$ and $\psi \in \text{Irred}(B_\phi)$. Then

1. $\phi \times \psi \in \text{Irred}(A \rtimes B_\phi)$.

2. $(\phi \times \psi)^G \in \text{Irred}(G)$.

**Proof:**

1. We have $\langle \phi \times \psi, \phi \times \psi \rangle = \frac{1}{|A| |B_\phi|} \sum_{a \in A} \phi(a)\psi(b)\overline{\phi(a)\psi(b)}$

$$
= \left( \frac{1}{|A|} \sum_{a \in A} \phi(a)\overline{\phi(a)} \right) \left( \frac{1}{|B_\phi|} \sum_{b \in B_\phi} \psi(b)\overline{\psi(b)} \right) = \langle \phi, \phi \rangle \cdot \langle \psi, \psi \rangle = 1.
$$

2. Let $H = A \ltimes B_\phi \leq G$, so that $\phi \times \psi \in \text{Irred}(H)$ by part 1. By 4 of previous section, it suffices to show that if $x = ab \in G \setminus H$, $a \in A, b \in B$ (i.e. if $b \not\in B_\phi$), then $(\phi \times \psi)|_{H \cap xHx^{-1}}$ and $(\phi \times \psi)^|_{A}$ have no common constituents. We show that, a fortiori, $(\phi \times \psi)|_{A}$ and $(\phi \times \psi)^|_{A}$ have no common constituents (note that $A = ax^{-1} \leq H \cap xHx^{-1}$). We have $(\phi \times \psi)|_{A} = \psi(1) \cdot \phi$, so that the constituents of $(\phi \times \psi)|_{A}$ are $\psi(1)$ copies of $\phi$. Next, $(\phi \times \psi)^|_{A}(a_1) = \phi \times \psi(b^{-1}a^{-1}a_1ab) = \psi(1)\phi(b^{-1}a^{-1}a_1ab) = \psi(1)\phi^b(a^{-1}a_1a) = \psi(1)\phi^b(a_1)$ $\forall a_1 \in A$. This shows that $(\phi \times \psi)^|_{A} = \psi(1)\phi^b$, so that its constituents are $\psi(1)$ copies of $\phi^b$. Finally, since $b \not\in B_\phi$, we have $\phi \neq \phi^b$ and hence we are done.

(4) Let $\phi, \theta \in \text{Irred}(A)$, $\psi \in \text{Irred}(B_\phi)$ and $\eta \in \text{Irred}(B_\theta)$.

1. If $\phi \not\in O_B(\theta)$ (the orbit of $\theta$ under the action of $B$ on $\text{Irred}(A)$), then $(\phi \times \psi)^G \neq (\theta \times \eta)^G$. 


2. If $\phi = \theta$ and $\psi \neq \eta$, then $(\phi \times \psi)^G \neq (\theta \times \eta)^G$.

**Proof:** We shall use Mackey’s result given by (25). Let $K = AB_\phi$ and $H = AB_\theta$. Let \( \{b_1, b_2, \ldots, b_s\} \) be a set of $B_\phi$-$B_\theta$ double coset representatives in $B$. We claim that \( \{b_1, \ldots, b_s\} \) is also a set of $K$-$H$ double coset representatives in $G = A \times B$. Indeed, assume \( b_i = (a_1c_1)b_j(a_2c_2) \) where \( a_1, a_2 \in A \), \( c_1, c_2 \in B_\phi \). Since \( A \leq G \), we can "move" the $a$'s to the left to get $b_i = ac_1b_jc_2$ where $a = a_1(c_1b_j)(c_1b_j)^{-1} \in A$. Hence, \( a \in A \cap B = \{1\} \) and $b_i = c_1b_jc_2$, i.e. $B_\phi b_i B_\theta = B_\phi b_j B_\theta$. Conversely, if $b_i = c_1b_jc_2$ with $c_1 \in B_\phi, c_2 \in B_\theta$ then clearly $AB_\phi b_i AB_\theta = AB_\phi b_j AB_\theta$, so the claim holds. Now for \( 1 \leq i \leq s \), note that $K \cap b_i H b_i^{-1} = AB_\phi \cap A(b_i B_\phi b_i^{-1}) = AB_i$, where $B_i = B_\phi \cap b_i B_\theta b_i^{-1}$. By (25), we have

