

HÖLDER EMBEDDINGS OF A DISK INTO THE HEISENBERG GROUP

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ABSTRACT. We present a proposed construction of a Hölder embedding of the closed planar disk into the first Heisenberg group, equipped with the Carnot–Carathéodory metric, with exponent $1/2 + 1/1800$. If verified, this construction would disprove Gromov’s conjecture that no such embedding exists with exponent greater than $1/2$. The construction is inspired by [11] and uses convex integration, proceeding through smooth embedded approximations whose contact defects converge uniformly to zero.

The construction was developed through an extended collaboration with OpenAI’s ChatGPT 5.6 Pro and GPT-6 Astra Max.

The calculations in this manuscript were carried out and written by GPT-6 Astra Max, guided by ideas supplied by the human authors and repeated requests for further details, explanations, clarifications, corrections, and simplifications. Repeated checks using GPT-6 Astra Max on two separate accounts did not identify any errors. **However, these checks do not establish correctness: the calculations have not yet been verified by the human authors.** The manuscript also remains difficult to read and requires further work to improve its exposition. It is therefore preliminary and is not yet ready for submission to arXiv.

Preliminary status. This is an AI-assisted preliminary manuscript. The proposed proof and its calculations have not yet been verified by the human authors.

1. INTRODUCTION

The Heisenberg group is a basic example of a space in which topology and metric regularity place different restrictions on a map. As a topological space, the first Heisenberg group $\mathbb{H}^1$ is $\mathbb{R}^3$. Its Carnot–Carathéodory distance, however, measures motion in the central direction differently from motion in the two horizontal directions. In particular, a smooth Euclidean embedding defined on a neighborhood of a closed planar disk is $1/2$ -Hölder into $\mathbb{H}^1$ on that disk. The question is whether an embedding can have better regularity.

This question belongs to the program initiated by Gromov [5] concerning Hölder maps between Euclidean and Carnot–Carathéodory spaces. For a planar disk in $\mathbb{H}^1$, his non-embedding theorem gives the upper bound $\alpha \leq 2/3$. The conjecture usually associated with this problem predicts the stronger bound $\alpha \leq 1/2$. Hajłasz and Schikorra [7, Theorems 1.1–1.2] gave a proof of the non-embedding theorem using differential forms and obtained refinements involving both Euclidean and Heisenberg regularity. These results

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provide obstructions to embeddings; they do not identify the largest exponent at which an embedding exists.

There is substantially more flexibility when injectivity is not required. Wenger and Young [12, Theorem 1.1] proved Hölder extension results into Carnot groups which, in $\mathbb{H}^1$, allow every exponent below $2/3$. Their constructions do not provide embedded disks. Thus obtaining a continuous map with a good Hölder exponent and obtaining an embedding are separate problems. In an iterative construction this distinction appears at the limit: even a sequence consisting entirely of embeddings can converge to a map which identifies distinct points.

A useful analytic formulation was developed by Hajłasz, Mirra, and Schikorra [6, Theorem 7.4]. For Euclidean Hölder maps of exponent greater than $1/2$, the contact equation can be interpreted distributionally. Their Theorem 7.4 characterizes Heisenberg Hölder regularity by the vanishing of this distributional pullback. This formulation makes it possible to approach the problem by correcting a differential form. In the present proof, uniform control of the defects of smooth approximations permits an elementary signed-area argument in place of the general distributional theorem.

The direct starting point for our calculations is Schikorra's paper [11]. Its Theorems 1.4 and 1.6 develop extension and approximation constructions for a general class of Pfaffian constraints. Applied to the Heisenberg groups, they give Hölder extensions above $1/2$ in every dimension and embeddings above $1/2$ of $\mathbb{B}^{n+1}$ into $\mathbb{H}^n$ for $n \geq 3$ [11, Corollaries 1.8–1.9]. Our proposed construction concerns the remaining problem of an embedded planar disk in $\mathbb{H}^1$. It adapts the oscillatory cancellation from that paper and supplies a different mechanism for retaining injectivity throughout the construction and at its limit.

We write α for the Hölder exponent and $\overline{\mathbb{B}^2} = \{a \in \mathbb{R}^2 : |a| \leq 1\}$ for the closed unit disk. The contact form used in the construction will be denoted by λ . Throughout, an embedding is a continuous injective map whose inverse on its image is continuous. Our proposed main result is the following.

Theorem 1.1. *There exist an embedding $F : \overline{\mathbb{B}^2} \rightarrow \mathbb{H}^1$ and a finite constant C such that*

$$(1.1) \quad d_{CC}(F(a), F(b)) \leq C|a - b|^\alpha \quad (a, b \in \overline{\mathbb{B}^2}), \quad \alpha = \frac{1}{2} + \frac{1}{1800}.$$

More generally, for every exponent

$$\frac{1}{2} < \alpha < \frac{4925}{9839} = \frac{1}{2} + \frac{11}{19678},$$

there exist an embedding $F : \overline{\mathbb{B}^2} \rightarrow \mathbb{H}^1$ and a finite constant C_α such that $d_{CC}(F(a), F(b)) \leq C_\alpha|a - b|^\alpha$ for all $a, b \in \overline{\mathbb{B}^2}$.

The conclusion is a counterexample to the $1/2$ -threshold conjecture in $\mathbb{H}^1$. The gain in the displayed exponent is small. It comes from deliberately generous bounds on derivatives of the changing contact coordinates. The proof does not approach the upper bound $2/3$, and the final section explains which inequality prevents the present estimates from doing so.

1.1. Injectivity and the role of dimension. The extra coordinates in $\mathbb{H}^n$ allow Schikorra to preserve injectivity directly. In suitable contact coordinates, the oscillations can be confined to b horizontal pairs and the central coordinate, leaving $2n - 2b$ coordinates unchanged [11, Theorem 1.6]. If at least $n + 1$ coordinates remain, choose $n + 1$ of them to equal the domain coordinates $a_1, \dots, a_{n+1}$. For the resulting map G , projection π onto these coordinates then satisfies

$$\pi \circ G(a) = a.$$

This identity gives both injectivity and a continuous inverse on the image. It is the argument in the proof of [11, Corollary 6.2].

At least one horizontal pair is needed for the correction. Taking $b = 1$ leaves enough unchanged coordinates precisely when

$$2n - 2 \geq n + 1, \quad \text{that is, } n \geq 3.$$

For example, in $\mathbb{H}^3$ the correction uses three of the seven coordinates, while the other four retain the four domain parameters. This is the precise sense in which higher dimension makes injectivity easier in that construction. In $\mathbb{H}^1$, one horizontal pair and the central coordinate already exhaust all three coordinates. The same argument therefore supplies no fixed projection that can recover the two domain parameters.

Our additional task is to replace this fixed inverse by estimates that remain useful while every ambient coordinate is allowed to change. The proposed construction does this by keeping contact coordinates around each corrected surface and controlling how much they can distort distances. The point is to make the contact defect small, keep the Hölder increments summable, and prevent distinct parameter points from merging, with one choice of iteration parameters.

1.2. The correction in the first Heisenberg group. We construct smooth embeddings $F_k : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ for which

$$F_k^* \lambda = dF_k^t - F_k^y dF_k^x \longrightarrow 0, \quad \lambda = dt - y dx,$$

uniformly on the disk. Superscripts denote components in the working coordinates (x, y, t) introduced in Section 2. The algebraic starting point is the contact cancellation in [11, Lemmas 3.3–3.4 and Proposition 3.5]: two horizontal oscillations of size ε and frequency comparable to ε^{-2} produce an order-one quadratic term in the contact form. A central-coordinate correction cancels its oscillatory part, leaving the desired reduction of the defect. The issue here is to realize this cancellation by an embedded graph and to prepare coordinates in which it can be repeated.

Each finite-stage map is written as

$$F_k = \Xi_k \circ \iota \circ P_k, \quad \iota(s, u) = (s, 0, u).$$

Here ι parametrizes the vertical plane $\{y = 0\}$, $P_k = (S_k, U_k) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a smooth diffeomorphism, and Ξ_k is a contact diffeomorphism on a slab around that plane. Thus the actual surface is the image of the vertical plane under Ξ_k . The planar map P_k changes its parametrization and need not take the disk onto itself. If $\Xi_k^* \lambda = \mu_k \lambda$, with $\mu_k > 0$, then

$$F_k^* \lambda = m_k dU_k, \quad m_k = \mu_k \circ \iota \circ P_k.$$

The defect has a single scalar phase U_k in these coordinates. In particular, a defect bound $|F_k^* \lambda| \leq r_k$ gives $m_k |DU_k| \leq r_k$, which controls how fast the next oscillation varies on the parameter disk.

