Chapter 6

Counting in the Presence of a Group

A mathematician, like a painter or poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made with ideas.

A Mathematician's Apology, G. H. HARDY

We discuss in this chapter a theory that originated in the algebraic works of Frobenius and Burnside [1] and culminated with those of Redfield [2], Pólya [3], and DeBruijn [4]. *The subject matter concerns counting the number of orbits (or orbits of a specific type) that arise by the action of a group on a set.* (When the set in question consists of functions the orbits are usually called patterns.)

6.1

Due to the highly intuitive nature of the subject it is best to introduce the general ques-

tions that arise by way of a simple example. Consider coloring the vertices of a square (labeled $\begin{smallmatrix}1&2\\3&4\end{smallmatrix}$) with colors A and B. There are 16 ways of doing this, for we have 4 vertices with 2 choices of color for each. However, one may wish to say that the colorations

$$\begin{matrix}A & B\\ & \\ B & B\end{matrix} \quad \text{and} \quad \begin{matrix}B & A\\ & \\ B & B\end{matrix}$$

are in effect the same, for the latter is obtained from the former by a mere (clockwise) rotation of 90 degrees.

To be specific, suppose we allow planar rotations of 0, 90, 180, and 270 degrees, and call two colorations *equivalent* if one is obtained from the other by such a rotation. We now raise the question: How many *nonequivalent* colorations are there? Upon some thought the reader will undoubtedly find the following six representatives:

$$\begin{matrix}A & A\\ & \\ A & A\end{matrix}, \quad \begin{matrix}B & B\\ & \\ B & B\end{matrix}, \quad \begin{matrix}A & B\\ & \\ B & B\end{matrix}, \quad \begin{matrix}B & A\\ & \\ A & A\end{matrix}, \quad \begin{matrix}A & A\\ & \\ B & B\end{matrix}, \quad \begin{matrix}A & B\\ & \\ B & A\end{matrix}.$$

By allowing this kind of equivalence (induced by the cyclic group Z_4 of planar rotations) we have only 6 (essentially different) colorations, and not 16. In addition, we become even more permissive and introduce another equivalence by saying that two colorations are in fact the *same* if one is obtained from the other by interchanging the colors A and B. Thus

$$\begin{matrix}A & B\\ & \\ B & B\end{matrix} \quad \text{and} \quad \begin{matrix}B & A\\ & \\ A & A\end{matrix}$$

become indistinguishable, and so do

$$\begin{matrix}A & A\\ & \\ A & A\end{matrix} \quad \text{and} \quad \begin{matrix}B & B\\ & \\ B & B\end{matrix}$$

Upon which performance we are left with only four nonequivalent representatives:

$$\begin{matrix} A & A \\ A & A \end{matrix}, \quad \begin{matrix} A & A \\ B & B \end{matrix}, \quad \begin{matrix} A & A \\ B & B \end{matrix}, \quad \begin{matrix} A & B \\ B & A \end{matrix}.$$

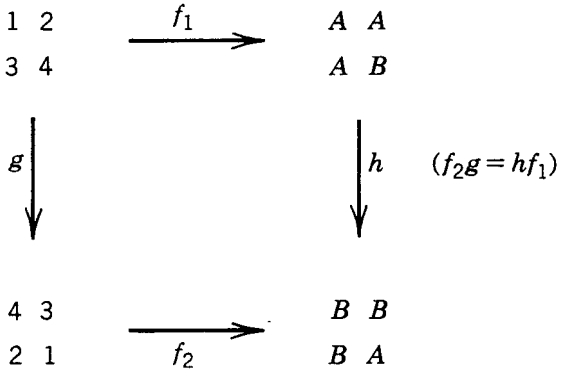
Generally we visualize a coloration as a function from a domain D (of the four vertices of the square - listed in the order $\begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix}$) to a range R of colors A and B. We allow a group G to act on D (the group Z_4 of planar rotations) and another group H to act on R (the group Z_2 that interchanges A and B). Call two colorations (or functions) f_1 and f_2 *equivalent* if

$$f_2 g = h f_1,$$

for some rotation g in G and some (possible interchange of colors) h in H. The fundamental question that we raise is: *How many nonequivalent functions (or colorations) are there?* The nonequivalent functions are called *patterns*. With the subject thus introduced, let us close our preliminary discussions by explicitly exhibiting the equivalence of colorations

$$\mathrm{f}_1 : \begin{matrix} A & A \\ A & B \end{matrix} \quad \text{and} \quad \mathrm{f}_2 : \begin{matrix} A & B \\ B & B \end{matrix}.$$

Indeed,



where the domain is D: $\begin{smallmatrix}1&2\\3&4\end{smallmatrix}$ (the vertices of the square displayed in this fixed order) and the range is R consisting of colors A and B. (A function from D to R is a specification of a color at each vertex, i.e., a coloration.)

1 THE GENERAL THEORY

We start out by reminding the reader of a few preliminaries on permutation groups.

6.2 Permutations on a Set

Let Ω be a set A bijection $\sigma : \Omega \to \Omega$ is called a *permutation.* By SymΩ we denote the set of all permutations on Ω. If $\Omega = \{1, 2, ..., n\}$ we write S_n for SymΩ; $\mid S_n \mid = n!$. Under composition of functions SymΩ is a group, called the *symmetric group.* In SymΩ we multiply (i.e., compose) permutations from right to left, just as we work with composite functions. That is, if α and β are permutations, then $\beta\alpha$ is the permutation that carries x into $\beta(\alpha(x))$.

Any permutation σ can be decomposed into disjoint cycles: pick a symbol (1, say), form the cycle (from right to left)

$$(\cdots\sigma(\sigma(\sigma(1)))\sigma(\sigma(1))\sigma(1)1)$$

and close it when 1 is reached again;repeat the process to the remaining symbols until none are left. For example,the permutation

$$\sigma : \begin{matrix}1&2&3&4&5&6\\3&2&4&1&6&5\end{matrix}$$

has cycle decomposition $(6\,5)(2)(4\,3\,1)$. Note that we list the elements of a cycle from *right to left.* Cycles of length 1 are usually omitted. Thus σ above is written $(6\,5)(4\,3\,1)$.

The decomposition into disjoint cycles is *unique*, up to the order in which the disjoint cycles are listed. Observe that disjoint cycles commute.

A *transposition* is a cycle of the form (ba). We can write $(m \cdots 3\,2\,1) = (m1) \cdots (3\,1)(2\,1)$, and thus express a cycle of length m as a product of $m-1$ transpositions. We now define the *parity* of a permutation. A permutation $\sigma = c_k \cdots c_2 c_1$ (with c_i disjoint cycles) is said to be *even* or *odd* according to the parity of the total number of transpositions obtained upon expanding the cycles. [A cycle of odd length being an even permutation and a cycle of even length being an odd permutation (see above), we conclude that the parity of an arbitrary permutation equals the parity of the number of cycles of even length which appear in its cycle decomposition.]

The *inverse* of $(m \cdots 3\,2\,1)$ *is* $(1\,2\,3 \cdots m)$. For two permutations α and β, we denote the product $\beta\alpha\beta^{-1}$ by α^β and call it the *conjugate* of α by β. Conjugation preserves cycle structure. In fact α^β can be obtained from α by replacing a position in α by its image under β, for example,

$$\begin{aligned} \alpha &= (8\,7)(9\,6\,5\,2)(4\,3\,1) \\ &\qquad\qquad \updownarrow \quad (8\,7\,4)(6\,1) = \beta \\ \alpha^\beta &= (4\,8)(9\,1\,5\,2)(7\,3\,6). \end{aligned}$$

Two permutations are conjugates in $\mathrm{Sym}\Omega$ if and only if they have the same cycle structure.

The set of even permutations is a subgroup of index 2 in $\mathrm{Sym}\Omega$. We call it the *alternating group* and denote it by $\mathrm{Alt}\Omega$. If $\Omega = \{1, 2, ..., n\}$ we write A_n instead of $\mathrm{Alt}\Omega$; $\mid A_n \mid = n!/2$.

6.3 Group Actions on Sets

If G is a subgroup of SymΩ we say that G *acts* on the set Ω. A homomorphism $g \to \hat{g}$ of an (abstract) group G into SymΩ is called a *permutation representation* of G on Ω. The representation is *faithful* if it has kernel 1, that is, if it is injective.

Let G act on Ω. Each element g of G can be therefore visualized as a permutation $\hat{g}$ on Ω. For notational simplicity we write $g(x)$ instead of $\hat{g}(x)$ for the image of x under the permutation $\hat{g}$.

The subset $\{g(x) : g \in G\}$ is called the *orbit* of x (under G) and is denoted by Gx. The orbit of x is a subset of Ω. Evidently Ω is a disjoint union of orbits. (To see this, pick a point in Ω and form its orbit; if this orbit does not exhaust Ω, pick a new point outside of the first orbit and form its orbit; proceed until every point in Ω is in an orbit. Two orbits are, of course, disjoint.)

The *stabilizer* of a point x in Ω is the subgroup of G defined as follows: {g : g(x) = x}; we denote it by G_x.

It turns out that the cardinality (or length) of an orbit divides the order of the group. The precise relationship is

$$|Gx|\,|G_x| = |G| \quad \textit{(or, in words, the length of an orbit equals the index of the stabilizer).} \tag{6.1}$$

Indeed, let y be a point in the orbit Gx of x. If h is an element of G that sends x into y, then a description of *all* elements that send x into y is the coset hG_x. We thus establish a *bijection* between the cosets of G_x in G and the points in orbit Gx of x. There are $|G|/|G_x|$ such cosets and hence $|Gx| = |G|/|G_x|$, as stated.

If x and y are points in the same orbit we naturally have $|Gx| = |Gy|$ =cardinality of the orbit. By (6.1) we conclude that also

$$|G_x| = \frac{|G|}{|Gx|} = \frac{|G|}{|Gy|} = |G_y|.$$

To summarize:

$$\textit{If } x \textit{ and } y \textit{ belong to the same orbit, then } |G_x| = |G_y|. \tag{6.2}$$

6.4 On Two-Way Counting

Throughout this chapter, and especially the next, a certain technique of counting becomes prevalent; we call it two-way counting.

Let S be a subset of the Cartesian product $V \times W$. Define

$$(v, \cdot) = \{w \in W : (v, w) \in S\}$$

and

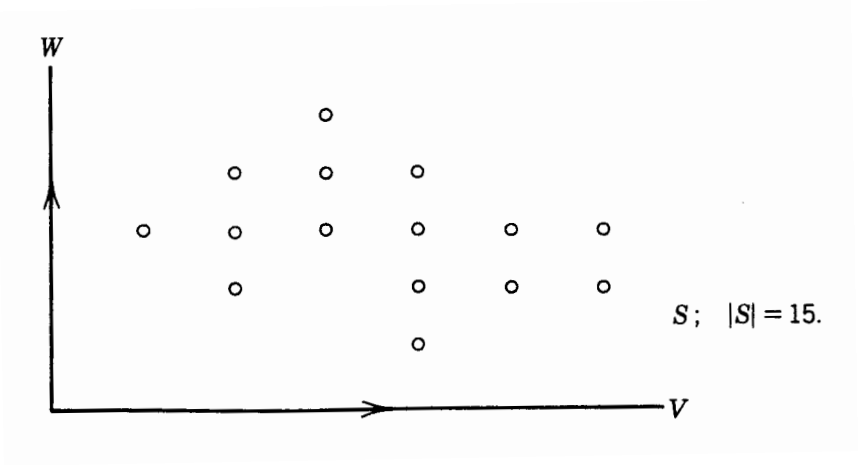
$$(\cdot, w) = \{v \in V : (v, w) \in S\}.$$

Then

$$\sum_{v \in V} |(v, \cdot)| = |S| = \sum_{w \in W} |(\cdot, w)|.$$

This self-evident fact expresses the cardinality of S in two ways: first by initially fixing the first coordinate and summing over the second, then by initially fixing the second and summing over the first.

An example illustrates this:



Here $\sum_{v\in V}|(v,\cdot)| = 1+3+3+4+2+2 = |S| = 1+4+6+3+1 = \sum_{w\in W}|(\cdot,w)|$.

Often the region S is rectangular, in which case we have

$$|(v,\cdot)| = r\,(say), \qquad \text{for all } v \in V$$

and

$$|(\cdot,w)| = k\,(say), \qquad \text{for all } w \in W.$$

Then

$$|V|r = |S| = k|W|.$$

6.5 Frobenius' Results

Let the group G act on the set Ω. Denote by t the number of orbits induced by G. In addition, let $F(g) = \{x \in \Omega : g(x) = x\}$; that is, $F(g)$ is the set of points fixed by the group element g.

Burnside's lemma informs us that:

Burnside's Lemma.

$$t = \frac{1}{|G|}\sum_{g\in G}|F(g)|.$$

(*Or, in words, the number of orbits equals the average number of points left fixed by the elements of the group.*)

Proof. We count in two ways the cardinality of the set of pairs $B = \{(g, x) : g(x) = x\}$, with g in G and x in Ω.