\[
\langle (\phi \times \psi)^G, (\theta \times \eta)^G \rangle = \sum_{i=1}^{s} \langle (\phi \times \psi)|_{AB_i}, (\theta \times \eta)^b_i |_{AB_i} \rangle
\]

\[
= \sum_{i=1}^{s} \left( \frac{1}{|AB_i|} \sum_{a \in A \atop b \in B_i} \phi(a) \psi(b) \cdot \theta^{b_i}(a) \eta^{b_i}(b) \right)
\]

\[
= \sum_{i=1}^{s} \langle \phi, \theta^{b_i} \rangle \cdot \langle \psi|_{B_i}, \eta^{b_i}|_{B_i} \rangle.
\]

In case 1, the last sum is 0, since $\phi$ and $\theta^{b_i}$ are distinct irreducible characters of $A$, $\forall 1 \leq i \leq s$. Hence \( \langle (\phi \times \psi)^G, (\theta \times \eta)^G \rangle = 0 \) and $(\phi \times \psi)^G, (\theta \times \eta)^G \in \text{Irred}(G)$ are distinct.

In case 2, we have \( \langle \phi, \theta^{b_i} \rangle = \langle \phi, \theta^b \rangle = 0 \), unless \( b_i \in B_\phi \), i.e. unless $i = 1$. But for $i = 1$, we have \( b_1 = 1 \) and \( B_1 = B_\phi \) and \( \langle \psi|_{B_1}, \eta^{b_1}|_{B_1} \rangle = \langle \psi, \eta \rangle = 0 \), since $\psi \neq \eta$. So, as in 1 above, \( \langle (\phi \times \psi)^G, (\theta \times \eta)^G \rangle = 0 \) and $(\phi \times \psi)^G \neq (\theta \times \eta)^G$.

We conclude with the following theorem, which characterizes the irreducible character of certain semidirect products.

**Theorem 15:** Let $G = A \times B$ with $A \leq G$ and $A$ - abelian. Let $\phi_1, \ldots, \phi_m$ be a complete set of orbit representatives for the action of $B$ on $\text{Irred}(A)$ described in 1. For $1 \leq i \leq m$, let \( \text{Irred}(B_{\phi_i}) = \{\psi_{ij} \mid 1 \leq j \leq m_i\} \). Then

\[
\text{Irred}(G) = \{(\phi_i \times \psi_{ij})^G \mid 1 \leq i \leq m, 1 \leq j \leq m_i\}.
\]

**Proof:** By 3 and 4, the characters $(\phi_i \times \psi_{ij})^G$ are all irreducible and distinct. To prove they are all irreducibles of $G$, we verify that the sum of squares of their degrees is $|G|$. Indeed, note that $|G : AB_{\phi_i}| = |AB|/|AB_{\phi_i}| = |B : B_{\phi_i}|$ and $(\phi_i \times \psi_{ij})^G(1) = |B : B_{\phi_i}| \cdot (\phi_i \times \psi_{ij})(1) = |B : B_{\phi_i}| \cdot (\phi_i \times \psi_{ij})(1)$. Note also that $|B : B_{\phi_i}| = |\mathcal{O}_B(\phi_i)|$ (the size of the
orbit of $\phi_i$ is the index of its stabilizer in $B$). Thus, we have

$$\sum_{1 \leq i, j \leq m} [(\phi_i \times \psi_{ij})^G(1)]^2 = \sum_{1 \leq i, j \leq m} |B : B_{\phi_i}|^2 \cdot \psi_{ij}(1)^2 = \sum_{i=1}^m \left( |B : B_{\phi_i}|^2 \cdot \sum_{j=1}^{m} \psi_{ij}(1)^2 \right)$$

$$= \sum_{i=1}^m |B : B_{\phi_i}|^2 \cdot |B_{\phi_i}| = |B| \cdot \sum_{i=1}^m |B : B_{\phi_i}|$$

$$= |B| \sum_{i=1}^m |\mathcal{O}_B(\phi_i)| = |B| \cdot |Irred(A)| = |B| \cdot |A| = |G| \blacksquare$$