We first adjust Ξ_k and P_k without changing F_k . The vectors to be controlled are

$$B_{1,k} = \partial_x \Xi_k^h(s, 0, u), \quad B_{2,k} = \partial_y \Xi_k^h(s, 0, u), \quad \Xi_k^h = (\Xi_k^x, \Xi_k^y).$$

They are the horizontal components of the derivatives along the plane in the x -direction and transverse to it in the y -direction. The contact identity fixes their determinant at $\mu_k(s, 0, u)$, but by itself does not bound either length: the vectors can be long and almost parallel. A planar stretch makes the first length comparable to $\sqrt{\mu_k}$. A contact shear then removes, up to a controlled error, the component of the second vector parallel to the first. [Proposition 5.1](#) proves that these changes give

$$|B_{1,k}| + |B_{2,k}| \leq 8\sqrt{\mu_k(s, 0, u)} \quad ((s, u) \in P_k(\overline{\mathbb{B}^2})).$$

This normalization has a specific analytic purpose. The vectors $B_{1,k}, B_{2,k}$ multiply the leading oscillations after composition with Ξ_k . Their bound gives amplitudes of size $C\varepsilon\sqrt{m_k}$, while $m_k |DU_k| \leq r_k$ controls the phase. Together these yield a half-Hölder bound $C\sqrt{r_k}$ for the leading term, with a constant independent of the changing coordinates. Including the remainders, [Proposition 3.2](#) and [lemma 6.3](#) give

$$\|F_{k+1} - F_k\|_{C^{1/2}(\overline{\mathbb{B}^2})} \leq C(\sqrt{r_k} + \varepsilon_k \Gamma_k^2).$$

Here ε_k is the oscillation size, and Γ_k bounds the coordinate quantities specified in [Definition 3.1](#), including derivatives, inverse estimates, and the reciprocal slab width. A separate derivative estimate allows interpolation above $1/2$. Controlling the growth of Γ_k is essential: small oscillations alone would not ensure summability of these stronger Hölder norms.

For one correction, suppress the index k and choose a small reduction parameter $0 < \theta \leq 1/4$. The vertical plane is replaced by

$$\mathcal{G}(s, u) = (s + \varepsilon\eta(Lu), \varepsilon\zeta_\theta(Lu), \psi(u)), \quad L = (1 - \theta)/\varepsilon^2,$$

where $\psi(u) = u + \varepsilon^2\chi_\theta(Lu)$ and the periodic profiles are constructed in [Lemma 6.1](#). They are chosen so that

$$\psi' \geq 7\theta/8 > 0, \quad \mathcal{G}^* \lambda = \theta du - \varepsilon\zeta_\theta(Lu) ds.$$

The second identity reduces the original defect du to a small multiple of itself and an error of size ε . The first ensures that the same correction remains an embedded graph over the vertical (x, t) -plane, namely

$$y = \varepsilon\zeta_\theta(L\psi^{-1}(t)).$$

Thus the cancellation and embeddedness hold for the same profiles.

The corrected map in the ambient space is $F_{k+1} = \Xi_k \circ \mathcal{G} \circ P_k$. An explicit contact diffeomorphism maps the graph back to the vertical plane; its inverse supplies contact coordinates around the new surface ([Lemma 6.2](#)). Bounds on their derivatives, inverse, and domain of definition ensure that the next graph fits inside this coordinate neighborhood. Applying the stretch and shear again restores the normalization. This quantitative passage from one embedded surface to the next is the main additional ingredient beyond the oscillatory contact cancellation.

Finally, inverse estimates for Ξ_k and P_k give a concrete replacement for the fixed projection in higher dimensions. The iteration in [Proposition 8.2](#) yields

$$\begin{aligned} |F_k(a) - F_k(b)| &\geq \Gamma_k^{-2} |a - b|, \\ \|F - F_k\|_{L^\infty(\overline{\mathbb{B}^2})} &\leq C\Gamma_k^{-3}, \quad \Gamma_k \longrightarrow \infty. \end{aligned}$$

For a fixed pair $a \neq b$, the second quantity is eventually less than half the lower bound in the first line. All later corrections therefore move the two image points by too little to identify them. The final injectivity argument is short; the work lies in producing these separation and displacement bounds together with the contact and Hölder estimates at every stage.

1.3. Relation to convex integration. The method has its origins in the C^1 isometric embedding theorems of Nash [\[10\]](#) and Kuiper [\[9\]](#). Its quantitative forms have led to much finer regularity results. Examples include Isett's resolution of the flexible side of Onsager's conjecture [\[8, Theorem 1\]](#), the $C^{1,\alpha}$ isometric immersions for $\alpha < 1/5$ of De Lellis, Inauen, and Székelyhidi [\[4, Theorems 0.1–0.2\]](#), and the more recent immersion constructions of Cao, Hirsch, and Inauen [\[1, Theorem 1.1 and Remark 1.2\]](#), which in dimension two reach $\alpha < 1/3$. These achievements depend on the structure of the equation as well as on the choice of parameters. Here the contact condition and the requirement of injectivity dictate the particular corrections that we can use.

Three aspects of those arguments helped organize the present proof. Following De Lellis and Inauen [\[3, Proposition 4.1 and Sections 6–8\]](#), we state one complete correction proposition before proving its ingredients. This makes explicit what the next stage needs from the current one. The composition estimates in [\[4, Proposition 2.1\]](#) and [\[1, Lemma A.1\]](#) also motivate retaining derivative orders separately. In our setting only two derivatives of the ambient coordinates are needed. Finally, we choose coordinates before estimating the new error. The particular stretch, contact shear, and phase coordinates used below are derived directly for this problem; none is an application of an isometric immersion theorem.

There is also a connection with the rigidity side of the isometric embedding problem. For positively curved surfaces, the exponent $2/3$ appears in a curvature and degree argument involving the Hölder Gauss map. In [\[2, Proposition 7.5, equation \(63\)\]](#), a quadratic commutator of size $\ell^{2\alpha-1}$ is paired with a differentiated test function of size $\ell^{\alpha-1}$, leaving an error of order $\ell^{3\alpha-2}$. This is the relevant analogy with distributional-Jacobian obstructions in the Heisenberg problem. We cite this rigidity argument for context; it is not a construction lemma in the proof below. The connection threshold $1/2$ studied in [\[3, Theorem 1.2\]](#) concerns a different question from this classical curvature threshold.

1.4. Organization of the proof. We include the details needed to read the construction without prior knowledge of convex integration. [Section 2](#) fixes the metric and contact conventions. [Section 3](#) lists the coordinate data and states the main correction proposition. The analytic estimates, normalization, and explicit graph are proved in [Sections 4 to 6](#); they are assembled in [Section 7](#). [Section 8](#) checks the numerical inequalities and proves that the limit is injective. [Section 9](#) supplies the Heisenberg Hölder estimate. The final

section discusses the limitations of the present estimates and the further ideas from convex integration which would require new arguments here.

The reader may first read the main correction proposition and the iteration, then return to the coordinate calculations. Within the longer proofs, a short explanation precedes each calculation to indicate what that calculation is meant to achieve.

2. THE HEISENBERG GROUP AND ITS METRIC

The first Heisenberg group $\mathbb{H}^1$ is $\mathbb{R}^3$, with coordinates (x, y, t) , equipped with the multiplication

$$(2.1) \quad (x, y, t) \cdot (x', y', t') = (x + x', y + y', t + t' + 2(yx' - xy')).$$

The identity is 0, and the inverse of (x, y, t) is $(-x, -y, -t)$. Bilinearity of the central cross term proves associativity: both bracketings of a triple product have the same three pairwise cross terms. The group operations are smooth, so this is a Lie group. Here x, y are the horizontal coordinates and t is the central, or vertical, coordinate.

The horizontal plane at each point is the kernel of the contact form

$$\beta = dt + 2(x dy - y dx).$$

It is spanned by the vector fields

$$X = \partial_x + 2y \partial_t, \quad Y = \partial_y - 2x \partial_t,$$

which we declare orthonormal. A curve is horizontal when its velocity lies in this plane. The Carnot–Carathéodory distance is the infimum of the horizontal lengths of curves joining the two given points. These conventions agree with [7, Section 2.2]. The central rescaling $(x, y, t) \mapsto (x, y, t/4)$ gives the normalization used in [11, Section 1, equation (1.1)].

2.1. The matrix model used in the proof. For the calculations we use an equivalent model in which the contact form has only one product. Consider the upper triangular matrices

$$M(x, y, t) = \begin{pmatrix} 1 & y & t \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix}, \quad x, y, t \in \mathbb{R}.$$

Their multiplication satisfies

$$M(x, y, t)M(x', y', t') = M(x + x', y + y', t + t' + yx').$$

Thus the matrix group $\tilde{\mathbb{H}}^1$ is $\mathbb{R}^3$ with law

$$(2.2) \quad (x, y, t) \odot (x', y', t') = (x + x', y + y', t + t' + yx').$$

We use the coordinate names (x, y, t) in both models; the symbols $\cdot$ and $\odot$ distinguish their multiplications. The map

$$\phi : \tilde{\mathbb{H}}^1 \longrightarrow \mathbb{H}^1, \quad \phi(x, y, t) = (x, y, 4t - 2xy)$$

has smooth inverse $\phi^{-1}(x, y, t) = (x, y, (t + 2xy)/4)$ and is a group isomorphism. Indeed, the central coordinate of $\phi(q \odot q')$ is

$$4(t + t' + yx') - 2(x + x')(y + y') = 4t - 2xy + 4t' - 2x'y' + 2(yx' - xy'),$$

which is the central coordinate of $\phi(q) \cdot \phi(q')$; the two horizontal coordinates agree as well.

In the matrix model put

$$(2.3) \quad \lambda = dt - y dx, \quad \widetilde{X} = \partial_x + y \partial_t, \quad \widetilde{Y} = \partial_y.$$

The fields $\widetilde{X}, \widetilde{Y}$ span $\ker \lambda$ and are declared orthonormal. The identity $\lambda \wedge d\lambda = dt \wedge dx \wedge dy \neq 0$ verifies the contact condition. The two models have the same horizontal geometry:

$$\phi^* \beta = d(4t - 2xy) + 2(x dy - y dx) = 4(dt - y dx) = 4\lambda.$$

More precisely,

$$D\phi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2y & -2x & 4 \end{pmatrix}$$

gives $D\phi(\widetilde{X}) = X \circ \phi$ and $D\phi(\widetilde{Y}) = Y \circ \phi$. Hence ϕ preserves horizontal lengths and is an isometry for the corresponding distances. We henceforth identify the two metric groups and use $\mathbb{H}^1$ for the matrix model (2.2). Every subsequent ambient calculation uses (x, y, t) and $\lambda = dt - y dx$.