Observe that in this situation $(g, \cdot)$ is simply $F(g)$ and that $(\cdot, x)$ is G_x. Therefore,

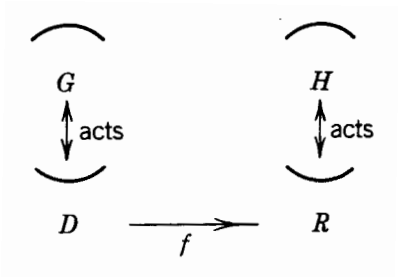
$$\begin{aligned}\sum_{g\in G} |F(g)| &= |B| = \sum_{x\in\Omega} |G_x| = \{\text{by (6.2), upon sorting by orbits}\} \\ &= |G_{x_1}||Gx_1| + |G_{x_2}||Gx_2| + \cdots + |G_{x_t}||Gx_t| \\ &= \{\text{by (6.1)}\} = |G| + |G| + \cdots + |G| = t|G|.\end{aligned}$$

Here the x_i's are representatives from distinct orbits. This ends the proof.

6.6 Groups Acting on Sets of Functions

What was described in the introductory passage (Section 6.1) we now consider in more generality. The reader should refer to Section 6.1 whenever (if ever) the theory seems difficult to follow.

Let D (called the *domain*) and R (called the *range*) be sets. Let also G and H be groups; assume that G acts on D and H acts on R. Denote by $F(D, R)$ the set of functions from D to R. [If $|D| = m$ and $|R| = n$, then $|F(D, R)| = n^m$.] We summarize:



The actions of G on D and of H on R induce a natural action of the direct sum $G \oplus H$

on *the set of functions* $F(D, R)$ as follows:

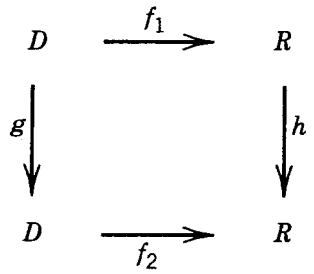
$$(gh)(f) = hfg^{-1}, \tag{6.3}$$

with $gh \in G \oplus H$ [$g \in G$, $h \in H$, and $f \in F(D, R)$].

We call the functions f_1 and f_2 *equivalent*, and write$f_1 \sim f_2$, if they are in the *same orbit* of the action of $G \oplus H$ on $F(D, R)$. Thus $f_1 \sim f_2$ if $f_2 = (gh)(f_1) = hf_1g^{-1}$, for some $g \in G$ and $h \in H$. This can also be written as follows:

$$f_1 \sim f_2 \qquad \text{if} \qquad f_2 g = h f_1,$$

for some g in G and h in H (see figure below and compare to that in Section 6.1).



By $f_2 g = h f_1$ we mean $f_2(g(d)) = h(f_1(d))$, for all d, in the domain D.

It is, of course, important to check that what we have in (6.3) is indeed an *action*. We are required to show that

$$g_2\widehat{h_2 g_1} h_1 = \widehat{g_2 h_2}\,\widehat{g_1 h_1}; \qquad\qquad g_1, g_2 \in G \text{ and } h_1, h_2 \in H$$

where, in general, g denotes the permutation representation of the group element g. This is indeed true, for $g_2 h_2 g_1 h_1$ equals $g_2 g_1 h_2 h_1$ and it sends f into $h_2 h_1 f(g_2 g_1)^{-1} = h_2 h_1 f g_1^{-1} g_2^{-1}$. On the other hand $g_1 h_1$, sends f into $h_1 f g_1^{-1}$, and $g_2 h_2$ sends $h_1 f g_1^{-1}$ into $h_2 h_1 f g_1^{-1} g_2^{-1}$; hence $g_2\widehat{h_2 g_1} h_1 = \widehat{g_2 h_2}\,\widehat{g_1 h_1}$ The orbits of $G \oplus H$ on $F(D, R)$ are called *patterns*. Our principal task is to count how many patterns there are.

6.7 DeBruijn's Results

We do actually want something more than just to count the number of patterns. If possible, we would prefer to know the number of patterns of a certain type, for example. Or maybe the number of patterns that involve only a certain number of elements of the range R (i.e., in Section 6.1 the number of colorations with precisely two colors - there are two such). All this indicates that we should perhaps separate the patterns by *weight*.

Let $W : F(D, R) \to A$ be a function that satisfies $W(f_1) = W(f_2)$ whenever $f_1 \sim f_2$. [Observe that W is by its definition *well-defined on the set of patterns*. For a pattern T we can therefore unambiguously write $W(T)$, where $W(T) = W(f)$ for any f in T. The function W is called a *weight function* and $W(T)$ is the *weight* of pattern T.] One would have expected the range of W to be the set of rational numbers, but it is better to make A consist of polynomials in indeterminates $y_1, ..., y_s$ (say) with rational or real coefficients. The weight of a pattern could therefore be something like $9y_6^7 - \frac{2}{3}y_1^4 y_5$ (it could, of course, also be its length, or whatever rational number one prefers). The choice of polynomials offers the opportunity to introduce generating functions, a potential tool.

We now prove a result of DeBruijn:

DeBruijn's Result.

$$\sum_{\substack{T \\ (Tpattern)}} W(T) = \frac{1}{|G||H|} \sum_{g \in G} \sum_{h \in H} \sum_{\substack{f \\ (gh)(f)=f}} W(f) \tag{6.4}$$

(The very last sum is over all functions f fixed by gh.)

Proof. Let y be an element in the range of W. Look at all patterns T of weight y; denote this collection by T_y. The group $G \oplus H$ acts on T_y since T_y is a union of orbits (or

patterns). By Burnside's lemma we have:

$$|T_y| = \frac{1}{|G \oplus H|} \sum_{gh \in G \oplus H} |F_y(gh)| = \frac{1}{|G||H|} \sum_{g \in G} \sum_{h \in H} |F_y(gh)|,$$

with $F_y(gh)$ the set of functions f in T_y fixed by gh. Sum now over y in the range of W to obtain

$$\begin{aligned} \sum_{\substack{T \\ (Tpattern)}} W(T) &= \sum_y y|T_y| = \sum_y \frac{y}{|G||H|} \sum_{g \in G} \sum_{h \in H} |F_y(gh)| \\ &= \frac{1}{|G||H|} \sum_{g \in G} \sum_{h \in H} \sum_y y|F_y(gh)| \\ &= \frac{1}{|G||H|} \sum_{g \in G} \sum_{h \in H} \sum_{\substack{f \\ (gh)(f)=f}} W(f). \end{aligned}$$

The last sign of equality is explained as follows:

$$\begin{aligned} \sum_y y|F_y(gh)| &= \sum_y y \text{ (number of functions } f \text{ fixed by } gh \text{ in } T_y) \\ &= \sum_{\substack{f \\ (gh)(f)=f}} W(f), \end{aligned}$$

with summation over all f in F(D, R) fixed by gh. This proves DeBruijn's result.

6.8 Results by Redfield and Pólya

We gradually specialize DeBruijn's result. First, let $H = 1$ (i.e., there is no group acting on the range). The group G (visualized as $G \oplus 1$, if necessary) acts on $F(D, R)$ as follows:

$$g(f) = fg^{-1},$$

for g an element of G [a special case of (6.3)]. Two functions f_1 and f_2 are equivalent (written still $f_1 \sim f_2$) if $f_2 g = f_1$, for some g in G.

Secondly, we restrict attention to special kinds of weight functions W. Let $w : R \to A$ be a function from the range R into A, the set of polynomials in indeterminates $y_1, ..., y_s$

with rational coefficients. For r an element of the range R we call $w(r)$ its *weight*. The function w induces a weight function W on $F(D,R)$ by setting

$$W(f) = \prod_{d\in D} w(f(d)). \tag{6.5}$$

[If $f_1 \sim f_2$ then $f_1 = f_2 g$, for some g in G, and then

$$\begin{aligned} W(f_1) &= W(f_2 g) = \prod_d w(f_2(g(d))) = \{\text{since } g \text{ is a bijection}\} \\ &= \prod_d w(f_2(d)) = W(f_2), \end{aligned}$$

which shows that W is indeed well defined on patterns.]

DeBruijn's result (6.4) can now be written

$$\sum_{\substack{T \\ (Tpattern)}} W(T) = \frac{1}{|G|} \sum_{g\in G} \sum_{\substack{f \\ fg=f}} W(f), \tag{6.6}$$

where W is as defined in (6.5).

The last sum $\sum_{f,fg=f} W(f)$ takes, in fact, a more explicit form determined chiefly by the cycle structure of the group element g in its action on D. Write $g = c_k \cdots c_2 c_1$, with c_i's the disjoint cycles of g. The condition $f = fg$ means (for all d in D) that $f(d) = f(g(d)) = f(g^2(d)) = f(g^3(d)) = \cdots$, which implies that f is constant on each cycle c_i of g (but it may take different values on different cycles). [We remind that $g^2(d)$ means $g(g(d))$, etc.] Conversely, every function f constant on the cycles c_i of g obviously satisfies $fg = f$. (In what follows by $|c_i|$we denote the length, or cardinality, of the cycle c_i.)

With this characterization of the functions f that satisfy $fg = f$ we can write

$$\prod_{i=1}^{k} \sum_{r\in R} (w(r))^{|c_i|}$$

$$= \left(\sum_r (w(r))^{|c_1|}\right)\left(\sum_r (w(r))^{|c_2|}\right)\cdots\left(\sum_r (w(r))^{|c_k|}\right)$$

$$= \left[\begin{array}{l}\text{Expand by picking a term from each of the k factors and sum over all such choices; visualize now the } i\text{th factor above as the } i\text{th cycle } c_i \text{ of } g\text{; then picking a term from each factor means defining a function } f \text{ on D, which is } \textit{constant} \text{ on each of the cycles } c_i \text{ and with range included in R.}\end{array}\right]$$

$$= \sum_{\substack{(f \text{ is a constant on} \\ \text{each cycle } c_i \text{ of } g)}} \prod_{d\in D} w(f(d)) = \sum_{\substack{f \\ fg=f}} W(f)$$

(the next to last step takes some thinking . . .).

Let $|D| = m$ and suppose that g is of type $1^{\lambda_1}2^{\lambda_2}\cdots m^{\lambda_m}$(with $\sum_{i=1}^{m} i\lambda_i = m$, that is, g has λ_i cycles of length i. Some of the λ_i's could be 0; we still assume that $g = c_k \cdots c_2 c_1$ with $|c_i| \geq 1$, $1 \leq i \geq k$. Rewriting the previous equation gives

$$\begin{aligned}\sum_{\substack{f \\ fg=g}} W(f) &= \prod_{i=1}^{k} \sum_r (w(r))^{|c_i|} \\ &= \left(\sum_r w(r)\right)^{\lambda_1}\left(\sum_r (w(r))^2\right)^{\lambda_2}\cdots\left(\sum_r (w(r))^m\right)^{\lambda_m}. \qquad (6.7)\end{aligned}$$

Substituting (6.7) in (6.6) we obtain Pólya's result:

$$\sum_T W(T) = \frac{1}{|G|}\sum_{g\in G}\left[\prod_{i=1}^{m}\left(\sum_{r\in R}(w(r))^i\right)^{\lambda_i}\right] \qquad (6.8)$$

where$1^{\lambda_1}2^{\lambda_2}\cdots m^{\lambda_m}$ *denotes the cycle type of the group element g in its action on the domain.*

Expression (6.8) strongly suggests that the *cycle structure* of the group elements (represented on the domain) is the determining feature of the group that comes into play. We take advantage of this by cleverly choosing our notation.

6.9 Just Notation

Recall that $|D| = m$. To the group G (in its action on the domain D) we associate now a polynomial P_G in m indeterminates $x_l, x_2, ..., x_m$ called the *cycle index* of G. Specifically,

$$P_G(x_1, x_2, \ldots, x_m) = \frac{1}{|G|} \sum_{g \in G} x_1^{\lambda_1(g)} x_2^{\lambda_2(g)} \cdots x_m^{\lambda^m(g)},$$

where $\lambda_i(g)$ denotes the number of cycles of length i of g.

By substituting $(w(r))^i$ for x_i in P_G we can rewrite (6.8) as follows:

Pólya's Theorem.

$$\sum_{\substack{T \\ (Tpattern)}} W(T) = P_G\left(\sum_r w(r), \sum_r (w(r))^2, ..., \sum_r (w(r))^m\right) \tag{6.9}$$

where P_G is the cycle index of G in its action on the domain D.

In particular, if $w(r) = 1$ for all r in the range R, we obtain:

$$\textit{The total number of patterns} = P_G(|R|, |R|, \ldots, |R|). \tag{6.10}$$

There is a lot of information locked in Pólya's Theorem theorem (6.9) and even more in DeBruijn's result given in (6.4). In the remaining pages of this chapter we have the opportunity to unleash the power of these results in many interesting special cases. To a large extent this also enhances the understanding of some of the finer points in the proof we just gave.

REMARK. It is instructive to compare Burnside's lemma with Pólya's enumeration theorem. At first it appears that the latter is essentially a special case of the former when the group acts on the set of functions from D to R. And this is indeed true, as the proof we just saw indicates. Due to the large cardinality of the set of functions it

is usually difficult to apply Burnside's lemma directly to this set and calculate for each group element the number of functions that it fixes. Pólya's theorem reduces drastically the size of the problem by only studying the group in its original representation on the *domain* D of these functions (a much smaller set). One needs to know the representation on the domain rather well, however: knowing the number of points fixed by each group element on the domain is not enough; rather, a list of the actual cycle structure of each group element (the cycle index) must be made available. This is the price paid by working on the domain only.

Pólya's theorem may be helpful, however, in understanding certain "naturally" induced representations of the group in question on subsets of a fixed cardinality of the domain. Specifically, let G be a group of permutations on the elements of D. Then G acts on the subsets of D of cardinality k in a natural way, that is, $g\{d_1, \ldots, d_k\} = \{g(d_1), \ldots, g(d_k)\}$.

