# Faculty Research Interest Seminar
# Department of Biostatistics, GSPH
# University of Pittsburgh

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# Introduction

- Joined the department in 2001.
- Teach two courses:
  - *Elements of Stochastic Processes (Biostat 2040).*
  - *Analysis of Incomplete Data (Biostat 2065)*.
- Some research interests:
  - *Statistical Analysis of Missing Data.*
  - *Analysis of Correlated Outcomes.*
  - *Semi-Parametric Statistics.*
- Statistician of the Biostatistical Center of NSABP (www.nsabp.pitt.edu).
  - Work on cancer clinical trials.
  - Recently involved in studies to investigate the association between genomic data and clinical outcomes (with Drs. John Bryant and Joe Costantino).

# An Interesting Phenomenon

- Pseudolikelihood method
  - Likelihood function of $\theta$: $L(\theta;\alpha) = L(\theta \mid \alpha)$.
  - $\theta$: parameter of interest; $\alpha$: nuissance parameter.
  - $\hat{\alpha}$ is a consistent estimator of $\alpha$; $\alpha_0$ is the true value.
  - Consider $\hat{\theta} = \arg\max_\theta L(\theta;\alpha_0)$ and

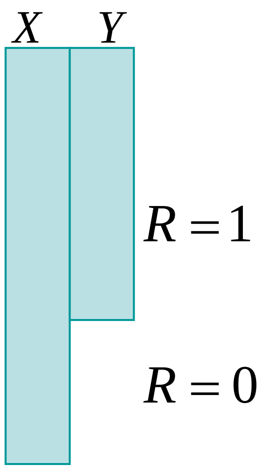
    $\tilde{\theta} = \arg\max_\theta L(\theta;\hat{\alpha})$ <pseudolikelihood estimate>.
- In some circumstances, $\mathrm{var}(\tilde{\theta}) < \mathrm{var}(\hat{\theta})$.
- Demonstrate through a missing-data problem.
- Question: how to improve the efficiency when $\alpha_0$ is actually known, from other source?

# Bivariate data with outcome-dependent nonresponse

- Model assumptions:

  $(a)\ X \sim f(x;\alpha),\ \ [Y \mid X] \sim g(y \mid x, \theta).$

  $(b)\ \Pr[\,R=1 \mid X, Y\,] = w(y;\psi).$




- A conditional likelihood method

$$L(\theta;F) = \prod_{i=1}^{m} p(x_i \mid y_i;\theta,F) \qquad (\text{for } x_i \perp R_i,\ \text{given } y_i)$$

$$\propto \prod_{i=1}^{m} \frac{g(y_i \mid x_i,\theta)}{\int g(y_i \mid x,\theta)\,dF(x)}, \qquad m = \#\text{ of c.c.}$$

where $F(x) = F(x;\alpha_0)$ represents the *true* CDF of $X$.

– $\hat{\theta} = \arg\ \max_{\theta} L(\theta;F)$

– $\tilde{\theta} = \arg\max_{\theta} L(\theta;F(x;\hat{\alpha}))$, where $\hat{\alpha} = \arg\max_{\alpha} \prod_{i=1}^{n} f(x;\alpha).$

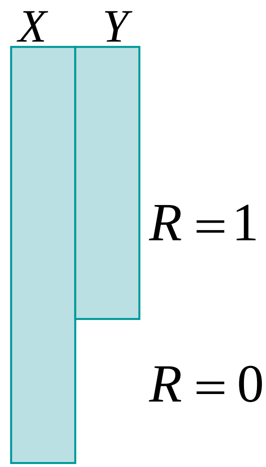
# Full likelihood for 2-pattern data

- Model assumptions:

  $(a)\ X \sim f(x|\alpha),\ \ [Y|X] \sim g(y|x,\theta).$

  $(b)\ \Pr[\ R=1|\ X,Y\ ] = w(y|\psi).$

- Full likelihood:




$$L_{Full}(\alpha,\theta,\psi) = \prod_{i=1}^{m} f(x_i|\alpha)\ g(y_i|x_i,\theta)\ w(y_i|\psi)$$

$$\bullet \prod_{\mathrm{i=m+1}}^{\mathrm{n}} f(x_i|\alpha) \int g(y|x_i,\theta)\{1-w(y|\psi)\}dy$$

$$= \{\prod_{i=1}^{n} f(x_i|\alpha)\ \} \bullet [\{\prod_{\mathrm{i=1}}^{\mathrm{m}} g(y_i|x_i,\theta) w(y_i|\psi)\} \prod_{\mathrm{i=m+1}}^{\mathrm{n}} \int g(y|x_i,\theta)\{1-w(y|\psi)\}dy]$$

$$= L(\alpha) L(\theta,\psi)$$

# Asymptotic Variance of $\hat{\theta}$

- Let $l(\theta) = \log\ L(\theta;\alpha)$, from $\hat{\theta} = \arg\ \max_\theta L(\theta;\alpha_0)$,

$$0 = l_\theta(\hat{\theta};\alpha_0) \approx l_\theta(\theta_0;\alpha_0) + l_{\theta\theta}(\theta_0;\alpha_0)(\hat{\theta} - \theta_0)$$

then

$$\sqrt{\text{n}}(\hat{\theta} - \theta_0) \approx -\{\frac{l_{\theta\theta}(\theta_0;\alpha_0)}{n}\}^{-1}\sqrt{\text{n}}\,\frac{l_\theta(\theta_0;\alpha_0)}{n}$$

$$\approx -E(l_{\theta\theta,1})^{-1}\sqrt{\text{n}}\,\frac{1}{n}\sum_{i=1}^{n} l_{\theta,i}(\theta_0;\alpha_0).$$

Therefore,

$$Var(\sqrt{n}(\hat{\theta} - \theta_0)) \approx E(l_{\theta\theta,1})^{-1} Var(l_{\theta,1}(\theta_0;\alpha_0)) E(l_{\theta\theta,1})^{-1}$$

$$= E(l_{\theta\theta,1})^{-1} E(l_{\theta,1} l_{\theta,1}{}^T) E(l_{\theta\theta,1})^{-1},$$

(A sandwich-type estimator).

# Asymptotic Variance of $\tilde{\theta}$ (I)

- Let $S(\alpha) = \log f(x;\alpha)$, $\hat{\alpha} = \arg\max_{\alpha} \prod_{i=1}^{n} f(x_i;\alpha)$ is MLE of $\alpha$,

$$\sqrt{\text{n}}(\hat{\alpha} - \alpha_0) \approx -E(S_{\alpha\alpha})^{-1}\sqrt{n}\ \frac{1}{\text{n}}\sum_{i=1}^{n} S_{\alpha,i} \ \rightarrow N(0, -E(S_{\alpha\alpha})^{-1}).$$

- From $\tilde{\theta} = \arg\max_{\theta} L(\theta;\hat{\alpha}) = \arg\max_{\theta} l(\theta;\hat{\alpha})$,

$$0 = l_{\theta}(\tilde{\theta};\hat{\alpha}) \approx l_{\theta}(\theta_0;\hat{\alpha}) + l_{\theta\theta}(\theta_0;\hat{\alpha})(\tilde{\theta} - \theta_0),$$

then

$$\sqrt{\text{n}}(\tilde{\theta} - \theta_0) \approx -\{\frac{l_{\theta\theta}(\theta_0;\hat{\alpha})}{n}\}^{-1}\sqrt{\text{n}}\,\frac{l_{\theta}(\theta_0;\hat{\alpha})}{n}$$

$$\approx -E(l_{\theta\theta,1})^{-1}\sqrt{\text{n}}\,\frac{l_{\theta}(\theta_0;\alpha_0) + l_{\theta\alpha}\ (\hat{\alpha} - \alpha_0)}{n}$$

$$\approx -E(l_{\theta\theta,1})^{-1}\sqrt{\text{n}}\,\frac{1}{n}\sum_{i=1}^{n}\{l_{\theta,i} - E(l_{\theta\alpha,1})E(S_{\alpha\alpha})^{-1}S_{\alpha,i}\}.$$