This model is also the first jet space $J^1(\mathbb{R})$: the coordinates (x, y, t) record the independent variable, derivative, and function value. Its contact form is $dt - y dx$, and (2.2) is the case $n = k = 1$ of [13, Section 2.2, equation (1)]. For example, the jet of a smooth function is the horizontal curve $x \mapsto (x, f'(x), f(x))$.

2.2. Horizontal curves and distance. A piecewise C^1 curve $\gamma(\tau) = (x(\tau), y(\tau), t(\tau))$, $0 \leq \tau \leq 1$, is horizontal precisely when

$$(2.4) \quad t' = yx'.$$

The prime here denotes differentiation with respect to τ . Since $\gamma' = x' \widetilde{X}(\gamma) + y' \widetilde{Y}(\gamma)$, its horizontal length is

$$\ell(\gamma) = \int_0^1 \sqrt{x'(\tau)^2 + y'(\tau)^2} d\tau.$$

Write $d_{CC}(p, q)$ for the infimum of these lengths over horizontal curves joining p to q . The next lemma proves that every pair can be joined and that the resulting distance is a metric.

Left translation by (a, b, c) sends a curve to $(a + x, b + y, c + t + bx)$. Horizontality is preserved, because

$$(t' + bx') - (b + y)x' = t' - yx'.$$

The horizontal derivatives remain x', y' , so lengths are preserved. Applying this also to the inverse translation gives

$$d_{CC}(p, q) = d_{CC}(0, p^{-1} \odot q).$$

The inverse is $(x, y, t)^{-1} = (-x, -y, -t + xy)$. Consequently, for $p = (x_a, y_a, t_a)$ and $q = (x_b, y_b, t_b)$,

$$(2.5) \quad p^{-1} \odot q = (x_b - x_a, y_b - y_a, t_b - t_a - y_a(x_b - x_a)).$$

The extra product in the central coordinate will combine with the contact equation in [Proposition 9.2](#).

Equation (2.4) explains how a loop in the horizontal (x, y) -plane can produce a vertical displacement. A square loop gives the elementary metric bounds needed in the proof.

Lemma 2.1 (Elementary bounds for the Heisenberg distance). *In the matrix model (2.2), let $q = (x, y, t) \in \mathbb{H}^1$ and $h = (x, y) \in \mathbb{R}^2$. Then*

$$(2.6) \quad \max\{|h|, \sqrt{|t|}\} \leq d_{CC}(0, q) \leq |x| + |y| + 4\sqrt{|t|}.$$

In particular d_{CC} is a finite metric and induces the Euclidean topology on $\mathbb{R}^3$.

Proof. Let γ be a horizontal curve from 0 to q , of length ℓ . Its horizontal endpoint satisfies $|h| \leq \ell$, and $|y(\tau)| \leq \ell$ along the curve. Integrating (2.4) gives

$$|t| = \left| \int_0^1 y(\tau) x'(\tau) d\tau \right| \leq \ell \int_0^1 |x'(\tau)| d\tau \leq \ell^2.$$

Taking the infimum proves the lower bounds.

For the upper bound, first move from 0 to $(x, 0, 0)$ and then to $(x, y, 0)$. Both segments are horizontal by (2.4), and their total length is $|x| + |y|$. For the counterclockwise boundary of the square $[0, a]^2$ in the horizontal plane, direct integration gives $\oint y dx = -a^2$. Its horizontal lift starting at 0 therefore ends at $(0, 0, -a^2)$ and has length $4a$. Reversing orientation gives central displacement a^2 . Choose $a = \sqrt{|t|}$ and the appropriate orientation, and left translate the lifted curve to $(x, y, 0)$. It ends at $(x, y, 0) \odot (0, 0, t) = (x, y, t)$, proving the upper bound. When $t = 0$ the loop is unnecessary.

Finiteness for every pair follows by left translation. Reversing curves proves symmetry, and concatenating curves and taking infima proves the triangle inequality. The lower bound applied to $p^{-1} \odot q$ shows that distance zero forces $p = q$. Finally, (2.5) shows that $p^{-1} \odot q \rightarrow 0$ precisely when $q \rightarrow p$ in Euclidean coordinates. The bounds in (2.6) therefore prove that the metric and Euclidean topologies agree. $\square$

These bounds also justify the observation at the beginning of the introduction. If f is smooth near a closed disk, then Df and f^y are bounded there. For a, b in that disk, (2.5) gives horizontal increments bounded by $C|a - b|$ and a central increment bounded by $C|a - b|$. The upper bound in (2.6) therefore gives $d_{CC}(f(a), f(b)) \leq C|a - b|^{1/2}$, after adjusting C for the diameter of the disk. Injectivity is needed only to call this map an embedding.

The square root explains the stronger estimates needed below. To prove that a map is α -Hölder into $\mathbb{H}^1$, we will bound the horizontal coordinates of its group differences by $C|a - b|^\alpha$ and their central coordinate by $C|a - b|^{2\alpha}$.

2.3. Contact defects and the vertical plane. For a smooth map $F = (F^x, F^y, F^t)$ from a planar domain, its *contact defect* is the one-form

$$F^* \lambda = dF^t - F^y dF^x.$$

It vanishes precisely when DF takes values in the horizontal planes. We construct embedded smooth surfaces whose defects tend uniformly to zero while their Euclidean Hölder

norms stay bounded. [Proposition 9.2](#) converts these bounds into Heisenberg regularity of the limit.

A smooth ambient diffeomorphism Ξ is *contact* if $\Xi^*\lambda = \mu\lambda$ for a smooth nowhere-zero function μ . We use positive factors μ throughout. For every p and v ,

$$\lambda_{\Xi(p)}(D\Xi(p)v) = \mu(p)\lambda_p(v).$$

Thus $D\Xi$ maps each horizontal plane onto the horizontal plane at the image point. It also implies

$$\Xi^*(\lambda \wedge d\lambda) = \mu^2 \lambda \wedge d\lambda, \quad \det D\Xi = \mu^2.$$

For the first equality, differentiate $\Xi^*\lambda = \mu\lambda$ and note that the term containing $\lambda \wedge d\mu \wedge \lambda$ vanishes. The second uses $\lambda \wedge d\lambda = dx \wedge dy \wedge dt$.

Superscripts x, y, t on a vector-valued map denote its components, and $\partial_x, \partial_y, \partial_t$ denote coordinate derivatives. Thus $\Xi^h = (\Xi^x, \Xi^y)$ is the horizontal projection of Ξ . The Euclidean coordinate vectors e_x, e_y, e_t are distinct from the horizontal vector fields $\tilde{X}, \tilde{Y}$. Thus α denotes a Hölder exponent, β denotes the contact form in the original group model, and $\lambda = dt - y dx$ is the contact form used in every subsequent calculation.

The starting surface is the vertical plane $\{y = 0\}$. Using planar parameters (s, u) , we parametrize it by

$$\iota(s, u) = (s, 0, u).$$

Thus s is a horizontal parameter and u a vertical parameter, and $\iota^*\lambda = du$. The ambient coordinate t is kept separate from these parameter names. The plane contains the central t -direction, which is why it is called vertical. Corrections move off the plane, so the ambient coordinate maps are defined on the slabs

$$U_\delta = \mathbb{R} \times (-\delta, \delta) \times \mathbb{R}, \quad \delta > 0.$$

Every ambient triple in the construction is ordered (x, y, t) .

2.4. Notation for the estimates. All derivatives in the construction are ordinary Euclidean derivatives. A map described as smooth on the closed disk $\overline{\mathbb{B}^2}$ is the restriction of a smooth map on an open neighborhood of the disk. Derivative norms are supremum operator norms on the indicated domains. The norm of a one-form on the parameter domain is the Euclidean norm of its two coefficients; this specifies, in particular, the norm used for the contact defect.

For a Euclidean-valued function f on $\overline{\mathbb{B}^2}$, we write

$$[f]_\alpha = \sup_{\substack{a, b \in \overline{\mathbb{B}^2} \\ a \neq b}} \frac{|f(a) - f(b)|}{|a - b|^\alpha}, \quad \|f\|_{C^\alpha} = \|f\|_\infty + [f]_\alpha.$$

Unless the Heisenberg target metric is specified, Hölder norms in the proof refer to this Euclidean norm. A C^j norm includes the function and its derivatives through order j ; when only derivatives are being bounded, they are written explicitly.

Constants C, c are independent of the iteration parameters and may change between estimates. We write $A \lesssim B$ when $A \leq CB$ for such a constant. After small fixed constants have been chosen, later constants may depend on those choices.

3. THE COORDINATE DATA AND THE MAIN CORRECTION

3.1. What is retained from one stage to the next. The map $P = (S, U) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a smooth diffeomorphism of the parameter plane. The contact diffeomorphism Ξ is defined on an ambient slab and maps the vertical plane $\{y = 0\}$ onto the current surface. The parametrization is the composition

$$F = \Xi \circ \iota \circ P, \quad a \mapsto P(a) = (S(a), U(a)) \mapsto (S(a), 0, U(a)) \mapsto F(a).$$

Thus the coordinates describe an entire neighborhood of the surface. This neighborhood is needed because the next correction moves off the plane before it is mapped by Ξ .

Here is a guide to the notation used in one correction.