Assume that the cycle index of G on D is known and denote it by $P_G(x_1, x_2, \ldots, x_m)$. We make the following assertion:

$*$ *The number of orbits of G in its action on the subsets of cardinality k of D equals the coefficient of y^k in $P_G(1+y, 1+y^2, \ldots, 1+y^m)$.*

To demonstrate this, look at the set of functions from D to $R = \{0, 1\}$. A *function* from D to R is easily identified with the *subset* of elements of D mapped into l. Let G act on the set of functions as follows:

$$g(f(d)) = f(g(d)).$$

Select a weight function w on R by setting $w(0) = 1$ and $w(1) = y$. Then f has weight

y^k if and only if f maps k elements of D into 1 [see (6.5), if necessary]. It is now clear that the number of orbits that G induces on the subsets of cardinality k of D equals the number of patterns of weight k that thus arise. Pólya's theorem informs us now that the number of such patterns is simply the coefficient of y^k in the polynomial

$$P_G(1+y, l+y^2, ..., 1+y^m).$$

In particular, the number of orbits of G on the elements of D (which is what Burnside's lemma gives) equals the coefficient of y in this polynomial.

2 RECIPE FOR PÓLYA'S THEOREM

6.10

Consider again the problem of coloring the vertices of the square $\begin{smallmatrix}1&2\\3&4\end{smallmatrix}$ with colors A and B (as in Section 6.1). Our domain consists therefore of the four vertices of the square, and the range consists of the two colors A and B. Call two colorations equivalent if they are the same apart from a planar rotation. In Section 6.1 we have found six patterns, a representative from each being listed below:

$$\begin{matrix}A&A\\A&A\end{matrix}, \quad \begin{matrix}B&B\\B&B\end{matrix}, \quad \begin{matrix}A&B\\B&B\end{matrix}, \quad \begin{matrix}B&A\\A&A\end{matrix}, \quad \begin{matrix}A&A\\B&B\end{matrix}, \quad \begin{matrix}A&B\\B&A\end{matrix}. \tag{6.11}$$

In particular observe that precisely two patterns (the last two, as displayed above) are colored with two $A's$ and two $B's$.

Let us show how we can derive this information from Pólya's theorem.

First, extract the cycle structure of the group G of the four planar rotations (of 0, 90, 180, and 270 degrees) on the domain. The identity (i.e., rotation by 0 degrees)

fixes all vertices of the square; thus its cycle structure is $1^4 2^0 3^0 4^0$, which we abbreviate by the monomial $x_1^4 x_2^0 x_3^0 x_4^0 = x_1^4$. The rotation by 90 degrees is (4 3 2 1), which we write $x_1^0 x_2^0 x_3^0 x_4^1 = x_4$, and the rotation by 270 degrees is (1 2 3 4) with same cycle structure$x_1^0 x_2^0 x_3^0 x_4^1 = x_4$. Lastly, the rotation by 180 degrees is (4 2)(3 1), described by the monomial $x_1^0 x_2^2 x_3^0 x_4^0 = x_2^2$. The cycle index of the group G of rotations of the square is therefore

$$P_G(x_1, x_2, x_3, x_4) = \frac{1}{4}(x_1^4 + x_2^2 + 2x_4).$$

(Now toss away the group; we do not need it anymore.)

Pólya's theorem informs us that the total number of patterns equals $P_G(|R|, |R|, |R|, |R|) = P_G(2, 2, 2, 2) = \frac{1}{4}(2^4 + 2^2 + 2 \cdot 2) = \frac{1}{4}24 = 6$, as expected.

Further, suppose we become interested in the number of patterns with precisely two $A's$ and two $B's$. The ability to select weights as general in nature as indeterminates comes in very handy at this time. We wish to clearly distinguish between the $A's$ and the $B's$ that occur in a pattern and thus assign weights $w(A) = y_l$ and $w(B) = y_2$, with y_l and y_2 indeterminates. Then the weight of a pattern containing a $A's$ and b $B's$ is $y_1^a y_2^b$, with $a + b = 4$ [see (6.5)]. We can therefore read the number of patterns with a $A's$ and b $B's$ in the coefficient of $y_1^a y_2^b$. Pólya's theorem (6.9) explicitly gives

$$\begin{aligned}
\sum_{\substack{T \\ (Tpattern)}} W(T) &= P_G(\sum_{r\in R} w(r), \sum_{r\in R} \Big(w(r))^2, ldots, \sum_{r\in R} (w(r))^m \Big) \\
&= P_G(y_1 + y_2, y_1^2 + y_2^2, y_1^3 + y_2^3, y_1^4 + y_2^4) \\
&= \frac{1}{4}\left((y_1 + y_2)^4 + (y_1^2 + y_2^2)^2 + 2(y_1^4 + y_2^4)\right).
\end{aligned}$$

We seek the number of patterns with two $A's$ and two $B's$, and upon inspecting the coefficient of $y_1^2 y_2^2$ find the answer $\frac{1}{4}(6 + 2) = 2$, in agreement with what (6.11) displays.

What we have achieved in the case of the square can be carried over to more general circumstances. We describe the procedure in Section 6.11.

6.11 The Recipe

We have before us a domain D with elements $d_1, d_2, \ldots, d_m$ and a range R of colors $r_1, r_2, \ldots, r_n$. Available before us is also a group G, which allows itself to be represented as a group of permutations on the domain D.

Interest is manifested in coloring the elements of the domain D with colors from the range R. Furthermore, two such colorations are said to be *equivalent* if one is carried into the other by a group element g in G. That is, colorations f_1 and f_2 are equivalent if $f_1 = f_2 g$, for some g in G.

The nonequivalent colorations are called *patterns*, and we ask:

(i) How many patterns are there?

(ii) How many patterns are there with

a_1 occurrences of color r_l

a_2 occurrences of color r_2

$\vdots$

a_n occurrences of color r_n? $(\sum_{i=1}^{n} a_i = m = |D|.)$

Pólya's theorem offers the answers in three steps, the first of which is by far the most difficult and time consuming.

Step 1. Find the cycle index of the group G in its action on the domain D.

[We remind ourselves that the cycle index is a polynomial P_G in indeterminates

$x_l, x_2, \ldots, x_m$ defined as follows:

$$P_G(x_1, x_2, \ldots, x_m) = \frac{1}{|G|} \sum_{g \in G} x_1^{\lambda_1(g)} x_2^{\lambda_2(g)} \cdots x_m^{\lambda^m(g)},$$

where $\lambda_i(g)$ is the number of cycles of length i of the group element g in its representation as a permutation on the domain D. (*Aside*: Upon extracting the cycle index throw away the group; we do not need it anymore.)]

Step 2. Obtain the total number of patterns by substituting the number of colors (i.e., $|R|$) for each of the variables $x_l, x_2, \ldots, x_m$ in the cycle index P_G. That is:

$$\text{The total number of patterns} = P_G(|R|, |R|, \ldots, |R|).$$

Step 3. The number of patterns with

a_1 occurrences of color r_l

a_2 occurrences of color r_2

$\vdots$

a_n occurrences of color $r_n (\sum_{i=1}^n a_i = m)$

is the coefficient of $y_l^{a_1} y_2^{a_2} \cdots y_n^{a_n}$ in the polynomial

$$P_G\left(\sum_{i=1}^n y_i, \sum_{i=1}^n y_i^2, \ldots, \sum_{i=1}^n y_i^m\right).$$

[This is indeed the case, for if we assign weight y_i to color r_i (with y_i indeterminates), then the weight of a pattern in which r_i occurs a_i times, $1 \leq i \leq n$, is $\prod_{i=1}^n y_i^{a_i}$ (conform (6.5)). Thus the coefficient of this monomial is the answer we seek, and Pólya's theorem (6.9) leads us to it by way of suitable substitution (i.e., substitute $\sum_{i=1}^n y_i^k$ for x_k) in the cycle index.]

We now illustrate the recipe by several examples.

3 EXAMPLES FOLLOWING THE RECIPE

6.12 The Case of the Trivial Group

Assume that we have a domain D consisting of m elements and a range R of n colors that we denote by $r_1, r_2, \ldots, r_n$. We begin with the simplest possible situation: that in which the group G acting on D consists of only one element (the identity). Two colorations are now equivalent if they are absolutely *identical*.

It is obvious that there are n^m patterns in all (for we have n choices of color for each of the m elements of the domain). The number of patterns with a_1 elements of color r_1, a_2 elements of color $r_2, \ldots, a_n$ elements of color r_n (with $\sum_{i=1}^{n} a_i = m$) is just the multinomial coefficient

$$\frac{m!}{a_1!a_2!\ \cdots\ a_n!}.$$

Our recipe indeed gives:

Step 1. The cycle index of G is simply

$$P_G(x_1, \ldots, x_m) = \frac{1}{1}(x_1^m + 0 + \cdots + 0) = x_1^m$$

Step 2. The total number of patterns is

$$P_G(n, n, \ldots, n) = n^m.$$

Step 3. The number of patterns with a_i elements of color r_i $(i = l, 2, \ldots, n)$ is the coefficient $y_1^{a_1} y_2^{a_2}\ \cdots\ y_n^{a_n}$ in

$$P_G(\sum y_i, \sum y_i^2, \cdots) = (\sum y_i)^m.$$

This coefficient is indeed the multinomial number written above. [For example, if $n = 2$ we obtain the binomial coefficient $\binom{m}{a_1}$ for an answer, as expected.]

6.13 Coloring the Cube

Who can possibly forget the pleasant childhood pastime of rolling colored cubes? Or the quiet frustration of observing a nicer color pattern displayed on your playmate's cube? To those whose interest in the charming enterprise was at the time less than mathematical we offer the opportunity for a change of emphasis.

Let us dust off our favorite childhood cube (or any ordinary cube, really) and equip ourselves with a selection of attractive colors. Naturally we proceed to try our hand at coloring the object. Guided by principles of good taste we can color either: (a) its 8 vertices, (b) its 12 edges, or (c) its 6 faces. In each case, however, we say that two colorations are essentially the same (i.e., equivalent) if one can be obtained from the other by a *rotation* of the cube.

The (abstract) group G that induces equivalence is therefore that of the $6 \times 4 = 24$ rotations. [There are indeed 24 rotations of the cube, for we have 6 places to send a face to and 4 further (independent) choices for an adjacent face.]

We examine in detail the essentially different colorations in each of the three cases mentioned above.

Let us make a list (by way of a geometrical description) of the 24 rotations of the cube:

(i) The identity.

(ii) Three 180 rotations around lines connecting the centers of opposite faces.

(iii) Six 90 rotations around lines connecting the centers of opposite faces.

(iv) Six 180 rotations around lines connecting the midpoints of opposite edges.

(v) Eight 120 rotations around lines connecting opposite vertices.

This may not appear as the most natural way in the world to classify rotations but it is if the object is computation of cycle indexes.

Coloring the Vertices of the Cube

The group G of rotations acts now on the eight vertices of the cube, which form our domain D.

Step 1. Extract the cycle index of G in its action on the vertices of the cube. The cycle structure of the group elements as outlined in (i) through (v) is summarized below:

$$\begin{aligned}
(i) &\leftrightarrow x_1^8\\
(ii) &\leftrightarrow 3x_2^4\\
(iii) &\leftrightarrow 6x_4^2\\
(iv) &\leftrightarrow 6x_2^4\\
(v) &\leftrightarrow 8x_1^2x_3^2
\end{aligned}$$

And so $P_G = \frac{1}{24}(x_1^8 + 9x_2^4 + 6x_4^2 + 8x_1^2x_3^2)$.

Step 2. If we are equipped with n colors the total number of essentially different colorations is the value of P_G at $x_1 = x_2 = \ldots = x_8 = n$. That is, we have $\frac{1}{24}(n^8 + 9n^4 + 6n^2 + 8n^4)$ distinct patterns in all. [Note that irrespective of n this number is always an integer, an amusing feature in itself. In the event of two available colors, for example, there are precisely $\frac{1}{24}(2^8 + 9 \cdot 2^4 + 6 \cdot 2^2 + 8 \cdot 2^4) = 13$ distinct color patterns.]

Step 3. Let the colors be $r_1, \ldots, r_n$. The number of patterns with a_i vertices colored with color r_i $(i = 1, 2, \ldots, 8)$ is the coefficient of $y_1^{a_1}y_2^{a_2}\cdots y_n^{a_n}$ in

$$P_G(\sum y_i, \sum y_i^2, \ldots, \sum y_i^8)$$

$$= \frac{1}{24}\left[\left(\sum y_i\right)^8 + 9\left(\sum y_i^2\right)^4 + 6\left(\sum y_i^4\right)^2 + 8\left(\sum y_i\right)^2\left(\sum y_i^3\right)^2\right]$$

[If we color with just colors red and blue, and wish to know how many essentially distinct colorations (patterns) there are with three red vertices and five blue, we look at the coefficient of $y_1^3 y_2^5$ in

$$\frac{1}{24}\left[(y_1+y_2)^8 + 9(y_1^2+y_2^2)^4 + 6(y_1^4+y_2^4)^2 + 8(y_1+y_2)^2(y_1^3+y_2^3)^2\right].$$

This coefficient is

$$\frac{1}{24}\left[\binom{8}{3} + 8\cdot 2\right] = \frac{1}{8\cdot 3}(8\cdot 7 + 8\cdot 2) = 3.$$

The reader can easily verify that this is true by exhibiting geometrically the three configurations on the cube.]