# Asymptotic Variance of $\tilde{\theta}$ (II)

- From $\sqrt{\mathrm{n}}(\tilde{\theta}-\theta_0) \approx -E(l_{\theta\theta,1})^{-1}\sqrt{\mathrm{n}}\frac{1}{n}\sum_{i=1}^{n}\{l_{\theta,i}-E(l_{\theta\alpha,1})E(S_{\alpha\alpha})^{-1}S_{\alpha,i}\}$,

$$Var(\sqrt{n}(\tilde{\theta}-\theta_0)) \approx E(l_{\theta\theta,1})^{-1}\{E(l_{\theta,1}l_{\theta,1}{}^T) - E(l_{\theta,1}S_{\alpha,1}{}^T)E(S_{\alpha\alpha})^{-1}E(l_{\theta\alpha,1})^T$$
$$-E(l_{\theta\alpha,1})E(S_{\alpha\alpha})^{-1}E(S_{\alpha,1}l_{\theta,1}{}^T) - E(l_{\theta\alpha,1})E(S_{\alpha\alpha})^{-1}E(l_{\theta\alpha,1})^T\}E(l_{\theta\theta,1})^{-1}.$$

- On the other hand,

$$0 = E(l_{\theta,1}) = \int l_{\theta,1}(\alpha,\theta)\, f(x_1;\alpha)g(y_1 \mid x_1;\theta)\,p(r_1 \mid x_1,y_1;\psi)\,dx_1dy_1dr_1$$
$$= \int l_{\theta,1}(\alpha,\theta)L(\alpha)L(\theta,\psi)\; d\mu$$

So, $0=\frac{\partial}{\partial\alpha}E(l_{\theta,1})^T = E(l_{\alpha\theta,1}) + \int S_{\alpha,1}l_{\theta,1}{}^T L(\alpha)L(\theta,\psi)\; d\mu$

$$=E(l_{\alpha\theta,1}) + E(S_{\alpha,1}l_{\theta,1}{}^T) = E(l_{\theta\alpha,1})^T + E(S_{\alpha,1}l_{\theta,1}{}^T).$$

- Therefore,

$$Var(\sqrt{n}(\tilde{\theta}-\theta_0)) \approx E(l_{\theta\theta,1})^{-1}\{E(l_{\theta,1}l_{\theta,1}{}^T) - E(l_{\theta\alpha,1})E(-S_{\alpha\alpha})^{-1}E(l_{\theta\alpha,1})^T\}E(l_{\theta\theta,1})^{-1}$$

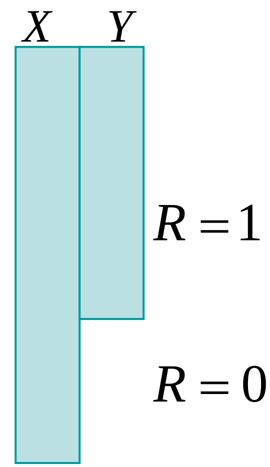
# Another Pseudolikelihood Estimator

- When the functional form of $F(x)$ is unknown, let

$$\breve{\theta} = \arg\ \max_{\theta} L(\theta; F_n)$$

$$= \arg\ \max_{\theta} \prod_{i=1}^{m} \frac{g(y_i \mid x_i, \theta)}{\int g(y_i \mid x, \theta)\, dF_n(x)},$$

$$= \arg\ \max_{\theta} \prod_{i=1}^{m} \frac{g(y_i \mid x_i, \theta)}{\frac{1}{n}\sum_{j=1}^{n} g(y_i \mid x_j, \theta)}, \qquad \text{(PL2)}$$




where, $F_n(x) = \frac{1}{n}\sum_{i=1}^{n} I(x_i \le x)$ is the empirical distribution of $X$.

- Simulation studies suggest that $\breve{\theta}$ is even more efficient even though it has no assumption on $F(x)$.

# Auxiliary Information

- If $F(x) = F(x;\alpha_0)$ is known, for example, in some survey studies, can we get more efficient estimator?
- Answer: yes, with empirical likelihood.