Symbol	Meaning and role
$a \in \overline{\mathbb{B}^2}$	A point in the original parameter disk.
$(s, u) = P(a)$	Parameters in the current plane; $\iota(s, u) = (s, 0, u)$.
(x, y, t)	An ambient coordinate point in the slab U_δ .
Ξ, μ	The ambient diffeomorphism and its positive contact factor, defined by $\Xi^* \lambda = \mu \lambda$.
$m(a)$	The factor on the surface: $m(a) = \mu(\iota(P(a)))$.
r	An upper bound for the contact defect $ F^* \lambda $ on $\overline{\mathbb{B}^2}$.
Γ	A common upper bound for the coordinate quantities listed below. During a calculation the individual derivative bounds are retained.
θ, ε, L	The principal defect reduction, the oscillation size, and its frequency $L = (1 - \theta)/\varepsilon^2$.
$\Phi = K^{-1}, B$	The ambient graph coordinates and the change of planar parameters, constructed in Lemma 6.2 .

In the normalization proof, $m(s, u) = \mu(s, 0, u)$ is a function on the unparametrized plane. It becomes the preceding $m(a)$ after composition with P . The planar stretch in that proof is denoted by $\mathcal{L}$, and its inverse by $\mathcal{J}$.

Evaluating $\Xi^* \lambda = \mu \lambda$ on the three coordinate vectors gives the scalar equations

$$(3.1) \quad \begin{aligned} \partial_x \Xi^t - \Xi^y \partial_x \Xi^x &= -\mu y, \\ \partial_y \Xi^t - \Xi^y \partial_y \Xi^x &= 0, \\ \partial_t \Xi^t - \Xi^y \partial_t \Xi^x &= \mu. \end{aligned}$$

Pulling this identity back by $\iota \circ P$ gives $F^* \lambda = m dU$. Therefore, if the contact defect satisfies $\|F^* \lambda\|_{L^\infty(\overline{\mathbb{B}^2})} \leq r$, then

$$(3.2) \quad F^* \lambda = m dU, \quad m |DU| \leq r.$$

The second inequality will control differentiation of the oscillation.

Definition 3.1 (Coordinate data and normalized coordinate data). Let $\Gamma \geq 2$. A set of coordinate data of size at most Γ consists of a number $\delta > 0$, smooth diffeomorphisms

$$\Xi : U_\delta \longrightarrow \Xi(U_\delta), \quad P = (S, U) : \mathbb{R}^2 \longrightarrow \mathbb{R}^2,$$

and a smooth positive function $\mu : U_\delta \rightarrow (0, \infty)$. They must satisfy the following four conditions.

(i) *Slab width and contact identity.*

$$\Gamma^{-1} \leq \delta \leq 1, \quad \Xi^* \lambda = \mu \lambda \quad \text{on } U_\delta.$$

(ii) *Bounds for the ambient coordinates and contact factor.*

$$(3.3) \quad \begin{aligned} \|D\Xi\|_{L^\infty(U_\delta)} &\leq \Gamma, & \|D^2\Xi\|_{L^\infty(U_\delta)} &\leq \Gamma, \\ \Gamma^{-1} &\leq \mu \leq \Gamma \quad \text{on } U_\delta, & \|D\mu\|_{L^\infty(U_\delta)} &\leq \Gamma. \end{aligned}$$

(iii) *Bounds for the planar coordinates.*

$$\|DP\|_{L^\infty(\mathbb{R}^2)} \leq \Gamma, \quad \sup_{a \in \mathbb{B}^2} |P(a)| \leq \Gamma.$$

(iv) *Inverse estimates for every pair of points.*

$$(3.4) \quad \begin{aligned} |p - q| &\leq \Gamma |\Xi(p) - \Xi(q)| \quad (p, q \in U_\delta), \\ |a - b| &\leq \Gamma |P(a) - P(b)| \quad (a, b \in \mathbb{R}^2). \end{aligned}$$

These data represent the map

$$F = \Xi \circ \iota \circ P : \mathbb{R}^2 \longrightarrow \mathbb{R}^3, \quad \iota(s, u) = (s, 0, u).$$

Thus P supplies coordinates on the vertical plane, and Ξ maps that plane onto the parametrized surface. The bound on the values of P is required only on $\mathbb{B}^2$; its derivative and inverse estimates hold on the entire plane. We impose no bound on the values of Ξ on its unbounded domain.

Normalized coordinate data. The data are called *normalized* if they also satisfy

$$(3.5) \quad |\partial_x \Xi^h(s, 0, u)| + |\partial_y \Xi^h(s, 0, u)| \leq 8\sqrt{\mu(s, 0, u)}$$

for every $(s, u) \in P(\mathbb{B}^2)$, where $\Xi^h = (\Xi^x, \Xi^y)$. All four conditions above remain part of the definition of normalized data.

The inequalities for pairs are essential: a uniform limit of embeddings need not be injective. In these coordinates each finite stage satisfies

$$(3.6) \quad |F(a) - F(b)| \geq \Gamma^{-2} |a - b|.$$

The proof will compare this separation with the remaining corrections. Indeed, applying the two inverse inequalities successively gives

$$|F(a) - F(b)| \geq \Gamma^{-1} |\iota(P(a)) - \iota(P(b))| = \Gamma^{-1} |P(a) - P(b)| \geq \Gamma^{-2} |a - b|.$$

The equality uses the fact that ι preserves Euclidean distances.

The additional condition (3.5) controls the two derivative vectors that multiply the leading oscillations. The contact identity fixes their determinant at $\mu(s, 0, u)$; this is proved

in (5.5). The normalization bounds both lengths by a constant times $\sqrt{\mu(s, 0, u)}$. In the correction, this produces amplitudes of size $C\sqrt{m}$. Combined with $m|DU| \leq r$, it gives the term $C\sqrt{r}$ in the half-Hölder estimate. Lemma 6.3 proves this estimate when m varies along a segment.

3.2. The main correction proposition.

Proposition 3.2 (One correction and the coordinates for the next step). *There are constants $c > 0$ and $C \geq 1$, depending only on the fixed profiles and cutoffs used below, with the following property. Let (δ, Ξ, P, μ) be normalized coordinate data of size at most $\Gamma \geq 2$, and put $F = \Xi \circ \iota \circ P$. Suppose $0 < r \leq 1$ and*

$$\sup_{a \in \mathbb{R}^2} |F^y(a)| \leq 1, \quad \|F^* \lambda\|_{L^\infty(\overline{\mathbb{B}^2})} \leq r.$$

Choose parameters

$$0 < \theta \leq 1/4, \quad 0 < \varepsilon \leq c\Gamma^{-2}.$$

Then there exist a smooth embedding $G : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ and normalized coordinate data $(\delta_+, \Xi_+, P_+, \mu_+)$ such that

$$G = \Xi_+ \circ \iota \circ P_+ \quad \text{on } \mathbb{R}^2.$$

They have the following quantitative properties.

(i) Bounds for the new coordinate data. All conditions of Definition 3.1, including normalization and the inverse estimates for every pair of points, hold with a bound Γ_+ satisfying

$$(3.7) \quad \Gamma_+ \leq C\Gamma^{22}(\varepsilon^{-2} + \varepsilon^{-1}\theta^{-2})^{22}.$$

If $\varepsilon \leq \theta^2$, the same output data have a bound satisfying

$$(3.8) \quad \Gamma_+ \leq C\Gamma^{22}\varepsilon^{-44}.$$

(ii) Defect, displacement, and half-Hölder estimates. Without any additional condition on ε, θ, r , one has

$$(3.9) \quad \begin{aligned} \|G^* \lambda\|_{L^\infty(\overline{\mathbb{B}^2})} &\leq 2\theta r + C\varepsilon\Gamma^2, \\ \|G - F\|_{L^\infty(\mathbb{R}^2)} &\leq C\Gamma\varepsilon, \\ \|G - F\|_{C^{1/2}(\overline{\mathbb{B}^2})} &\leq C(\sqrt{r} + \varepsilon\Gamma^2). \end{aligned}$$

The displacement estimate holds on the whole parameter plane; the defect and Hölder estimates are required only on $\mathbb{B}^2$.

(iii) Estimates above exponent $1/2$. If, in addition, $\varepsilon\Gamma^2 \leq \sqrt{r}$, then

$$(3.10) \quad \boxed{\|D(G - F)\|_{L^\infty(\overline{\mathbb{B}^2})} \leq C \frac{r\sqrt{\Gamma}}{\varepsilon}.}$$

Under this same additional hypothesis, for every $1/2 < \alpha < 1$,

$$(3.11) \quad \boxed{\|G - F\|_{C^\alpha(\overline{\mathbb{B}^2})} \leq C_\alpha \left(\Gamma\varepsilon + r^\alpha \left(\frac{\sqrt{\Gamma}}{\varepsilon} \right)^{2\alpha-1} \right).}$$

Here C_α may also depend on α . All constants are independent of the input maps and of $\Gamma, r, \theta, \varepsilon$.

What the proposition gives. Part (ii) bounds the new defect and the displacement. In the iteration, we choose θ and ε so that the defect decreases and the displacements are summable. Part (iii) gives the stronger Hölder estimate. Part (i) supplies the coordinate maps, contact factor, and slab needed to apply the proposition again. The bound $|G^y| \leq 1$ is not part of the output: in the iteration it will follow by summing the global supremum estimates from $F_0^y = 0$.