Coloring the Edges of the Cube

The group of rotations permutes the 12 edges of our cube. Edges are now the elements of our domain.

Step 1. Extract the cycle index of the group of rotations in its action on the 12 edges of the cube. We summarize again, from (i) to (v):

$$\begin{aligned}
(i) &\leftrightarrow x_1^{12}\\
(ii) &\leftrightarrow 3x_2^6\\
(iii) &\leftrightarrow 6x_4^3\\
(iv) &\leftrightarrow 6x_1^2x_2^5\\
(v) &\leftrightarrow 8x_3^4
\end{aligned}$$

The cycle index is

$$P_G = \frac{1}{24}(x_1^{12} + 3x_2^6 + 6x_4^3 + 6x_1^2x_2^5 + 8x_3^4).$$

Step 2. The total number of patterns is

$$P_G(n, n, ..., n) = \frac{1}{24}(n^{12} + 3n^6 + 6n^4 + 6n^7 + 8n^4),$$

if we use n colors. [In case of two colors we obtain $P_G(2, 2, ..., 2) = 218$ patterns in all.]

Step 3. The number of patterns with a_i edges colored r_i $(i = 1, 2, \ldots, 12)$ equals the coefficient of $\prod_{i=1}^{12} y_i^{a_i}$ in the expansion of $P_G(\sum y_i, \sum y_i^2, \ldots, \sum y_i^{12})$. For example, there are 27 essentially distinct colorations with four blue and eight red edges.

Coloring the Faces of the Cube

Step 1. The rotation group certainly permutes the six faces and its cycle structure is

$(i) \leftrightarrow x_1^6$

$(ii) \leftrightarrow 3x_1^2x_2^2$

$(iii) \leftrightarrow 6x_1^2x_4^1$

$(iv) \leftrightarrow 6x_2^3$

$(v) \leftrightarrow 8x_3^2$

Hence the cycle index of the rotation group represented on the six faces of the cube is

$$P_G = \frac{1}{24}(x_1^6 + 3x_1^2x_2^2 + 6x_1^2x_4 + 6x_2^3 + 8x_3^2).$$

Step 2. The total number of color patterns on faces possible to generate with n colors is

$$P_G(n, n, \ldots, n) = \frac{1}{24}(n^6 + 3n^4 + 6n^3 + 6n^3 + 8n^2).$$

[If two colors are used we obtain $P_G(2, 2, \ldots, 2) = 10$ patterns in all.]

Step 3. Let us become curious and ask how many patterns there are with three distinct colors per pattern and with each of the three colors occurring twice? Assume that we have n colors available; $n \geq 3$.

Choose three out of the n available colors [call them 1, 2, 3 (say)]. Give weight y_l to color l, with y_l an indeterminate. Then the number of patterns colored with colors 1,2,3, with each color occurring twice, is the coefficient of $y_1^2\, y_2^2\, y_3^2$ in $P_G(\sum y_i, \sum y_i^2, \ldots, \sum y_i^6)$. The intrinsic symmetry of the problem and the symmetric nature of the substitution made in the cyclic index P_G allow us to conclude that there are $\binom{n}{3}$ times the coefficient of $y_1^2\, y_2^2\, y_3^2$ such patterns. The coefficient of $y_1^2\, y_2^2\, y_3^2$ in $\frac{1}{24}[(\sum y_i)^6 + 3(\sum y_i)^2(\sum y_i^2)^2 + 6(\sum y_i)^2(\sum y_i^4) + 6(\sum y_i^2)^3 + 8(\sum y_i^3)^2]$ is

$$\frac{1}{24}\left[\frac{6!}{2!\,2!\,2!} + 3(2+2+2) + 0 + 6\frac{3!}{1!\,1!\,1!} + 0\right] = 6,$$

and the answer to our question is therefore $6\binom{n}{3}$. Check this result for $n = 3$ by a direct examination of patterns.

6.14 Patterns on a Chessboard

How many essentially distinct configurations can we make by placing 16 checkers on an ordinary chessboard? Two configurations are called essentially distinct if one cannot be obtained from the other by a *rotation* of the chessboard. The alternating black-white coloring of the squares of the chessboard plays no role in what we do; the reader should thus ignore it and assume that the whole chessboard is white. Apart from the initial question we raised, the techniques developed so far answer other natural questions as well, and we shall point these out as we go along.

Step 1. The group in question is that of rotations by 0, 90, 180, and 270. Our domain consists of the $8 \times 8 = 64$ squares of the chessboard. In its action on the domain our group has cycle structure as indicated below:

Identity $\leftrightarrow$ x_1^{64}

Rotation by 180 $\leftrightarrow$ $x2^{32}$

Rotation by 90 $\leftrightarrow$ x_4^{16}

Rotation by 270 $\leftrightarrow$ x_4^{16}

We conclude that the cycle index of the group G (of rotations of the board), in its action on the 64 squares of the chessboard, has cycle index:

$$P_G = \frac{1}{4}(x_1^{64} + x_2^{32} + 2x_4^{16}).$$

Step 2. The act of placing checkers on the chessboard should be viewed as a coloration of the domain with two colors: *checker = red*, and *no checker = blue*.

Pólya's theorem informs us now that *the total number of patterns* $= P_G(2,2,2,2) = \frac{1}{4}(2^{64} + 2^{32} + 2 \cdot 2^{16})$.

Step 3. Assign weight y_1 to the red color (i.e., to a checker) and weight y_2 to blue (i.e., to no checker). Our original question asks for the number of patterns with precisely 16 red and $64 - 16$ blue colors. The answer can be found in the coefficient of $y_1^{16}y_2^{64-16}$ in

$$\begin{aligned} &P_G(y_1 + y_2,\, y_1^2 + y_2^2,\, y_1^3 + y_2^3,\, y_1^4 + y_2^4) \\ &\quad = \frac{1}{4}\left((y_1 + y_2)^{64} + (y_1^2 + y_2^2)^{32} + 2(y_1^4 + y_2^4)^{16}\right). \end{aligned}$$

Upon collecting terms we find this coefficient to be

$$\frac{1}{4}\left(\binom{64}{16} + \binom{32}{8} + 2\binom{16}{4}\right)$$

More generally, assume that we have checkers of $n-1$ colors and ask for the number of patterns with a_1 checkers of color r_1, a_2 checkers of color $r_2, \ldots, a_{n-1}$ checkers of color

r_{n-1}. Upon assigning weight y_i to checkers of color r_i $(i = 1, 2, \ldots, n-1)$ and weight y_n to *no checker*, the answer is found in the coefficient of

$$y_1^{a_1}\, y_2^{a_2}\, \cdots\, y_{n-1}^{a_{n-1}}\, y_n^{n} \qquad \left(a_n = 64 - \sum_{i=1}^{n-1} a_i\right)$$

upon expanding

$$\begin{aligned} &P_G(\textstyle\sum y_i, \sum y_i^2, \sum y_i^3, \sum y_i^4) \\ &\quad = \frac{1}{4}\left(\left(\textstyle\sum y_i\right)^{64} + \left(\sum y_i^2\right)^{32} + 2\left(\sum y_i^4\right)^{16}\right). \end{aligned}$$

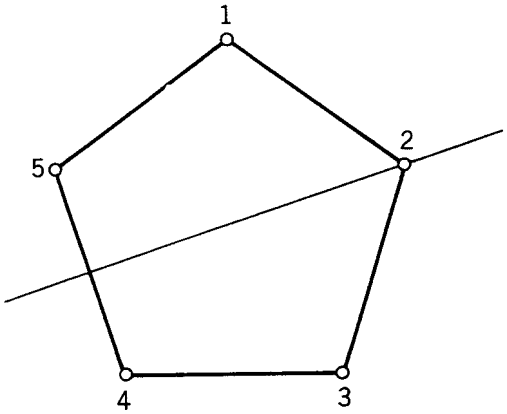
(In the summation signs above the index i runs between 1 and n.)

We leave to the reader the general case of placing a_i checkers of color r_i $(i = 1, 2, \ldots, n-1)$ on a $m \times m$ chessboard and of answering how many patterns arise. (Note that for odd m the cycle index changes its form, because the square in the center remains fixed under all four rotations.)

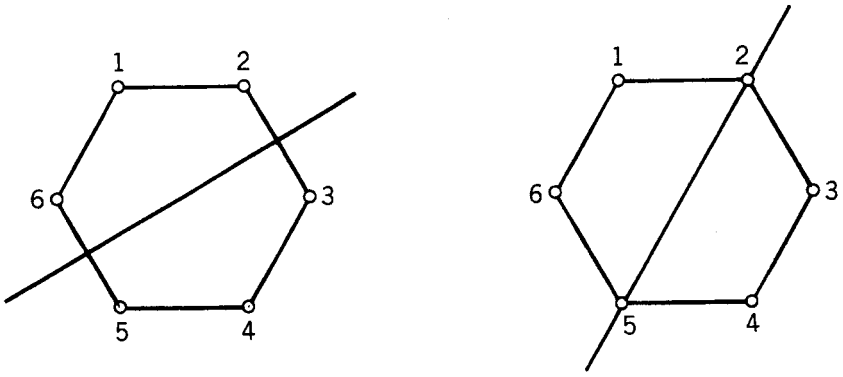
6.15 Necklaces

Equipped with n colors and m beads we ask for the number of essentially different necklaces of m colored beads that we can make. Two necklaces are said to be equivalent if one can be obtained from the other by a rotation and/or a flip.

The group that induces equivalence is in this case the dihedral group D_{2m} (of order $2m$) consisting of the m planar rotations plus m flips. For example, D_{10} consists of planar rotations with angles $2\pi k/5\,(k = 0, 1, 2, 3, 4)$ and 5 flips across lines joining a vertex to the middle of the edge completely opposite:



This remains a valid description of the elements of D_{2m} for any odd number m. When m is even we have $m/2$ flips across lines joining the middle points of completely opposite edges, and $m/2$ flips fixing a pair of totally opposite vertices (see below), in addition to the m planar rotations of angles $2\pi k/m\,(k = 0, 1, ..., m-1)$.



A necklace should be envisioned as a regular m-gon with beads as vertices and the group that induces equivalence is D_{2m} whose elements we just described.

Step 1. The cycle index of D_{2m} in its action on the vertices of the m-gon (i.e., on the m beads of the necklace) depends on the parity of m.

If m is odd, say $m = 2s + 1$, then we have m flips, each of which fixes one vertex and pairs up the remaining $2s$ vertices into s disjoint transpositions. The cyclic subgroup of rotations is generated by a rotation with angle $2\pi/m$. If a rotation has order d, then it

consists of m/d disjoint cycles of length d each; and there are $\phi(d)$ such rotations, where $\phi(d)$ denotes the number of positive integers less than d and relatively prime to d. The cycle index of D_{2m} for odd m is therefore

$$P_{D_{2m}} = \frac{1}{2m}\left[m x_1\, x_2^{(m-1)/2} + \sum_{\substack{d \\ d\backslash m}} \phi(d) x_d^{m/d}\right].$$

Analogously, for m even we obtain

$$P_{D_{2m}} = \frac{1}{2m}\left[\frac{m}{2}\, x_2^{m/2} + \frac{m}{2} x_1^2\, x_2^{(m-2)/2} + \sum_{\substack{d \\ d\backslash m}} \phi(d) x_d^{m/d}\right].$$

Step 2. The total number of patterns of necklaces with m beads colored with n colors is

$$P_{D_{2m}}(n, ..., n) = \frac{1}{2m}\left[m\, n\, n^{(m-1)/2} + \sum_{\substack{d \\ d\backslash m}} \phi(d) n^{m/d}\right], \qquad \text{for } m \text{ odd}$$

and

$$P_{D_{2m}}(n, ..., n) = \frac{1}{2m}\left[\frac{m}{2} n^{m/2} + \frac{m}{2} n^2\, n^{(m-2)/2} + \sum_{\substack{d \\ d\backslash m}} \phi(d) n^{m/d}\right], \qquad \text{for } m \text{ even.}$$

Step 3. By assigning weight y_i to color r_i $(i = 1, \ldots, n)$ we can, as usual, answer how many (patterns of) necklaces with m beads in all there are such that a_l beads are colored r_l, a_2 are colored $r_2, \ldots,$ and a_n are colored r_n. The answer is the coefficient of $y_1^{a_1} y_2^{a_2} \cdots y_n^{a_n}$ in

$$P_{D_{2m}}(\textstyle\sum y_i, \sum y_i^2, ..., \sum y_i^m)$$

If $m = 6$, for example, and if we have two colors (green and red), the number of patterns of necklaces with two green and four red beads is the coefficient of $y_1^2 y_2^4$ in

$$\frac{1}{2 \cdot 6}[3(y_1^2 + y_2^2)^3 + 3(y_1 + y_2)^2(y_1^2 + y_2^2)^2 + \phi(1)(y_1 + y_2)^6$$
$$+\phi(2)(y_1^2 + y_2^2)^3 + \phi(3)(y_1^3 + y_2^3)^2 + \phi(6)(y_1^6 + y_2^6)].$$

This coefficient is

$$\frac{1}{2\cdot 6}\left[3\binom{3}{2}+3(1+2)+\phi(1)\binom{6}{2}+\phi(2)\binom{3}{2}+0+0\right]$$

$$=\frac{1}{2\cdot 6}\left[3\cdot 6+\binom{6}{2}+1\binom{3}{2}\right]=3.$$

Indeed, the patterns are:

$$\begin{array}{ccc} r & r & \\ g & g & \end{array} \qquad \begin{array}{ccc} r & r & \\ g & r & \end{array} \qquad \begin{array}{ccc} r & g & \\ r & r & \end{array}$$

$$\begin{array}{ccc} r & & \\ r & & \\ g & & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} r & & \\ g & & \\ r & & \end{array}$$

4 THE CYCLE INDEX

Most of the effectiveness of counting in the presence of a group rests with the ability of understanding the cycle structure of the group as a permutation group on the domain. Having available a list of cycle indexes will thus enhance our computational power. On the other hand, the same abstract group has different cycle indexes when represented on different domains; this latter piece of intelligence renders the idea of a comprehensive list pretty much meaningless.