# Empirical Likelihood

- "Empirical likelihood is a nonparametric method of inference based on a data-driven likelihood ratio function" (Art Owen).
- Suppose $\{x_i\}_{i=1}^{n}$ is a random sample of $X \sim F(x)$, then the non-parametric estimator for $F(x)$ is $F_n(x) = \dfrac{1}{n}\sum_{i=1}^{n} I\{x_i \le x\}$, or $\prod_{i=1}^{n} p_i$ is maximized subject to $\sum_{i=1}^{n} p_i = 1$, where $p_i = pr\{X = x_i\}$.
- With auxiliary information $E(w(x)) = 0$ available, the empirical likelihood estimator maximizes $\prod_{i=1}^{n} p_i$ subject to constraints

$$\sum_{i=1}^{n} p_i = 1 \text{ and } \sum_{i=1}^{n} p_i w(x_i) = 1.$$

# Incorporate Auxiliary Information

- For example, $E(X) = \mu_0$ is known.
- The PL2 estimator solves the estimating equations

$$0 = l_\theta(\theta; F_n) = \sum_{i=1}^{n} l_{\theta,i}(\theta; F_n).$$

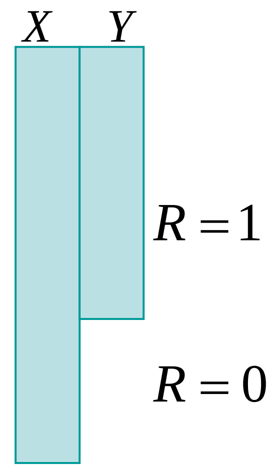
- New estimator:

(i) Maximizes $\prod_{i=1}^{n} p_i$ subject to constraints

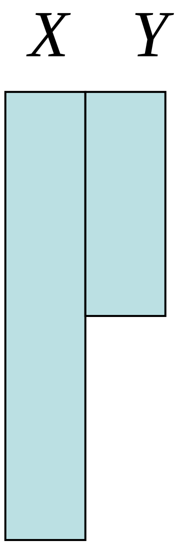
$$\sum_{i=1}^{n} p_i = 1 \text{ and } \sum_{i=1}^{n} p_i(x_i - \mu_0) = 1. \text{ Get } \{\hat{\mathrm{p}}_{\mathrm{i}}\}_{\mathrm{i}=1}^{\mathrm{n}}.$$

(ii) Let $\bar{\theta}$ solves $0 = \sum_{i=1}^{n} \hat{\mathrm{p}}_{\mathrm{i}} l_{\theta,i}(\theta; F_n)$, then

$$Var(\bar{\theta}) = (El_{\theta\theta})^{-1}\{Var(l_\theta(\theta; F_n)) - E(l_\theta x)Var(x)^{-1}E(xl_\theta)\}(El_{\theta\theta})^{-1}.$$



# A simulation study




- Complete data:

  (1) $[x] \sim N(0,1)$,

  (2) $[y \mid x] \sim N(1+x,1)$.

- Missing-data mechanism:

$$pr[R=1 \mid x, y] = \Phi(y-1).$$

- Compare the performance of the PL and EL estimates for the regression model (2), where $\theta = (\beta_0, \beta_1, \sigma^2) = (1,1,1)$.

Note: sample size n=300, average # of c.c.=150.

# Simulation results

**Table**: empirical biases and standard deviations of two estimators over 1000 replicates

| Methods | $\beta_0$ | $\beta_1$ | $\sigma^2$ |
|---|---|---|---|
| PL | 0.012 (0.133) | 0.019 (0.115) | 0.015 (0.197) |
| EL | 0.012 (0.129) | 0.017 (0.11) | 0.009 (0.186) |

# Reference


- Tang, Little and Raghunathan. Analysis of multivariate missing data with nonignorable nonresponse. *Biometrika*, vol. 90, pp. 747-764, 2003.
- Tang, Little and Raghunathan. Analysis of multivariate monotone missing data by a pseudolikelihood method.

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