The construction behind the proposition consists of four explicit operations. We first replace the plane by the embedded graph $\mathcal{G}$ of [Lemma 6.2](#), and then place it in the old coordinates:

$$G = \Xi \circ \mathcal{G} \circ P.$$

The same lemma constructs K, B with $K \circ \mathcal{G} = \iota \circ B$, so the corrected surface can be rewritten as

$$G = (\Xi \circ K^{-1}) \circ \iota \circ (B \circ P).$$

Finally, [Proposition 5.1](#) normalizes these coordinates without changing G . Put $W = \varepsilon^{-2} + \varepsilon^{-1}\theta^{-2}$. Here a coordinate bound means the common bound in [Definition 3.1](#). Flattening and normalization give these successive bounds:

$$\Gamma \xrightarrow{\text{flattening}} C\Gamma W \xrightarrow{\text{normalization}} C\Gamma^{22}W^{22}.$$

The useful compatibility condition $\varepsilon \leq \theta^2$ makes $W \leq 2\varepsilon^{-2}$. It will be checked explicitly in the iteration. This explains the otherwise unmotivated powers in [\(3.7\)](#); both estimates are proved below.

What must be checked in the infinite construction. Let F_k be the maps obtained by applying the proposition repeatedly. There are three independent limiting requirements:

$$(3.12) \quad \sum_k \|F_{k+1} - F_k\|_{C^\alpha(\overline{\mathbb{B}^2})} < \infty, \quad r_k \longrightarrow 0,$$

for every $d > 0$, some k satisfies $2 \sum_{j \geq k} \|F_{j+1} - F_j\|_{L^\infty(\overline{\mathbb{B}^2})} < \Gamma_k^{-2}d$.

The first gives a Euclidean C^α limit. The second permits the metric argument in [Proposition 9.2](#). For the third, apply the triangle inequality to any pair $|a - b| = d$ and use [\(3.6\)](#); their images remain distinct in the limit. [Section 8](#) checks these requirements with explicit parameters.

4. ELEMENTARY ESTIMATES USED BELOW

We record the elementary analytic estimates, including their proofs, so that the later coordinate calculations have no unstated regularity requirements. Keeping the derivative orders separate is the practice used in [\[4, Proposition 2.1\]](#) and [\[1, Lemma A.1\]](#). For this construction the following second-order formulas suffice.

For smooth maps f, g whose composition is defined, the chain rule is $D(f \circ g) = (Df \circ g)Dg$. Differentiating in directions v, w gives

$$D^2(f \circ g)[v, w] = (D^2f \circ g)[Dg v, Dg w] + (Df \circ g)D^2g[v, w].$$

Taking supremum norms on the domain of g and its image yields

$$(4.1) \quad \begin{aligned} \|D(f \circ g)\| &\leq \|Df\| \|Dg\|, \\ \|D^2(f \circ g)\| &\leq \|D^2f\| \|Dg\|^2 + \|Df\| \|D^2g\|. \end{aligned}$$

For a scalar function v bounded away from zero,

$$D(v^{-1}) = -v^{-2}Dv, \quad D^2(v^{-1}) = 2v^{-3}(Dv)^2 - v^{-2}D^2v.$$

The second formula denotes a bilinear map, with $(Dv)^2 = Dv \otimes Dv$. The coordinate estimates below use these rules through order two. Higher derivatives of the auxiliary scalar functions will be obtained from smoothing and estimated separately.

If V is convex, integration along a segment gives

$$(4.2) \quad |f(u) - f(v)| \leq \|Df\|_{L^\infty(V)} |u - v| \quad (u, v \in V).$$

Also, for $0 < \gamma < \alpha < 1$ on the disk,

$$(4.3) \quad [f]_\alpha \leq [f]_\gamma^{(1-\alpha)/(1-\gamma)} \|Df\|_\infty^{(\alpha-\gamma)/(1-\gamma)}.$$

Indeed an increment at distance d is bounded both by $[f]_\gamma d^\gamma$ and by $\|Df\|_\infty d$. The minimum of two positive numbers is at most their weighted geometric mean. Choose the weights making the power of d equal to α . The same argument gives $[f]_{1/2} \leq (2\|f\|_\infty \|Df\|_\infty)^{1/2}$.

Lemma 4.1 (Smoothing with an absolute or relative error). *Fix a nonnegative function $\varphi \in C_c^\infty(\mathbb{R}^2)$, supported in the unit ball and of integral one. For $\ell > 0$, set $\varphi_\ell(x) = \ell^{-2}\varphi(x/\ell)$. Let $a \in C^1(\mathbb{R}^2; \mathbb{R})$ have bounded value and first derivative, and put $A = a * \varphi_\ell$. Then A is smooth and*

$$\begin{aligned} \|A - a\|_{L^\infty(\mathbb{R}^2)} &\leq \ell \|Da\|_{L^\infty(\mathbb{R}^2)}, \\ \|DA\|_{L^\infty(\mathbb{R}^2)} &\leq \|Da\|_{L^\infty(\mathbb{R}^2)}, \\ \|D^j A\|_{L^\infty(\mathbb{R}^2)} &\leq C_j \ell^{1-j} \|Da\|_{L^\infty(\mathbb{R}^2)} \quad (j \geq 1). \end{aligned}$$

The constants C_j depend only on j and the chosen mollifier. If $m \leq a(x) \leq M$ for every x , then $m \leq A(x) \leq M$ for every x . For any constant a_* , one also has

$$\text{supp}(A - a_*) \subset \{x : \text{dist}(x, \text{supp}(a - a_*)) \leq \ell\}.$$

If, in addition, $a(x) > 0$ and $|Da(x)| \leq L_a a(x)$ for every $x \in \mathbb{R}^2$, where $L_a \geq 0$, then

$$(4.4) \quad e^{-L_a \ell} a(x) \leq A(x) \leq e^{L_a \ell} a(x), \quad |DA(x)| \leq L_a A(x).$$

In particular, $L_a \ell \leq 1$ implies $|A(x) - a(x)| \leq 2L_a \ell a(x)$ for every x .

Proof. Integration along a segment gives $|a(x - \ell y) - a(x)| \leq \ell |y| \|Da\|_\infty$. Averaging proves the first estimate. Put one derivative on a and the remaining $j - 1$ on the mollifier, whose corresponding L^1 norm is $C_j \ell^{1-j}$. This proves the derivative estimates. The value and support assertions also follow from averaging.

For the relative estimate, $|D \log a| \leq L_a$. Consequently $|\log a(x - \ell y) - \log a(x)| \leq L_a \ell$ for $|y| \leq 1$. Exponentiate and average to get the two value bounds in (4.4). Also $|DA| \leq |Da| * \varphi_\ell \leq L_a(a * \varphi_\ell) = L_a A$. Finally $e^\tau - 1 \leq 2\tau$ for $0 \leq \tau \leq 1$. $\square$

The relative estimate is useful when the function to be smoothed can be small: the smoothing error is compared with its value at the same point, instead of with its smallest value on the whole plane.

Lemma 4.2 (A Taylor remainder using only two derivatives). *Let $V \subset \mathbb{R}^n$ and $\Omega \subset \mathbb{R}^d$ be open, $f \in C^2(V; \mathbb{R}^N)$, and $u, w \in C^1(\Omega; \mathbb{R}^n)$. Suppose $\|D^2 f\|_{L^\infty(V)} \leq M_2$ and $u(a) + \tau w(a) \in V$ for $a \in \Omega$ and $0 \leq \tau \leq 1$. Define the remainder by*

$$R(a) = f(u(a) + w(a)) - f(u(a)) - Df(u(a))w(a).$$

Then $R \in C^1(\Omega; \mathbb{R}^N)$, and at every $a \in \Omega$,

$$(4.5) \quad |R| \leq \frac{1}{2} M_2 |w|^2, \quad |DR| \leq M_2 |w| (2|Du| + |Dw|).$$

Proof. The first inequality follows by integrating the second derivative along the segment. For a unit direction v , differentiate the difference defining R :

$$\begin{aligned} DR[v] &= [Df(u + w) - Df(u)]Du[v] - D^2 f(u)[Du[v], w] \\ &\quad + [Df(u + w) - Df(u)]Dw[v]. \end{aligned}$$

The difference of the first derivatives has norm at most $M_2 |w|$. Bound the three terms separately to obtain (4.5). The derivative estimate loses one power of $|w|$. This is acceptable in our application and avoids asking for a third derivative of f . $\square$

All smooth bumps and cutoffs in the proof can be obtained from $b(\tau) = e^{-1/\tau}$ for $\tau > 0$, extended by zero for $\tau \leq 0$. Its derivatives at zero vanish, because any polynomial in $1/\tau$ times $e^{-1/\tau}$ tends to zero. The function $b(\tau)/(b(\tau) + b(1 - \tau))$ is a smooth transition from zero to one. Products, translations, and dilations give the cutoffs used below. Normalizing a nonnegative radial bump gives the even mollifier above.

Lemma 4.3 (A small perturbation of the identity). *Let $h : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be smooth with $\|Dh\|_{L^\infty(\mathbb{R}^n)} \leq 1/4$, and set $H(x) = x + h(x)$. Then H is a smooth diffeomorphism from $\mathbb{R}^n$ onto $\mathbb{R}^n$. For every $x, x' \in \mathbb{R}^n$,*

$$\frac{3}{4}|x - x'| \leq |H(x) - H(x')| \leq \frac{5}{4}|x - x'|.$$

Its inverse $J = H^{-1}$ satisfies

$$\|DJ\|_{L^\infty(\mathbb{R}^n)} \leq \frac{4}{3}.$$

If $\|D^2 H\|_{L^\infty(\mathbb{R}^n)} < \infty$, then also

$$\|D^2 J\|_{L^\infty(\mathbb{R}^n)} \leq \left(\frac{4}{3}\right)^3 \|D^2 H\|_{L^\infty(\mathbb{R}^n)}.$$

Proof. The point of the small derivative assumption is that solving for the inverse requires no bound on the displacement h itself.