6.16

We nonetheless mention several cycle indexes that occur fairly often.

The Identity

When represented on a domain with m elements the identity group has cycle index x_1^m.

The Cyclic Group

The cyclic group of the m (planar) rotations of a regular m-gon, represented as a permutation group on the vertices of the m-gon, has cycle index

$$\frac{1}{m}\sum_{\substack{d\\ d\backslash m}} \phi(d)\, x_d^{m/d},$$

where $\phi(d)$ denotes the positive integers less than d and relatively prime to d.

[The cyclic group discussed above consists of the cycle $(m\cdots 3\,2\,1)$ and all its powers. An examination of the cycle structure of its powers readily reveals the formula for its cycle index written above.]

The Dihedral Group

By the appellation of a dihedral group of order $2m$ we understand the m rotations and m flips of a regular m-gon. These groups appeared in Section 6.15 where we also computed their cycle indexes:

$$\frac{1}{2m}\left[m x_1\, x_2^{(m-1)/2} + \sum_{\substack{d\\ d\backslash m}} \phi(d)\, x_d^{m/d}\right], \qquad \text{for } m \text{ odd}$$

and

$$\frac{1}{2m}\left[\frac{m}{2} x_2^{m/2} + \frac{m}{2} x_1^2\, x_2^{(m-2)/2} + \sum_{\substack{d\\ d\backslash m}} \phi(d)\, x_d^{m/d}\right] \qquad \text{for } m \text{ even}$$

The Symmetric Group

The cycle index of the $m!$ permutations on m elements (which form the symmetric group on the m elements) is

$$\frac{1}{m!}\left[\sum \frac{m!}{(1!)^{\lambda_1}\cdots(m!)^{\lambda_m}\,\lambda_1!\cdots\lambda_m!} x_1^{\lambda_1}\, x_2^{\lambda_2}\, \cdots\, x_m^{\lambda_m}\right],$$

where the sum extends over all vectors $(\lambda_1,\ldots,\lambda_m)$ that satisfy $1\lambda_1+2\lambda_2+\cdots+m\lambda_m = m$.

[Indeed, we counted on several occasions the number of permutations of cycle type $1^{\lambda_1}2^{\lambda_2}\cdots m^{\lambda_m}$ (see, e.g., Section 1.8(a)).]

The Alternating Group

As the reader recalls, a cycle of even length is an odd permutation. And hence, a permutation of cycle type $1^{\lambda_1}2^{\lambda_2}\cdots m^{\lambda_m}$ is even if and only if it has an even number of cycles of even length (or, written in symbols, if and only if $\lambda_2 + \lambda_4 + \lambda_6 + \cdots$ is even).

The alternating group consists of all even permutations on m symbols; its order is $m!/2$. By the remark we just made its cycle index is

$$\frac{m!}{2}\sum \frac{m!}{(1!)^{\lambda_1}\cdots(m!)^{\lambda_m}\lambda_1!\cdots\lambda_m!}x_1^{\lambda_1}\cdots x_m^{\lambda_m},$$

where the sum is taken over all vectors $(\lambda_1, ..., \lambda_m)$ with $\lambda_2 + \lambda_4 + \lambda_6 + \cdots$ being even, and $\sum_{i=1}^{m} i\lambda_i = m$.

The Direct Sum Acting on the Disjoint Union of Sets

We let G and H be (abstract) groups and form their direct sum $G \oplus H$. If G acts on the set X and H acts on the set Y (with X and Y *disjoint*), then we have a natural action of $G \oplus H$ on $X \cup Y$, namely

$$(gh)(z) = \begin{cases} g(z) & \text{if } z \in X \\ h(z) & \text{if } z \in Y \end{cases}$$

A permutation gh so represented on $X \cup Y$ consists therefore of the permutation of g on X followed by the permutation of h on Y. When translated into cycle indexes this simply says that

$$P_{G\oplus H} = P_G \times P_H.$$

[Or, in words, the cycle index of a direct sum acting on the union of two disjoint sets equals the product of the respective cycle indexes.]

The Direct Sum Acting on the Cartesian Product

If G acts on X and H on Y, we can define an action of $G \oplus H$ on $X \times Y$ by letting

$$(gh)(x, y) = (g(x), h(y)).$$

For r and s natural numbers denote by $(r; s)$ their greatest common divisor and by $[r; s]$ their least common multiple. The cycle index of $G \oplus H$ acting in this fashion on $X \times Y$ can now be written as follows:

$$P_{G \oplus H} = \frac{1}{|G||H|} \sum_{g,\, h} \sum_{r,\, s} x_{[r;\, s]}^{(r;\, s)\lambda_r(g)\lambda_s(h)}.$$

[To understand this formula, suppose x belongs to a cycle α of G on X, and that y belongs to a cycle β of H on Y. Let r and s denote the lengths of α and β, respectively. Then (x, y) belongs to a cycle of gh of length $[r; s]$. The cycles α and β induce $(r; s)$ cycles of gh on $X \times Y$. Since there are $\lambda_r(g)$ cycles of length r of g and $\lambda_s(h)$ cycles of length s of h, the formula is proved. We suggest that the reader verify this on a small example.]

The Wreath Product

Visualize five cubes as vertices of a regular pentagon. Generate permutations on the $5 \cdot 6 = 30$ positions by rotating each cube individually and rotating the whole pentagon as well. The group of permutations so obtained is called the wreath product of the group of rotations of the pentagon with the group of rotations of the cube. (The five cubes circularly placed form a configuration not unlike a wreath, and this suggests the general

nomenclature.) What we do in general the reader is invited to specialize to the case described above.

Let S and T be sets, and let G and H be groups acting on S and T, respectively. (The set S generalizes the pentagon and T is the cube.) We consider the Cartesian product $S \times T$ and construct special permutations on $S \times T$ as follows: Choose g in G, and to each s in S associate an element h_s of H. These elements determine a permutation on $S \times T$ defined by

$$(s, t) \to (g(s), h_s(t)).$$

There are $|G||H|^{|S|}$ such permutations; they form a group we call the *wreath product* of G with H, denoted by $G[H]$. Our objective is to express the cycle index of $G[H]$ in terms of those of G and H. Pólya proved the following result:

$*$ *The cycle index of* $G[H]$ is

$$P_{G[H]}(x_1, x_2, \ldots) = P_G(P_H(x_1, x_2, x_3, \ldots), P_H(x_2, x_4, x_6, \ldots), \ldots),$$

where the right-hand side is obtained by substituting

$$P_H(x_k, x_{2k}, x_{3k}, \ldots) \; for \; y_k \; in \; P_G(y_1, y_2, y_3, \ldots).$$

Proof (Following DeBruijn). Denote by m the cardinality of S. Fix an element γ of $G[H]$ by selecting g in G and $h_1, h_2, \ldots, h_m$ in H. We want to determine the cycle type of γ. Let $s_1, \ldots, s_k$ be a cycle of g (in S). We thus have

$$g(s_1) = s_2, \; g^2(s_1) = s_3, \ldots, g^{k-1}(s_1) = s_k, \; g^k(s_1) = s_1.$$

It follows that the set of pairs $\{(s_i, t) : i = 1, 2, \ldots, k \; and \; t \in T\}$ is mapped onto itself by γ. We call this set a *block*, and want to find out in what cycles this block splits under the

influence of γ. It turns out that this only depends on the product

$$h = h_{s_k} h_{s_{k-1}} \cdots h_{s_2} h_{s_1}$$

in the following way: If h has cycle structure $1^{\lambda_1} 2^{\lambda_2} 3^{\lambda_3} \cdots$, then the block splits into λ_1 cycles of length k, λ_2 cycles of length $2k$, and so on. This can be seen by writing out what the successive applications of γ do to an element (s_1, t) :

$$(s_1, t) \to (s_2, h_{s_1}(t)) \to (s_3, h_{s_2}(h_{s_1}(t))) \to \cdots \to (s_1, h(t)).$$

Now, if l is the order of h, then we close the cycle of (s_1, t) upon kl applications of γ. It follows that (s_1, t) generates a cycle of length kl. Noting that (s_1, t) and (s_1, t') generate the same cycle under γ if and only if t and t' generate the same cycle in T under h, we conclude that our block contains λ_l cycles of length kl, and this holds for each l.

We thus attach the monomial $|H|^{-k}\, x_k^{\lambda_1}\, x_{2k}^{\lambda_2}\, x_{3k}^{\lambda_3} \cdots$ to this block. With g and $s_1, \ldots, s_k$ still fixed, summing over all possible choices for $h_{s_1}, \ldots, h_{s_k}$ we obtain $P_H(x_k, x_{2k}, x_{3k}, \ldots)$. This follows from the fact that if $h_{s_1}, \ldots, h_{s_k}$ all run through H, then their product h runs $|H|^{k-1}$ times through H.

Next we consider all elements γ of $G[H]$ arising from a single g of type $1^{\mu_1}\, 2^{\mu_2}\, 3^{\mu_3} \cdots$. By what we have done it follows that the contribution of these elements to the cycle index $P_{G[H]}$ is

$$[P_H(x_1, x_2, \ldots)]^{\mu_1}\, [P_H(x_2, x_4, \ldots)]^{\mu_2}\, [P_H(x_3, x_6, \ldots)]^{\mu_3} \cdots.$$

Summing up over all g in G yields the result.

We conclude this section with an example. Reminiscing once again about the wonderful childhood games with cubes, one cannot fail to observe that rolling one cube may

be fun for some, but (as every self-respecting infant knows) it is the multitude of colored cubes that offers true excitement. Let us then take eight cubes, arrange them circularly to form a regular octagon, and use two colors to color their faces. Two colorations are said to form the same pattern if one is obtained from the other upon *rotating the individual cubes and cyclicly rotating all eight cubes* around the shape of the octagon. We ask for the number of patterns.

In this case the domain consists of the $8 \cdot 6 = 48$ faces of the eight cubes, the range consists of two colors (red and blue, say), and the group that induces equivalence is the wreath product $G[H]$, where G is the rotation group of the octagon (of order 8) and H is the familiar rotation group of the cube (of order 24).

Step 1. We just finished investigating the cycle index of $G[H]$. It is

$$P_{G[H]}(x_1, x_2, \ldots) = P_G(P_H(x_1, x_2, \ldots), P_H(x_2, x_4, \ldots), \ldots).$$

In this case

$$\begin{aligned} P_G(y_l, \ldots, y_8) &= \frac{1}{8}[\phi(1)y_1^8 + \phi(2)y_2^4 + \phi(4)y_4^2 + \phi(8)y_8] \\ &= \frac{1}{8}[y_1^8 + y_2^4 + 2y_4^2 + 4y_8], \end{aligned}$$

where $\phi(d)$ signifies (as usual) the number of positive integers less than and relatively prime to d. From Section 6.13 the cycle index of H is

$$P_H(x_1, \ldots, x_6) = \frac{1}{24}[x_1^6 + 3x_1^2x_2^2 + 6x_1^2x_4 + 6x_2^3 + 8x_3^2].$$

Thus

$$\begin{aligned} P_{G[H]} = P_G(&P_H(x_1, x_2, \ldots, x_6), \\ &P_H(x_2, x_4, \ldots, x_{12}), \ldots, P_H(x_8, x_{16}, \ldots, x_{48})). \end{aligned}$$

Step 2. The answer to our problem is $P_{G[H]}(2, 2, \ldots, 2)$. In Section 6.13 we found that $P_H(2, \ldots, 2) = 10$, and hence

$$\begin{aligned} P_{G[H]}(2, 2, \ldots, 2) &= P_G(10, 10, \ldots, 10) \\ &= \frac{1}{8}(10^8 + 10^4 + 2 \cdot 10^2 + 4 \cdot 10). \end{aligned}$$

This last (large) figure counts the total number of patterns.

Step 3. We leave it to anyone curious to decide how many patterns with 20 red and 28 blue faces are possible. To do this, an explicit computation of $P_{G[H]}$ is necessary. This is routine but time consuming.

EXERCISES

1. Give an example of two nonisomorphic groups that have the same cycle index.

2. Verify that the cycle index of the symmetric group S_n (in its action on n points) is the coefficient of z^n in the power series

$$exp\left(zx_1 + \frac{z^2x_2}{2} + \frac{z^3x_3}{3} + \cdots\right).$$

3. Equip yourself with n colors and a hollow cube with thin walls.

   (a) Represent the rotation group of the cube on the 12 edges of the cube and compute its cycle index.

   (b) Represent the rotation group of the cube on the 12 faces (the 6 on the outside plus the 6 on the inside) and compute its cycle index.

   (c) Are the representations in (a) and (b) isomorphic?

(d) Do we generate more patterns by coloring the 12 edges or by coloring the 12 (inside and outside) faces with the n available colors?

(e) How many patterns are there with two faces of one color, three faces of another, and seven faces of a third color? Answer the same when the n colors are used to color the edges of the cube.