Fix a target y , take $x_0 = y$, and define $x_{j+1} = y - h(x_j)$. Integration along a segment gives

$$|x_{j+1} - x_j| \leq \frac{1}{4}|x_j - x_{j-1}| \leq 4^{-j}|x_1 - x_0| \quad (j \geq 1).$$

For $m > j$, the triangle inequality therefore bounds $|x_m - x_j|$ by the tail of a convergent geometric series. The sequence converges in $\mathbb{R}^n$ to a point x ; continuity gives $x = y - h(x)$, or $H(x) = y$. If $H(x) = H(x')$, then $|x - x'| = |h(x') - h(x)| \leq |x - x'|/4$, so $x = x'$. This proves global existence and uniqueness of the inverse.

For arbitrary points the same estimate gives

$$|H(x) - H(x')| \geq |x - x'| - |h(x) - h(x')| \geq \frac{3}{4}|x - x'|.$$

The upper bound follows from the triangle inequality and $|h(x) - h(x')| \leq |x - x'|/4$. The lower bound shows that the inverse J is Lipschitz with constant $4/3$. Here is also a direct check of differentiability, so no local invertibility assertion is left implicit. Put $x = J(y)$ and $\Delta x = J(y + \Delta y) - J(y)$. Differentiability of H and its inverse identities give

$$\Delta y = DH(x)\Delta x + o(|\Delta x|).$$

We already know $|\Delta x| \leq 4|\Delta y|/3$. Also $|DH(x)v| \geq 3|v|/4$, so $DH(x)$ is invertible and its inverse has norm at most $4/3$. It follows that

$$\Delta x = (DH(x))^{-1}\Delta y + o(|\Delta y|), \quad DJ(y) = (DH(J(y)))^{-1}.$$

This formula makes DJ continuous. Since matrix inversion is smooth where the determinant is nonzero, repeated differentiation proves that J is smooth.

Differentiating $H \circ J = \text{id}$ twice gives

$$D^2J = -(DH \circ J)^{-1}(D^2H \circ J)[DJ, DJ].$$

The inverse matrix and both factors of DJ have norm at most $4/3$. Taking supremum norms gives the stated second derivative bound. $\square$

5. NORMALIZING THE CONTACT COORDINATES

Proposition 5.1 (Normalization of the coordinate representation). *There is a constant $C \geq 1$, independent of the input data, with the following property. Let (δ, Ξ, P, μ) be coordinate data of size at most $\Gamma \geq 2$, as specified in [Definition 3.1](#). Then there exist normalized coordinate data*

$$(\tilde{\delta}, \tilde{\Xi}, \tilde{P}, \tilde{\mu})$$

of size at most $\tilde{\Gamma} = C\Gamma^{22}$ which represent exactly the same map:

$$\tilde{\Xi} \circ \iota \circ \tilde{P} = \Xi \circ \iota \circ P \quad \text{on } \mathbb{R}^2.$$

The new data satisfy all four conditions in [Definition 3.1](#), with Γ replaced by $\tilde{\Gamma}$, and the normalization inequality (3.5) on $\tilde{P}(\mathbb{B}^2)$.

The estimate for $\tilde{\Gamma}$ uses only the quantitative bounds explicitly listed in [Definition 3.1](#). In particular, the input derivative bounds required are those for

$$D\Xi, \quad D^2\Xi, \quad D\mu, \quad DP.$$

Although all coordinate maps are smooth, no quantitative bounds on higher derivatives of the input maps are required.

The normalization changes the description of the surface, not the surface itself. Its geometric purpose is to balance the two columns that multiply the leading oscillations. Their determinant is fixed by the contact factor, but their individual lengths can be very different. A planar stretch first gives the first column length comparable to $\sqrt{m}$. A contact shear then removes the part of the second column parallel to the first. The determinant identity controls what remains.

The proof has two quantitative issues. The planar stretch must extend to a contact diffeomorphism of a whole slab, and both this extension and the shear must have usable inverse bounds. We establish the shear first, then construct the stretch and apply the shear with its precise amplitude and derivative bounds. The resulting coordinate maps are

$$\Xi_1 = \Xi \circ \mathcal{T}^{-1}, \quad \tilde{\Xi} = \Xi_1 \circ \Theta, \quad \tilde{P} = \mathcal{L} \circ P.$$

Here $\mathcal{T}$ extends the planar stretch $\mathcal{L}$, and Θ fixes the plane. These identities will also show directly that $\tilde{\Xi} \circ \iota \circ \tilde{P} = F$.

The original data provide only two derivatives of Ξ and one of μ . We smooth the scalar rescaling and shear coefficients before asking for their second and third derivatives. At each step the tables summarize the calculations immediately preceding or following them; their powers are not extra hypotheses.

5.1. A contact shear with separate amplitude and derivative bounds. The shear must subtract a possibly large multiple of one column from another. That large coefficient affects its first derivative. The width on which the map is used, however, is determined by derivatives of the coefficient. Keeping these two quantities separate saves many powers.

Lemma 5.2 (A contact shear fixing the vertical plane). *There are constants $c > 0$ and $C \geq 1$ with the following property. Let $q \in C_c^\infty(\mathbb{R}^2)$ and $1 \leq M \leq K$ satisfy*

$$\|q\|_{L^\infty(\mathbb{R}^2)} \leq M, \quad \|D^j q\|_{L^\infty(\mathbb{R}^2)} \leq K^j \quad (1 \leq j \leq 3),$$

and put $d = c/K$. Then there exist a smooth diffeomorphism $\Theta : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ and a smooth positive function $\kappa : \mathbb{R}^3 \rightarrow (0, \infty)$ such that $\Theta^ \lambda = \kappa \lambda$ on $\mathbb{R}^3$.*

At every point $(x, 0, t)$ of the vertical plane, the map and its differential satisfy

$$\begin{aligned} \Theta(x, 0, t) &= (x, 0, t), & \kappa(x, 0, t) &= 1, \\ D\Theta(x, 0, t)e_x &= e_x, & D\Theta(x, 0, t)e_t &= e_t, \\ D\Theta(x, 0, t)e_y &= e_y - q(x, t)e_x. \end{aligned}$$

Thus the map fixes the plane pointwise and changes its transverse derivative by the prescribed shear. On the whole space,

$$\begin{aligned} (5.1) \quad & \|D\Theta\|_{L^\infty(\mathbb{R}^3)} \leq CM, & \|D^2\Theta\|_{L^\infty(\mathbb{R}^3)} &\leq CMK, \\ & \|D\Theta^{-1}\|_{L^\infty(\mathbb{R}^3)} \leq CM, \\ & C^{-1} \leq \kappa \leq C, & \|D\kappa\|_{L^\infty(\mathbb{R}^3)} &\leq C. \end{aligned}$$

In addition, $\Theta(U_d) \subset U_{2d}$. The constants depend only on the fixed cutoff used in the proof; they are independent of q, M, K and of the size of $\text{supp } q$.

Proof. Step 1: the formulas.

The coefficient q should appear in the derivative transverse to the plane, while the map itself fixes the plane. We use a function which vanishes to second order there. The remaining components are chosen to make the unwanted transverse differential cancel in the contact form.

Fix $\rho \in C_c^\infty((-2, 2))$ equal to one on $[-1, 1]$, and put

$$f(y) = \frac{1}{2}y^2\rho(y/d), \quad H(y) = yf'(y) - f(y).$$

For an input (x, y, t) define, with $q = q(x, t)$,

$$(5.2) \quad \begin{aligned} \mathcal{A}(x, y, t) &= \left(x, \frac{y - f(y)q_s(x, t)}{1 + f(y)q_u(x, t)}, t \right), \\ \mathcal{B}(x, y, t) &= (x - f'(y)q(x, t), y, t - H(y)q(x, t)). \end{aligned}$$

Here q_s and q_u denote differentiation in the first and second arguments of q , respectively; those arguments are (x, t) in this formula. We will set $\Theta = \mathcal{B} \circ \mathcal{A}^{-1}$. The choice $H = yf' - f$ is useful because $H' = yf''$.

Step 2: the first map is close to the identity.

Every derivative of f satisfies $|f^{(i)}| \leq C_i d^{2-i}$; the same bound holds for $H^{(i)}$. For f' the exponent is $1 - i$. These functions vanish outside $|y| \leq 2d$. The middle component of $\mathcal{A} - \text{id}$ is

$$(5.3) \quad \frac{-fq_s - yfq_u}{1 + fq_u}.$$

The denominator lies between $3/4$ and $5/4$, since $|fq_u| \leq Cd^2K = Cc^2/K$. For a derivative of total order $j \leq 2$ of $f q_s$, put i derivatives on f and the other $j - i$ on q_s . Its bound is

$$Cd^{2-i}K^{1+j-i} = Cc^{2-i}K^{j-1}.$$

Terms from $-yfq_u$ have an additional power of d . In order one these bounds are Cc or Cc^2 ; hence the first derivative of (5.3) is at most $C(c + c^2)$. For completeness, write the numerator as $E = -fq_s - yfq_u$ and put $v = -fq_u$. The preceding product estimates give

	value	first derivative	second derivative
E	Cc^2/K	$C(c + c^2)$	CK
v	Cc^2/K	$C(c + c^2)$	CK

For example, a second y derivative of $f q_s$ has size CK , one y and one base derivative have size $CdK^2 = CcK$, and two base derivatives have size $Cd^2K^3 = Cc^2K$. The denominator is handled by

$$D(1 - v)^{-1} = (1 - v)^{-2}Dv, \quad D^2(1 - v)^{-1} = 2(1 - v)^{-3}Dv \otimes Dv + (1 - v)^{-2}D^2v.$$

Applying the product rule to $E(1 - v)^{-1}$ now shows that its first derivative is at most $C(c + c^2)$ and its second derivative is at most CK . Hence

$$\|D\mathcal{A} - I\| \leq 1/4, \quad \|D^2\mathcal{A}\| \leq CK$$

when c is sufficiently small. The constant in the second derivative estimate may depend on the fixed choice of c , but not on q, M, K . By Lemma 4.3, $\mathcal{A}$ is a global diffeomorphism, and

$$(5.4) \quad \|D\mathcal{A}^{-1}\| \leq C, \quad \|D^2\mathcal{A}^{-1}\| \leq CK.$$

Step 3: the second map is inverted on each plane of fixed y .