4. Represent the group of rotations of the octahedron on its vertices, its edges, and then its faces. Derive the cycle index in each case. Then take n colors and find out how many patterns arise in each case. (Compare the results to those in Section 6.13 and explain the "duality.")

5. (a) In the usual setting of Pólya's theorem denote by p the number of patterns that arise under the action of the group G on the domain. Let H be a subgroup of G and denote by q the number of patterns that are induced by H. Show that $q \leq p$.

(b) As an example, examine the case of necklaces with m beads of two colors. Let G be the dihedral group D_{2m} and let H be its cyclic subgroup of M rotations. For what values of m do we have $p = q$?

6. Color the ten vertices of Petersen's graph red, white, and blue. Call two colorations the same if one is obtained from the other upon a permutation of vertices that preserves the edges. How many essentially distinct colorations (i.e., patterns) are possible? How many with two red, three white, and five blue vertices? Petersen's graph is displayed in Section 4.11.

7. The group S_4 of the 24 permutations on 4 vertices induces a natural action on the 6 available edges. Find the cycle index of S_4 in its action on the 6 edges.

8. Let $X = \{1, 2, 3, 4, 5, 6\}$ and $Y = \{7, 8, 9\}$ be disjoint sets. Let G be the cyclic group $Z_6 = \langle (1\,2\,3\,4\,5\,6) \rangle$ acting on X, and let H be the symmetric group S_3 acting on Y. Compute the cycle index of $G \oplus H$ in its action on $X \times Y$.

5 MORE THEORY

In the last three sections the focus was on special instances and on many examples connected to Pólya's theorem. We chose that course of exposition mostly for pedagogical reasons: It gave the reader the opportunity to develop a "feel" for what the theorem offers and to practice a little with the cycle index of a group. The main theme of this section is DeBruijn's generalization of Pólya's theorem.

6.17

We look once again at the set $F(D, R)$ of all functions from the domain D to the range R (of colors). A function f in $F(D, R)$ is commonly viewed as a coloration of D, in the sense that it colors element d of D with color $f(d)$. There are two groups present: a group G acting on the elements of D and a group H that permutes the colors of the range R.

The objects of our study are the two induced group actions on the set $F(D, R)$ of functions. Firstly, we let the group G act on $F(D, R)$ by setting $g(f) = fg^{-1}$, with g in G and f in $F(D, R)$. The orbits so induced on $F(D, R)$ are called *G-patterns*. We denote the G-pattern of f by fG.

Secondly, we look at the action of the direct sum $G \oplus H$ on $F(D, R)$ and define $(gh)(f) = hfg^{-1}$, for g in G, h in H, and f in $F(D, R)$. The orbits of this latter action are simply called *patterns* (and we carefully distinguish them from the G-patterns defined above). We denote by HfG the pattern of f.

In addition to these two actions we let H act on the G-patterns by defining $h(fG) = (hf)G$. This is indeed well-defined and indeed an action. Denote the orbit of the G-pattern fG under the action of H by $(Hf)G$. It is important to realize that *the orbits of H on the G-patterns coincide with the patterns that $G \oplus H$ induced on $F(D, R)$ by the action $(gh)(f) = hfg^{-1}$ described above.* [This follows from the law of associativity which composition of functions obeys:$(Hf)G = HfG = H(fG)$.]

6.18

Let w be a *weight function* from R into the set of polynomials with rational coefficients in indeterminates $y_l, y_2, y_3, \ldots$; denote by $w(r)$ the *weight* of color r. We allow w to induce a weight on the functions in $F(D, R)$ by defining

$$W(f) = \prod_{d \in D} w(f(d)). \tag{6.12}$$

[The weight function W is in fact well defined on G-patterns; for if f_1 and f_2 display the same G-pattern, then $f_2 = f_1 g^{-1}$, for some g in G, and $W(f_2) = W(f_l)$ follows from the fact that g is a bijection on D. Define therefore $W(fG) = W(f)$.]

We now fix a permutation h in H and ask: How many G-patterns remain invariant under h? In other words, how many G-patterns fG have the property that $hfG = fG$? (The condition $hfG = fG$ we occasionally express in its equivalent form $hf \in fG$.)

We devote Sections 6.19 through 6.21 to answering a slightly more general question. Specifically, *we propose to compute*

$$\sum_S W(S), \tag{6.13}$$

where the sum extends over all G-patterns S that are invariant under h. The weight function W is of the form described in (6.12).

6.19

Let G_f signify the subgroup $\{g \in G : fg^{-1} = f\}$, which we call the *stabilizer* of f in G. We can then write:

$$\begin{aligned}
\sum_{\substack{S \\ hS=S}} W(S) &= \sum_{\substack{f \\ hfG=fG}} \frac{W(f)}{|fG|} = \sum_{\substack{f \\ hfG=fG}} W(f)\frac{|G_f|}{|G|} \\
&= \frac{1}{|G|}\sum_{\substack{f \\ hfG=fG}} W(f)|G_f| = \frac{1}{|G|}\sum_{\substack{(g,f) \\ fg^{-1} \\ hfG=fG}} W(f) \\
&= \frac{1}{|G|}\sum_{g\in G}\sum_{\substack{f \\ fg^{-1} \\ hfG=fG}} W(f) = \frac{1}{|G|}\sum_{g\in G}\sum_{\substack{f \\ hfg^{-1}=f}} W(f).
\end{aligned}$$

Let us explain in some detail the chain of equalities written above. The first follows by recalling that the weight of the G-pattern fG equals that of any one of the $|fG|$ functions that it contains. Recalling that $|G_f||Gf| = |G|$ [i.e., that the length of an orbit equals the index of the stabilizer-see (6.1)] explains the second. The thud is clear. To understand the fourth and fifth equality signs look at the expression in the middle,

$$\frac{1}{|G|}\sum W(f),$$

where the sum is over the set of pairs

$$\{(g, f) : fg^{-1} = f \text{ and } hfG = fG\}.$$

First, by fixing f and summing over g one obtains $|G_f|$, and the sum over f explains the fourth sign of equality. By fixing g and summing over f first, one obtains

$$\sum_{\substack{f \\ fg^{-1} \\ hfG=fG}} W(f),$$

and then letting g run through G explains the fifth equality sign. Lastly, the last sign of equality is explained by observing that

$$fg^{-1} = f = hfg_0^{-1},$$

and that g runs over G if and only if g_0 does.

For a fixed permutation h of colors we therefore proved that

$$\sum_{\substack{S \\ h(S)=S}} = \frac{1}{|G|} \sum_{g \in G} \sum_{\substack{f \\ hfg^{-1}=f}} W(f). \tag{6.14}$$

6.20

Formula (6.14) suggests fixing g in G (in addition to h in H, which remains fixed) and then examining the weight $W(f)$ of a function f that satisfies $hfg^{-1} = f$. The weight function W has the form as written in (6.12). This is what we in fact do in this section.

Fix therefore g in G, and let $g = c_k \dots c_2\, c_1$ denote the decomposition of g into disjoint cycles in its action on the domain D. Write $|c_j|$ for the length of the jth cycle. Select an element in the jth cycle c_j of g; call it d_j. The list of the $|c_j|$ elements in the cycle c_j is

$$d_j,\ gd_j,\ g^2d_j,\ \dots,\ g^{|c_j|-1}d_j \tag{6.15}$$

[with g^2d_j an abbreviation for $g(g(d_j))$, and so on].

Assume now that f is fixed by gh and thus satisfies $f = (gh)(f) = hfg^{-1}$, or $fg = hf$. This iteratively implies $fg^s = fgg^{s-1} = hfg^{s-1} = hfgg^{s-2} = h^2fg^{s-2} = \cdots = h^sf$, for any power g^s of g.

Applying f to the list of elements in (6.15) we therefore obtain

$$fd_j,\ hfd_j,\ h^2fd_j,\ \ldots,\ h^{|c_j|-1}fd_j, \tag{6.16}$$

[with h^2fd_j signifying $h(h(f(d_j)))$, for notational simplicity]. [We wish to inform that, while the list in (6.15) consists of $|c_j|$ distinct elements, the list in (6.16) need not consist of $|c_j|$ distinct elements. In general it contains l distinct elements (say), where l is the length of the cycle generated by fd_j under the influence of the group element h. Observe in addition that l divides $|c_j|$, since $h^{|c_j|}fd_j = fg^{|c_j|}d_j = fd_j(= h^lfd_j)$.]

As with regard to the weight $W(f)$ of a function f that satisfies $fg = hf$, taking into account the information obtained so far, we may write

$$\begin{aligned} W(f) &= \prod_{d\in D} w(fd) = \prod_{j=1}^{k} w(fd_j)\, w(fgd_j)\cdots w(fg^{|c_j|-1}d_j) \\ &= \prod_{j=1}^{k} w(fd_j)\, w(hfd_j)\cdots w(h^{|c_j|-1}fd_j). \end{aligned}$$

Set

$$w(r)w(hr)w(h^2r)\cdots(h^{s-1}r) = p_s(r). \tag{6.17}$$

[The product $p_s(r)$ depends on h, of course, but we omit h as an index for notational simplicity.] The above expression for $W(f)$ takes now the simpler form

$$W(f) = \prod_{j=1}^{k} p_{|c_j|}(r_j), \tag{6.18}$$

where r_j stands for fd_j, $j = 1, 2, \ldots, k$.

6.21

As formula (6.14) demands, our next step is to compute $\sum_f W(f)$, where the sum is taken over all functions f that satisfy $(gh)(f) = f$. We still keep g and h fixed.

By selecting an element d_j from cycle c_j of g ($j = l, 2, \ldots, k$), and denoting $f(d_j)$ by r_j, we completely specify the function f on D. (For let d be an arbitrary element of the domain. Then d is in some cycle c_j of g and thus $d = g^s d_j$, for some power s of g. Consequently $fd = fg^s d_j = h^s f d_j = h^s r_j$.) There is one and only one restriction on the image r_j of d_j under f : If d_j belongs to the cycle c_j, then r_j should satisfy $h^{|c_j|} r_j = r_j$ (in other words r_j should belong to a cycle of h whose length divides that of c_j). This motivates defining

$$R_t = \{r \in R : h^t r = r\}.$$

(The set R_t depends on h, of course, but we omit h as an index for notational simplicity. Observe that the R_t's are *not disjoint* in general. They do cover, however, the whole range R, $t = 0, 1, 2 \ldots$.)

We intend to sum over all f that satisfy $(gh)(f) = f$. It will hence be convenient to fix elements d_1 in c_l, d_2 in $c_2, \ldots, d_k$ in c_k, and write r_j^f for $f(d_j)$, $j = 1, 2, \ldots, k$. The geometrically inclined may wish to think of r_j^f as the "projection" of f on the subset $R_{|c_j|}$ of R. We thus have a bijection

$$f \leftrightarrow (r_1^f, r_2^f, \ldots, r_k^f),$$

with r_j^f belonging to $R_{|c_j|}$.

Equipped conceptually in such a fashion, and letting f run over all functions that satisfy $(gh)(f) = f$, we now write

$$\begin{aligned} \sum_f W(f) &= \{\text{by (6.18)}\} = \sum_f \prod_{j=1}^{k} p_{|c_j|}(r_j^f) \\ &= [\text{obtain the left side by picking a term in each factor at right and summing}] \end{aligned}$$

$$
\begin{aligned}
&= \left(\sum_f p_{|c_1|}(r_1^f)\right)\left(\sum_f p_{|c_2|}(r_2^f)\right)\cdots\left(\sum_f p_{|c_k|}(r_k^f)\right)\\
&= \prod_{i=1}^{k}\left(\sum_f p_{|c_j|}(r_j^f)\right) = \{\text{by (6.17)}\}\\
&= \prod_{i=1}^{k}\sum_{r\in R_{|c_j|}} w(r)w(hr)\cdots w(h^{|c_j|-1}r)\\
&= [\text{upon denoting the above sum over } r \text{ in } R_{|c_j|} \text{ by } p_{|c_j|}(h)]\\
&= \prod_{i=1}^{k} p_{|c_j|}(h) = \{\text{sorting by the cycle length}\}\\
&= (p_1(h))^{\lambda_1(g)}\,(p_2(h))^{\lambda_2(g)}\cdots(p_m(h))^{\lambda_m(g)},
\end{aligned}
$$

where $\lambda_j(g)$ denotes the number of cycles of length j of the group element g, $j = 1, 2, \ldots, m$ (with $m = |D|$).

We proved that

$$\sum_{\substack{f\\ hfg^{-1}=f}} W(f) = (p_1(h))^{\lambda_1(g)}(p_2(h))^{\lambda_2(g)}\cdots(p_m(h))^{\lambda_m(g)}). \tag{6.19}$$

6.22

In view of what (6.19) displays, formula (6.14) becomes

$$
\begin{aligned}
\sum_{\substack{S\\ h(S)=S}} W(S) &= \frac{1}{|G|}\sum_{g\in G}\prod_{i=1}^{m}(p_i(h))^{\lambda_i(g)}\\
&= P_G(p_1(h), p_2(h), \ldots, p_m(h)),
\end{aligned}
$$

with P_G the cycle index of G in its action on D. We summarize:

DeBruijn's Result. *The sum of the weights of the G -patterns fixed by the permutation h of colors is*

$$\sum_{\substack{S\\ h(S)=S}} W(S) = P_G(p_1(h), p_2(h), \ldots, p_m(h)) \tag{6.20}$$

with $p_s(h) = \sum_{r,\, h^s r = r} w(r)w(hr)\cdots w(h^{s-1}r)$.

Observe that by taking $h = 1$ (the identity element of H) we have

$$p_s(1) = \sum_{r\in R} w(r)w(r)\cdots w(r) = \sum_{r\in R} w(r)^s.$$

Since the identity fixes all the G-patterns, the result (6.20) yields in this case the well-known result of PÓlya:

Pólya's Theorem

$$\sum_{\substack{T \\ (T\ G-pattern)}} W(T) = P_G\left(\sum_r w(r), \sum_r (w(r))^2, \ldots, \sum_r (w(r))^m\right).$$

6.23

Our next objective is to derive a formula for the weight $\sum W(T)$ of all the patterns T induced by the action $(gh)(f) = hfg^{-1}$ of $G \oplus H$ on $F(D, R)$. A first requirement is that the weight function W [defined on $F(D, R)$] should be well defined on patterns. Along these lines we remark that weight functions of the form (6.12) *are not well defined on patterns*, in general. [But there are weight functions of this form that are well defined. An important instance is obtained by taking $w(r) = 1$, for all r in R. Then the induced weight function W is well defined on patterns (specifically, $W(T) = 1$ for any pattern T). With this W the sum $\sum_T W(T)$ simply equals the total number of patterns.]

Assume in what follows that W is a weight function of the form (6.12) and that W *is well defined on patterns.*

In Section 6.17 we informed the reader that H acts on the G-patterns, and that the orbits induced by H coincide with the patterns of $G \oplus H$ on $F(D, R)$. At this time we take advantage of this point of view and thus *visualize the patterns as orbits of H in its*

action $h(fG) = (hf)G$ on the G-patterns. For notational simplicity let us denote by $\bar{\mathrm{f}}$ the G-pattern of f (normally written as fG). Then

$$\begin{aligned}\sum_{\substack{T\\(Tpattern)}} W(T) &= \sum_{\bar{f}} \frac{W(\bar{\mathrm{f}})}{|H\bar{f}|} = \sum_{\bar{\mathrm{f}}} W(\bar{\mathrm{f}}) \frac{|H_{\bar{\mathrm{f}}}|}{H} \\ &= \frac{1}{|H|} \sum_{\bar{\mathrm{f}}} W(\bar{\mathrm{f}}) |H_{\bar{\mathrm{f}}}| = \frac{1}{|H|} \sum_{(h,\bar{\mathrm{f}}) h\bar{\mathrm{f}}=\bar{\mathrm{f}}} W(\bar{\mathrm{f}}) = \{\text{by } 6.20\} \\ &= \frac{1}{|H|} \sum_{h\in H} P_G(p_1(h), p_2(h), \ldots, p_m(h)).\end{aligned}$$

In this chain of equalities $H\bar{f}$ denotes the orbit under H of the G-pattern $\bar{f}$, and $H_{\bar{f}}$ is the stabilizer of $\bar{f}$, that is, $H_{\bar{f}} = \{h \in H : h\bar{f} = \bar{f}\}$, a subgroup of H.

This result is attributed to deBruijn as well. Let us formally state:

DeBruijn's Result. *The sum of weights of the patterns is*

$$\sum_{\substack{T\\(T-pattern)}} W(T) = \frac{1}{|H|} \sum_{h\in H} P_G(p_l(h), P_2(h), \ldots, p_m(h)) \tag{6.21}$$

with $p_s(h) = \sum_{r,h^s r=r} w(r)w(hr)\cdots w(h^{s-l}r)$ (assuming that W is induced by w and that W is well defined on patterns.)

6 RECIPE FOR DEBRUIJN'S RESULT

6.24

We go back once again to Section 6.1 and'examine what information DeBruijn's results give in that simple situation. As the reader recalls, the problem is that of coloring (with colors A and B) the vertices of a square, labeled $\begin{smallmatrix}1&2\\3&4\end{smallmatrix}$.

In this example the group G (acting on the domain) consists of the four planar rotations of the square, and H is the group of order 2, which either fixes each of the two colors or

interchanges them. There are six G-patterns that emerge,

$$\begin{matrix} A & A \\ A & A \end{matrix}, \quad \begin{matrix} B & B \\ B & B \end{matrix}, \quad \begin{matrix} A & B \\ B & B \end{matrix}, \quad \begin{matrix} B & A \\ A & A \end{matrix}, \quad \begin{matrix} A & A \\ B & B \end{matrix}, \quad \begin{matrix} A & B \\ B & A \end{matrix}, \tag{6.22}$$

and four patterns,

$$\begin{matrix} A & A \\ A & A \end{matrix}, \quad \begin{matrix} A & B \\ B & B \end{matrix}, \quad \begin{matrix} A & A \\ B & B \end{matrix}, \quad \begin{matrix} A & B \\ B & A \end{matrix}. \tag{6.23}$$

We see, in particular, that among the six G-patterns two (the last two) are left invariant when colors A and B are interchanged.

It is possible to reach these conclusions by way of DeBruijn's results. The cycle index of the rotation group of the square is

$$P_G = \frac{1}{4}(x_1^4 + x_2^2 + 2x_4).$$

In this case the group permuting the colors is $H = \{(B)(A), (BA)\}$. For notational simplicity we write 1 (the identity) for $(B)(A)$ and denote (BA) by h, so that $H = \{1, h\}$.

We assign weight y_1 to A and weight y_2 to B, that is, $w(A) = y_1$ and $w(B) = y_2$. Then

$$\begin{aligned}
p_1(1) &= \sum_{\substack{r \\ 1r=r}} w(r) = w(A) + w(B) = y_1 + y_2, \\
p_2(1) &= \sum_{\substack{r \\ 1^2r=r}} w(r)w(1r) = w(A)^2 + w(B)^2 = y_1^2 + y_2^2, \\
p_3(1) &= \sum_{\substack{r \\ 1^3r=r}} w(r)w(1r)w(1^2r) = w(A)^3 + w(B)^3 = y_1^3 + y_2^3, \\
p_4(1) &= \sum_{\substack{r \\ 1^4r=r}} w(r)w(1r)w(1^2r)w(1^3r) = w(A)^4 + w(B)^4 = y_1^4 + y_2^4,
\end{aligned}$$

and

$$p_1(1) = \sum_{\substack{r \\ hr=r}} w(r) = 0 \qquad \text{(since } h \text{ fixes neither } A \text{ nor } B\text{)}$$

$$\begin{aligned} p_2(h) &= \sum_{\substack{r \\ h^2 r = r}} w(r)w(hr) = w(A)w(hA) + w(B)w(hB) \\ &= w(A)w(B) + w(B)w(A) = 2y_1y_2 \end{aligned}$$

(since $h(A) = B$ and $h(B) = A$)

$$p_3(h) = \sum_{\substack{r \\ h^3 r = r}} w(r)w(hr)w(h^2r) = 0$$

(since $h^3 = h$ and it does not fix anything)

$$\begin{aligned} p_4(h) &= \sum_{\substack{r \\ h^4 r = r}} w(r)w(hr)w(h^2r)w(h^3r) \\ &= w(A)w(B)w(A)w(B) + w(B)w(A)w(B)w(A) \\ &= 2y_1^2y_2^2. \end{aligned}$$

A G-pattern consisting of a_1 $A's$ and a_2 $B's$ ($a_1 + a_2 = 4$) has weight $y_1^{a_1}y_2^{a_2}$. Formula (6.20) informs us, therefore, that the number of G-patterns consisting of a_1 $A's$ and a_2 $B's$ that are invariant under h (the permutation that, interchanges colors A and B) equals the coefficient of $y_1^{a_1}y_2^{a_2}$ in $P_G(p_1(h), \ldots, p_4(h))$. In particular, the number of G-patterns with two $A's$ and two $B's$ invariant under h equals the coefficient of $y_1^2y_2^2$ in

$$\begin{aligned} P_G(p_1(h), \ldots, p_4(h)) &= P_G(0, 2y_1y_2, 0, 2y_1^2y_2^2) \\ &= \frac{1}{4}(0^4 + (2y_1y_2)^2 + 2(2y_1^2y_2^2)) = 2y_1^2y_2^2. \end{aligned}$$

The answer is 2, in agreement with what (6.22) displays.

Formula (6.20) tells us also that the total number of G-patterns invariant under h can be obtained by setting $w(A) = y_1 = 1$, $w(B) = y_2 = 1$ and then computing $P_G(p_1(h), \ldots, p_4(h))$. We obtain

$$\begin{aligned} P_G(p_1(h), \ldots, p_4(h)) &= P_G(0, 2y_1y_2, 0, 2y_1^2y_2^2) \\ &= 2y_1^2y_2^2 = 2 \cdot 1^2 \cdot 1^2 = 2, \end{aligned}$$

which compares favorably to display (6.22).

Finally, according to (6.21), the total number of patterns is obtained by taking $w(A) = y_1 = 1$, $w(B) = y_2 = 1$ and then computing

$$\frac{1}{|H|}\sum_{h\in H} P_G(p_1(h),\ldots,p_m(h)).$$

(Observe that the weight function involved is well defined on patterns.)

In this case we obtain

$$\begin{aligned}
&\frac{1}{2}P_G(p_1(1),\ldots,p_4(1)) + \frac{1}{2}P_G(p_1(h),\ldots,p_4(h)) \\
&\quad = \frac{1}{2}P_G(2,2,2,2) + \frac{1}{2}P_G(0,2,0,2) \\
&\quad = \frac{1}{2}\cdot\frac{1}{4}(2^4+2^2+2\cdot 2) + \frac{1}{2}\cdot\frac{1}{4}(0^4+2^2+2\cdot 2) = 4,
\end{aligned}$$

consistent with (6.23).

6.25 The Recipe

We have before us a domain D of m elements, a range R of colors $r_l, r_2, \ldots, r_n$ and two groups, G and H. The group G acts on D, and H acts on R. A function f from D to R is called a coloration of D [it colors element d of D with color $f(d)$]. The set of all colorations is denoted by $F(D,R)$.

We let G act on $F(D,R)$ by defining $g(f) = fg^{-1}$, with g in G and f in $F(D,R)$; the orbits of this action are called *G-patterns.*

We let $G \oplus H$ act on $F(D,R)$ by defining $(gh)(f) = hfg^{-1}$; with g in G, h in H, and f in $F(D,R)$. The orbits of this action are called *patterns.* (A pattern is in fact a union of G-patterns, as explained in Section 6.17.)

Our interest lies with investigating the patterns and G-patterns. A fair number of questions commonly asked can be answered by following the steps outlined below. [It is not unlikely that answers to many others lie with clever choices of weight functions. Whenever the recipe falls short of the readers' computational objectives we suggest (as a first recourse) a readjustment of the weight function.]

Step 1. Find the cycle index P_G of the group G in its action on the domain D.

[Recall that

$$P_G(x_l, \ldots, x_m) = \frac{1}{|G|} \sum_{g \in G} x_1^{\lambda_1(g)} x_2^{\lambda_2(g)} \cdots x_m^{\lambda_m(g)},$$

where $\lambda_j(g)$ denotes the number of cycles of length j of the group element g in its representation on D.]

Step 2. Assign weight $w(r_i)$ to color r_i $(i = 1, 2, \ldots, n)$, and denote (when convenient) $w(r_i)$ by y_i, with y_i an indeterminate.

Step 3. For a permutation of interest h in H, and each s $(s = 1, 2, \ldots, m)$, find the set $S(h, s) = \{r \in R : h^s r = r\}$. Compute

$$p_s(h) = \sum_{\substack{r \\ r \in S(h,s)}} w(r)w(hr)w(h^2r) \cdots w(h^{s-1}r).$$

If $S(h, s)$ is the empty set, then define $p_s(h)$ to be 0.

Step 4. The number of G-patterns with

a_1 *occurrences of color* r_l

a_2 *occurrences of color* r_2

$\vdots$

a_n *occurrences of color* r_n $(\sum_{i=1}^{n} a_i = m)$

that are invariant under a permutation h of colors equals the coefficient of $y_1^{a_1}y_2^{a_2}\cdots y_n^{a_n}$ in the polynomial

$$P_G = (p_1(h), p_2(h), \ldots, p_m(h)),$$

where the indeterminate y_i stands for $w(r_i)$, $i = 1, 2, \ldots, n$.

Step 5. The total number of G-patterns that are invariant under a permutation h of colors is obtained by setting $w(r_i) = y_i = 1$ (for all i) in the polynomial

$$P_G = (p_1(h), p_2(h), \ldots, p_m(h)).$$

Step 6. The total number of patterns is obtained by setting $w(r_i) = y_i = 1$ (for all i) in the polynomial

$$\frac{1}{|H|}\sum_{h\in H} P_G = (p_1(h), p_2(h), \ldots, p_m(h)).$$

[This step requires computation of $p_s(h)$, for all h in H - often quite a lengthy affair.]

End of recipe.

7 EXAMPLES FOLLOWING THE RECIPE

6.26 Coloring the Faces of a Cube

We have six colors $r_1, r_2, \ldots, r_6$ and take interest in coloring the six faces of a cube. Two colorations are equivalent if they differ by a rotation of the cube. We take special interest in the colorations that remain invariant under the permutation $h = (r_6r_5)(r_4r_3)(r_2r_1)$ of colors.

Step 1. The cycle index of the rotation group G of the cube in its action on the six faces is [cf. Section 6.13]

$$P_G = \frac{1}{24}(x_1^6 + 3x_1^2x_2^2 + 6x_1^2x_4 + 6x_2^3 + 8x_3^2).$$

Step 2. Let $w(r_i) = y_i,\ i = 1, 2, \ldots, 6$.