The map induced by $\mathcal{B}$ on (x, t) is the identity plus $(-f'q, -Hq)$. Its derivative in these two variables differs from the identity by at most $C(d^2 + d)K \leq 1/4$. It is therefore a global diffeomorphism for every fixed y . Here is a verification that these planar inverses fit together smoothly. Denote the planar map by $F_y(v)$, where $v = (x, t)$, and its inverse by J_y . Its inverse Lipschitz constant is at most $4/3$. Moreover

$$\partial_y F_y(v) = (-f''(y)q(x, t), -H'(y)q(x, t)), \quad |\partial_y F_y(v)| \leq CM.$$

Comparing $F_y(J_y(w)) = w$ and $F_{y'}(J_{y'}(w')) = w'$ gives

$$|J_y(w) - J_{y'}(w')| \leq \frac{4}{3}(|w - w'| + CM|y - y'|).$$

Thus the inverse is jointly continuous and locally Lipschitz. The difference-quotient argument in Lemma 4.3, now with the parameter y included, gives

$$D_w J_y = (D_v F_y \circ J_y)^{-1}, \quad \partial_y J_y = -(D_v F_y \circ J_y)^{-1}(\partial_y F_y \circ J_y).$$

Both right-hand sides are continuous. Differentiating these formulas repeatedly proves smoothness in all variables. Their bounds show $\|D\mathcal{B}^{-1}\| \leq CM$ on $\mathbb{R}^3$.

Direct differentiation of (5.2) gives

$$\|D^j \mathcal{B}\| \leq CMK^{j-1} \quad (1 \leq j \leq 2).$$

For example, if all j derivatives of $f'q$ fall on f' , the bound is $CMd^{1-j} \leq CMK^{j-1}$. If $j - i \geq 1$ derivatives fall on q , the bound is $Cd^{1-i}K^{j-i} \leq CK^{j-1}$, up to a fixed power of c . The central component has an extra power of d and is no larger. Composing with (5.4) proves the two derivative bounds for Θ . Since $\Theta^{-1} = \mathcal{A} \circ \mathcal{B}^{-1}$, it also proves the inverse bound.

Step 4: the contact identity and the slab.

The formulas were chosen so that the dy terms cancel. Keeping those terms visible gives

$$\begin{aligned} \mathcal{B}^* \lambda &= d(t - Hq) - y d(x - f'q) \\ &= dt - y dx + (-H' + yf'')q dy + (-H + yf') dq \\ &= dt - y dx + f dq. \end{aligned}$$

We used $H' = yf''$ and $H - yf' = -f$. Thus

$$\mathcal{B}^* \lambda = (1 + fq_u) dt - (y - fq_s) dx = (1 + fq_u) \mathcal{A}^* \lambda,$$

so $\kappa = (1 + fq_u) \circ \mathcal{A}^{-1}$. Before composition its first derivative has bound C : apply the derivative calculation in Step 2 with q_u in place of q_s . After composition, (5.4) gives the same bound. Positivity is already known. At $y = 0$, we have $f = f' = H = H' = 0$ and $f'' = 1$. Thus $\mathcal{A}$ and $\mathcal{B}$ fix the plane, $D\mathcal{A} = I$ there, and $D\mathcal{B}e_y = e_y - qe_x$. Their differentials fix e_t, e_x . This proves the stated differential of Θ and also gives $\kappa = 1$ on the plane. Finally (5.3) has size at most $Cd^2K \leq d/4$ after decreasing c . If $\mathcal{A}(x, y, t)$ has

middle coordinate of absolute value less than d , then $|y| < 5d/4$. The middle coordinate of $\mathcal{B}$ is y , so $\Theta(U_d) \subset U_{2d}$. $\square$

5.2. Proof of the normalization proposition.

Proof of Proposition 5.1. Step 1: choose and smooth the relative rescaling.

On the old plane put

$$B_1(s, u) = \partial_x \Xi^h(s, 0, u), \quad B_2(s, u) = \partial_y \Xi^h(s, 0, u), \quad m(s, u) = \mu(s, 0, u).$$

Differentiate the first equation of (3.1) in y and the second in x , subtract, and evaluate at $y = 0$. The mixed derivatives of Ξ^t and Ξ^x cancel, leaving

$$-(\partial_y \Xi^y)(\partial_x \Xi^x) + (\partial_x \Xi^y)(\partial_y \Xi^x) = -\mu.$$

Multiplying by -1 gives

$$(5.5) \quad \det(B_1, B_2) = m.$$

In particular, if $b_1 = |B_1|$, then $b_1 \geq m/\Gamma \geq \Gamma^{-2}$. The desired rescaling is $a_0 = b_1/\sqrt{m}$. Direct differentiation gives

$$(5.6) \quad \Gamma^{-3/2} \leq a_0 \leq \Gamma^{3/2}, \quad |Da_0| \leq C\Gamma^{7/2}, \quad |Da_0| \leq C\Gamma^3 a_0.$$

The value bounds retain the relation between b_1 and m :

$$\frac{\sqrt{m}}{\Gamma} \leq \frac{b_1}{\sqrt{m}} \leq \frac{\Gamma}{\sqrt{m}}.$$

Since $\Gamma^{-1} \leq m \leq \Gamma$, these are the two value bounds in (5.6). For the derivatives, $|Db_1| \leq |DB_1| \leq \Gamma$, so

$$|Da_0| \leq \Gamma m^{-1/2} + \frac{1}{2} b_1 m^{-3/2} |Dm| \leq C\Gamma^{7/2}.$$

For the relative derivative,

$$\frac{|Da_0|}{a_0} \leq \frac{|Db_1|}{b_1} + \frac{|Dm|}{2m} \leq \Gamma^3 + \frac{1}{2}\Gamma^2.$$

Only first derivatives of B_1, m have been used.

Let ζ equal one on $[-\Gamma, \Gamma]^2$, vanish outside $[-2\Gamma, 2\Gamma]^2$, and satisfy $0 \leq \zeta \leq 1$, $|D\zeta| \leq C/\Gamma$. Set $a = 1 + \zeta(a_0 - 1)$. Its values and its absolute first derivative have the bounds in (5.6). The relative derivative of a also satisfies $|Da| \leq C\Gamma^3 a$. To see this, use $a \geq \zeta a_0$ for the term ζDa_0 , and use

$$|D\zeta| |a_0 - 1| \leq C\Gamma^{1/2} \leq C\Gamma^2 a$$

for the other term, since $a \geq \Gamma^{-3/2}$.

We smooth at scale $\ell = c_1 \Gamma^{-3}$. By Lemma 4.1, choosing $c_1 > 0$ sufficiently small gives for $A = a * \varphi_\ell$

$$(5.7) \quad |A - a| \leq a/16, \quad |DA| \leq C\Gamma^3 A.$$

In particular, $15a_0/16 \leq A \leq 17a_0/16$ on $P(\overline{\mathbb{B}^2})$, and globally $A \leq C(1 + a_0)$. Positivity and the bounds for a are preserved exactly by averaging. The needed derivative bounds are

$$(5.8) \quad \begin{array}{c|ccccc} \text{quantity} & A & A^{-1} & DA & D^2A & D^3A \\ \hline \text{power of } \Gamma & 3/2 & 3/2 & 7/2 & 13/2 & 19/2. \end{array}$$

Here and below a table entry means an upper bound by a fixed constant times that power. For $j = 2, 3$, the bound for $D^j A$ is $C_j \ell^{1-j} \Gamma^{7/2}$. The relative estimate (5.7), rather than the absolute minimum of a , is what allows the smoothing scale Γ^{-3} .

Step 2: integrate the rescaling and control its inverse.

Define

$$h(s, u) = s + \int_{-\infty}^s (A(\xi, u) - 1) d\xi, \quad \mathcal{L}(s, u) = (h(s, u), u).$$

The function $A - 1$ is supported in a square of side CT . For each fixed u , the integral therefore involves an interval of length at most CT . Thus

$$(5.9) \quad \begin{array}{c|ccc} & j & 1 & 2 & 3 \\ \hline |\partial_u^j h| & \text{power} & 9/2 & 15/2 & 21/2 \\ |D^j h| & \text{power if an } s \text{ occurs} & 3/2 & 7/2 & 13/2. \end{array}$$

The first row integrates a derivative of A over length CT ; the second uses $h_s = A$. Since $h_s > 0$ and $h - s$ is bounded, $h(\cdot, u)$ is increasing and onto for every u . Write $\mathcal{J} = \mathcal{L}^{-1}$ as $\mathcal{J}(\xi, u) = (k(\xi, u), u)$. Before differentiating the inverse, we check its continuity. The lower bound $h_s = A \geq \Gamma^{-3/2}$ and the mean value estimate in u give

$$|k(\xi, u) - k(\xi', u')| \leq \Gamma^{3/2} (|\xi - \xi'| + \|h_u\|_\infty |u - u'|) \leq CT^6 |(\xi, u) - (\xi', u')|.$$

For the first inequality, compare $h(k(\xi', u'), u)$ with $h(k(\xi', u'), u') = \xi'$ and then invert the increasing function $h(\cdot, u)$. The equation $h(k(\xi, u), u) = \xi$ now gives

$$k_\xi = (A^{-1}) \circ \mathcal{J}, \quad k_u = -(h_u/A) \circ \mathcal{J}.$$

Indeed, expand the inverse equation at (ξ, u) ; the Lipschitz bound just proved controls the remainder in this expansion. Solving the resulting linear equation gives the two displayed derivatives. They are continuous, and repeated differentiation proves smoothness. In particular $|D\mathcal{J}| \leq CT^6$.