Step 3. We take interest in $h = (r_6 r_5)(r_4 r_3)(r_2 r_l)$. For this permutation of colors we have

$$\begin{aligned} p_1(h) &= 0 \\ p_2(h) &= 2y_1y_2 + 2y_3y_4 + 2y_5y_6 \\ p_3(h) &= 0 \\ p_4(h) &= 2y_1^2y_2^2 + 2y_3^2y_4^2 + 2y_5^2y_6^2 \\ p_5(h) &= 0 \\ p_6(h) &= 2y_1^3y_2^3 + 2y_3^3y_4^3 + 2y_5^3y_6^3. \end{aligned}$$

Step 4. The number of G-patterns with

Two occurrences of color r_1

Two occurrences of color r_2

One occurrence of color r_5

One occurrence of color r_6

that are invariant with respect to h is the coefficient of $y_1^2y_2^2y_5y_6$ in the polynomial

$$\begin{aligned} P_G(p_1(h), \ldots, p_6(h)) &= \frac{1}{24}\, 6(p_2(h))^3 \\ &= \frac{1}{4}(2y_1y_2 + 2y_3y_4 + 2y_5y_6)^3. \end{aligned}$$

This coefficient is $\frac{1}{4}(3!/2!1!)2^2 \cdot 2 = 6$, allowing us to conclude that there are six such colorations.

Step 5. The total number of colorations invariant under h is $P_G(p_1(h), \ldots, p_6(h))$, when $w(r_i) = y_i = 1$, for all i. We obtain

$$P_G(p_1(h), \ldots, p_6(h)) = \frac{1}{24}\, 6(p_2(h))^3$$

$$= \frac{1}{4}(2+2+2)^3 = 2 \cdot 27 = 54$$

invariant colorations in all.

6.27 Enumerating Nonisomorphic Graphs- Redfield and Pólya

The domain D consists of all the $\binom{v}{2}$ subsets of size 2 of a set with v elements. (For convenience we think of the $\binom{v}{2}$ subsets of size two as the edges of the complete graph K_v on v vertices. We take two colors, red (r) and blue (b), and proceed to color the edges of K_v. The range is thus $R = \{r, b\}$.

The group that acts on the domain is the symmetric group on the v vertices S_v in *its natural permutation representation on the edges of* K_v. (Note that S_v is not the symmetric group of D but a much smaller group.)

On the range the group that acts is $H = \{(b)(r), (br)\}$, the group of order 2 that eventually interchanges the two colors.

An S_v-pattern is an isomorphism class of (simple) graphs.

Step 1. We need to find the cycle index of S_v in its representation on the edges of K_v. To accomplish this we take an arbitrary permutation σ of type $x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_v^{\lambda_v}$ on the v vertices and proceed to calculate the cycle type that it induces on the $\binom{v}{2}$ edges of K_v. Select therefore an edge $\{i, j\}$ and investigate what happens to it under the repeated actions of the permutation σ. There are in fact only two cases to consider: (a) the contributions to the cycle index from edges $\{i, j\}$, with i and j in the same cycle of σ, and (b) contributions from edges $\{i, j\}$ with i and j in different cycles.

In case (a) suppose i and j belong to a cycle α of length k. We seek to understand what happens to this whole cycle under the action of σ. If k is odd, say $k = 2n + 1$,

then the representation of α on edges of K_v becomes a product of n cycles each of length $2n+1$. (The reader may wish to quickly check this for $k = 5$, say.) When $k = 2n$ the cycle a decomposes into $n-1$ cycles of length $2n$ and one cycle of length n on the edges of K_v. [If, e.g., $\alpha = (1\,2\,3\,4\,5\,6)$, then its representation on edges takes the form $(1\,2, 2\,3, 3\,4, 4\,5, 5\,6, 6\,1)(1\,3, 2\,4, 3\,5, 4\,6, 5\,1\,6\,2)(1\,4, 2\,5, 3\,6)$; we abbreviated $\{i, j\}$ by ij.] In summary

$$x_{2n+1} \to x_{2n+1}^n,$$

and

$$x_{2n} \to x_n x_{2n}^{n-1},$$

where the symbols left of the arrow describe the cycle structure on vertices and the symbols at the right describe the cycle structure on edges. Thus if there are λ_k cycles of length k in σ, the pairs of points lying in common cycles contribute as follows:

$$x_{2n+1}^{\lambda_{2n+1}} \to x_{2n+1}^{n\lambda_{2n+1}},$$

and

$$x_{2n}^{\lambda_{2n}} \to x_n^{\lambda_{2n}} x_{2n}^{(n-1)\lambda_{2n}},$$

for odd and even cycle lengths, respectively.

Let us address the situation in case (b), for edges $\{i, j\}$ with i and j in different cycles of σ. Consider two cycles of σ, α and β, of lengths r and s, respectively. On the set of edges $\{i, j\}$ with $i \in \alpha$ and $j \in \beta$, the cycles α and β induce exactly $(r; s)$ cycles of length $[r; s]$, where $(r; s)$ denotes the greatest common divisor and $[r; s]$ the least common multiple of r and s. In particular, when $r = s = k$, they contribute k cycles of length k.

[We suggest the reader examine the cycles $\alpha\beta = (1\,2\,3\,4\,5, 6)(7\,8\,9\,10)$. They induce two cycles of length 12 on edges ij, with $i \in \alpha$ and $j \in \beta$:

$$(1\,7, 2\,8, 3\,9, 4\,10, 5\,7, 6\,8, 1\,9, 2\,10, 3\,7, 4\,8, 5\,9, 6\,10)$$

$$(1\,8, 2\,9, 3\,10, 4\,7, 5\,8, 6\,9, 1\,10, 2\,7, 3\,8, 4\,9, 5\,10, 6\,7).]$$

Thus the contributions from cycles $\alpha\beta$ with $r \neq s$ is

$$x_r^{\lambda_r} x_s^{\lambda_s} \to x_{[r;s]}^{\lambda_r \lambda_s (r;s)},$$

and from cycles with $r = s = k$ these contributions amount to

$$x_k^{\lambda_k} \to x_k^{k\binom{\lambda_k}{2}}.$$

The cycle index of S_v, on the edges of K_v can now be written out. It is

$$\begin{aligned} P_{S_v}(x_1, x_2, \ldots, x_m) \;=\; & \frac{1}{v!} \sum_{\lambda} \frac{v!}{\prod_{k=1}^{v} (\lambda_k! k^{\lambda_k})} \prod_{n=0}^{[(v-1)/2]} x_{2n+1}^{n\lambda_{2n+1}} \\ & \cdot \prod_{n=1}^{[v/2]} x_n^{\lambda_{2n}} x_{2n}^{(n-1)\lambda_{2n}} \cdot \prod_{1 \leq r \leq s \leq p-1} x_{[r;s]}^{\lambda_r \lambda_s (r;s)} \cdot \prod_{k=1}^{[v/2]} x_k^{k\binom{\lambda_k}{2}}. \end{aligned}$$

Here the sum is over all vectors $\lambda = (\lambda_1, \ldots, \lambda_v)$ that satisfy $\sum_{i=1}^{v} i\lambda_i = v$, $[t]$ denotes the integral part of the fraction t, and m stands for $\binom{v}{2}$.

For example, $P_{S_4}(x_1, \ldots, x_6) = \frac{1}{24}(x_1^6 + 8x_3^2 + 9x_1^2 x_2^2 + 6x_2 x_4)$.

Step 2. Let $w(r) = y_1$ and $w(b) = y_2$.

Step 3. Suppose $h = (b)(r)$, that is, h is the identity. Then $S(h, s) = R$, for all $s = 1, 2, \ldots, m$.Consequently,

$$p_s(h) = w(r)^s + w(b)^s = y_1^s + y_2^s.$$

And if $h = (br)$,then $h^3 = h^5 = h^7 = \cdots = h$ and $h^0 = h^2 = h^4 = \cdots = 1$ (the identity in H). Thus

$$S(h, s) = \phi \qquad \text{if } s \text{ is odd}$$

and

$$S(h, s) = R \qquad \text{if } s \text{ is even.}$$

In this case $p_s(h) = 0$, for s odd, and $p_s(h) = w(r)w(hr)\cdots w(h^{s-1}r)+w(b)w(hb)\cdots w(h^{s-1}b) = y_1y_2y_1y_2\cdots + y_2y_1y_2y_1\cdots = 2y_1^{(s-1)/2}y_2^{(s-1)/2}$, for s even.

Step 4. The number of isomorphism classes of graphs with a_1 red edges and a_2 blue edges ($a_1 + a_2 = \binom{v}{2} = m$) is the coefficient of $y_1^{a_1}y_2^{a_2}$ in

$$P_{S_v}(y_1 + y_2, y_1^2 + y_2^2, \cdots, y_1^m + y_2^m).$$

The number of isomorphism classes of graphs with a_1 red edges, a_2 blue edges ($a_1 + a_2 = \binom{v}{2} = m$) isomorphic to their complements [which is what invariance with respect to $h = (br)$ means] equals the coefficient of $y_l^{a_1}y_2^{a_2}$ in

$$P_{S_v}(0,\ 2y_1y_2,\ 0,\ 2y_1^2y_2^2,\ 0,\ 2y_1^3y_2^3,\ \ldots).$$

Step 5. The total number of (isomorphism classes of) graphs obtained by setting $y_1 = y_2 = 1$ in Step 3 is

$$P_{S_v}(2, 2, \ldots, 2).$$

The total number of (isomorphism classes of) graphs isomorphic to their complements equals (again by setting $y_1 = y_2 = 1$)

$$P_{S_v}(0, 2, 0, 2, 0, 2, \ldots).$$

Step 6. The total number of patterns equals

$$\frac{1}{2}P_{S_v}(2,2,\ldots,2)+\frac{1}{2}P_{S_v}(0,2,0,2,0,2\ldots).$$

(Describe a pattern in graph theoretical language.)

EXERCISES

1. How many necklaces with eight beads of (at most) four colors are left invariant by an interchange of two colors? By a cyclic permutation of the four colors?

2. Count the number of patterns of injective functions between a domain D of m elements and a (specified a priori) range R of n colors.

3. A directed graph (digraph, for short) is a set of ordered pairs, called edges, with entries from a finite set of vertices. Two digraphs are isomorphic if one becomes the other upon a permutation of vertices.

   (a) Find the number of (isomorphism classes of) digraphs on v vertices.

   (b) Find the number of (isomorphism classes of) digraphs with e edges and v vertices.

   (c) How many (isomorphism classes of) digraphs on v vertices are isomorphic to their complements?

   (d) How many (isomorphism classes of) digraphs with e edges and v vertices are isomorphic to their complements?

   (e) Find the number of patterns of digraphs on v vertices.

4. Suppose the edges of the complete graph are colored with n colors. Go through the recipe for DeBruijn's result in this case.

5. Examine what happens if the domain D consists of one element only and the range R has n colors. What are the patterns if the group acting on R is the full symmetric group $\mathrm{Sym}(R)$?

6. How many nonisomorphic classes of spanning trees does the complete graph K_v have? Find at least a recurrence (see Pólya [3]).

7. Solve "le problème des ménages" by making use of Pólya's theorem.

8. Let the group G act transitively on the set Ω (this means that G has one orbit only on Ω). Let x be an element of Ω and G_x the stabilizer of x. Show that the number of orbits of G_x on Ω equals

$$\frac{1}{|G|}\sum_{g\in G}|F(g)|^2,$$

where $F(g)$ is the set of points in Ω fixed by the group element g.

9. Let group G act on set X; $|X| = m$. Define an equivalence relation on ordered pairs of disjoint subsets of $X(S_1, S_2)$ (with S_1 of cardinality r and S_2 of cardinality s) by writing

$$(S_1, S_2) \sim (S_3, S_4)$$

if $S_2 = g(S_1)$ and $S_4 = g(S_3)$ for some group element g of G. Show that the number of equivalence classes that result equals the coefficient of $x^r y^s$ obtained upon substituting $1 + x^k + y^k$ for x_k in the cycle index $P_G(x_1, \ldots, x_m)$ of G in its action on X.

10. Denote by $P_{S_v}(x_1, \ldots, x_m)$ the cycle index of the symmetric group S_v (of order $v!$) in its action on the $m = \binom{v}{2}$ subsets of size 2 of the set $\{1, 2, \ldots, v\}$. Prove that the number of graphs (with multiple edges allowed) on v vertices and r edges is the coefficient of x^r in the polynomial

$$P_{S_v}\left((1-x)^{-1}, (1-x^2)^{-1}, \ldots, (1-x^m)^{-1}\right), \qquad m = \binom{v}{2}.$$

NOTES

What we call Burnside's lemma is said to have been known to Frobenius. At any rate, it had been rediscovered by Burnside [1]. Redfield [2] introduced the notion of the group reduction function, which is what we now call the cycle index. His paper anticipates and implicitly contains Pólya's theorem. In his extensive work [3] Pólya explicitly states, demonstrates, and applies his theorem to a host of enumeration problems in chemistry, graph theory, and other fields. The same paper contains asymptotic results in enumeration, such as the asymptotic behavior of the number of isomorphism classes of spanning trees in complete graphs. Noteworthy extensions of Pólya's theorem were obtained by DeBruijn [4]. He introduces equivalence induced by a group on the range of the functions. Harary [5] draws attention to the great enumerative powers of these results (especially to the theory of graphs) by exploiting the notions of power group and exponentiation. Our presentation was influenced significantly by the accounts on this topic given in [4] and [6].

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