For the second derivative it is useful to divide by A before estimating. For constant directions v, w , write $k_v = Dk[v]$, $u_v = v_u$. Twice differentiating the inverse equation gives

$$Ak_{vw} = -h_{ss}k_vk_w - h_{su}(k_vu_w + k_wu_v) - h_{uu}u_vu_w,$$

where all coefficients on the right are evaluated at $\mathcal{J}(\xi, u)$. By (5.7), $|h_{ss}|/A + |h_{su}|/A \leq CT^3$, whereas $|h_{uu}|/A \leq CT^9$. For unit v, w the three terms are therefore bounded by CT^{15} , CT^9 , and CT^9 . We have proved

$$(5.10) \quad \|D\mathcal{J}\| \leq CT^6, \quad \|D^2\mathcal{J}\| \leq CT^{15}.$$

No third derivative of the inverse will be needed.

Step 3: extend the planar change to a contact map.

The planar stretch must act on a neighborhood of the plane. Keeping $\bar{t} = t$ and setting $\bar{x} = h(x, t)$ leaves only the middle coordinate to determine. If its new value is $\bar{y}$, the contact identity requires

$$dt - \bar{y}(A dx + h_u dt) = \kappa(dt - y dx).$$

Equating the dt and dx coefficients gives $\kappa = 1 - \bar{y}h_u$ and $\bar{y}A = \kappa y$. Here $A = h_s$ and h_u are evaluated at $(s, u) = (x, t)$; the subscripts continue to denote the first and second arguments. Solving for $\bar{y}$ gives the formula below. For small fixed $c_2, c_3 > 0$, set $d_0 = c_2\Gamma^{-6}$ and $d_1 = c_3\Gamma^{-15/2}$. On U_{d_0} define

$$(5.11) \quad \mathcal{T}(x, y, t) = \left(h(x, t), \frac{y}{A(x, t) + y h_u(x, t)}, t \right).$$

On U_{d_1} its inverse is

$$(5.12) \quad x = k(\bar{x}, \bar{t}), \quad y = \frac{\bar{y}A(x, t)}{1 - \bar{y}h_u(x, t)}, \quad t = \bar{t}.$$

The forward denominator is at least $3A/4$, since $d_0\|h_u\| \leq Cc_2\Gamma^{-3/2} \leq Cc_2A$. The inverse denominator lies between $3/4$ and $5/4$, since $d_1\|h_u\| \leq Cc_3\Gamma^{-3}$. Moreover $|y| \leq Cc_3\Gamma^{-6} < d_0/2$ if c_3 is sufficiently small relative to c_2 . Thus the inverse is defined on the whole new slab and takes values in $U_{d_0} \subset U_\delta$. The same formulas invert $\mathcal{T}$ on its full image: there the inverse denominator equals $A/(A + yh_u) > 0$.

Writing $\bar{y} = y/(A + yh_u)$ and $dh = A dx + h_u dt$, we find

$$d\bar{t} - \bar{y} d\bar{x} = dt - \frac{y}{A + yh_u}(A dx + h_u dt) = \frac{A}{A + yh_u}(dt - y dx).$$

Hence

$$(5.13) \quad \mathcal{T}^*\lambda = \frac{A}{A + yh_u}\lambda.$$

We give the derivative calculation on the whole inverse slab. Put $\mathcal{A}_0 = A \circ \mathcal{J}$ and $\mathcal{H}_0 = h_u \circ \mathcal{J}$, as functions of $(\bar{x}, \bar{t})$. Equivalently, the arguments (ξ, u) of $\mathcal{J}$ are $(\bar{x}, \bar{t})$ here, so Du below is the differential of the second argument. Their bounds are

$$(5.14) \quad \begin{array}{c|ccc} \text{order} & 0 & 1 & 2 \\ \hline \mathcal{A}_0 & 3/2 & 19/2 & 37/2 \\ \mathcal{H}_0 & 9/2 & 19/2 & 37/2. \end{array}$$

For $\mathcal{A}_0$, the chain rule gives the first power $7/2+6 = 19/2$ and the second powers $13/2+12 = 37/2$ and $7/2 + 15 = 37/2$. For $\mathcal{H}_0$, first use

$$D\mathcal{H}_0 = (A_u \circ \mathcal{J})Dk + (h_{uu} \circ \mathcal{J})Du.$$

Its powers are $7/2+6$ and $15/2$. At second order, we expand the two arguments separately. This matters because the second component of $\mathcal{J}$ is exactly u and has no second derivative.

With the directional notation from Step 2,

$$\begin{aligned} D^2\mathcal{H}_0[v, w] &= (A_{su} \circ \mathcal{J})k_vk_w \\ &\quad + (A_{uu} \circ \mathcal{J})(k_vu_w + k_wu_v) \\ &\quad + (h_{uuu} \circ \mathcal{J})u_vu_w + (A_u \circ \mathcal{J})k_{vw}. \end{aligned}$$

For unit v, w , the four terms have powers at most

$$\frac{13}{2} + 12 = \frac{37}{2}, \quad \frac{13}{2} + 6 = \frac{25}{2}, \quad \frac{21}{2}, \quad \frac{7}{2} + 15 = \frac{37}{2},$$

respectively. This proves the second row of (5.14), including all mixed derivatives.

The small transverse coordinate now offsets these large derivatives. Let

$$N = \bar{y}\mathcal{A}_0, \quad v = \bar{y}\mathcal{H}_0, \quad R = (1 - v)^{-1}.$$

On $|\bar{y}| < d_1$, the product and reciprocal rules give

order	0	1	2
N	−6	2	11
v	−3	9/2	11
R	0	9/2	11
NR	−6	2	11.

For example, two base derivatives of N have power $-15/2 + 37/2 = 11$, while one transverse and one base derivative have power $19/2$. The corresponding powers for v are the same at second order. At first order its transverse derivative has power $9/2$. Since $|v| \leq 1/4$, the second reciprocal derivative is bounded by $C(|Dv|^2 + |D^2v|)$, whose powers are 9 and 11. Finally the second derivative of NR uses powers 11, $2 + 9/2$, and $-6 + 11$, all at most 11. Here are also the lower-order entries. For a base index i ,

$$\begin{aligned} N_{\bar{y}} &= \mathcal{A}_0, & N_i &= \bar{y} \partial_i \mathcal{A}_0, & N_{\bar{y}\bar{y}} &= 0, \\ v_{\bar{y}} &= \mathcal{H}_0, & v_i &= \bar{y} \partial_i \mathcal{H}_0, & v_{\bar{y}\bar{y}} &= 0. \end{aligned}$$

The values of N, v have powers $-15/2 + 3/2 = -6$ and $-15/2 + 9/2 = -3$. The first derivatives of N have powers $3/2$ and $-15/2 + 19/2 = 2$; those of v have powers $9/2$ and 2 . Since R is bounded, $D(NR) = RDN + NDR$ has powers at most 2 and $-6 + 9/2 = -3/2$. This verifies every entry of the table. Combining (5.12) and (5.10) therefore gives

$$(5.16) \quad \|D\mathcal{T}^{-1}\| \leq C\Gamma^6, \quad \|D^2\mathcal{T}^{-1}\| \leq C\Gamma^{15}.$$

For later separation we need a forward Lipschitz bound as well. In the middle component of (5.11) the transverse derivative is $A/(A + yh_u)^2 \leq C/A$. The base derivatives are bounded by $C(|y||DA| + y^2|Dh_u|)/A^2$. Their powers are at most $-6 + 7/2 + 3 = 1/2$ and $-12 + 15/2 + 3 = -3/2$. The first component has derivative at most $C\Gamma^{9/2}$. Thus

$$(5.17) \quad \|D\mathcal{T}\| \leq C\Gamma^{9/2} \quad \text{on } U_{d_0}.$$

Step 4: record the stretched coordinates.

Define on U_{d_1}

$$\Xi_1 = \Xi \circ \mathcal{T}^{-1}, \quad P_1 = \mathcal{L} \circ P, \quad \mu_1 = (\mu \circ \mathcal{T}^{-1})R.$$

The factor R is the inverse contact factor by (5.13): substituting the inverse formula gives

$$\left(\frac{A + y h_u}{A} \right) \circ \mathcal{T}^{-1} = \frac{1}{1 - \bar{y} \mathcal{H}_0} = R.$$

Also $\mathcal{T} \circ \iota = \iota \circ \mathcal{L}$, so these data represent the original map on all of $\mathbb{R}^2$. The two chain rules give

$$(5.18) \quad \begin{aligned} \|D\Xi_1\| &\leq C\Gamma^7, & \|D^2\Xi_1\| &\leq C(\Gamma^{13} + \Gamma^{16}) \leq C\Gamma^{16}, \\ (2\Gamma)^{-1} &\leq \mu_1 \leq 2\Gamma, & \|D\mu_1\| &\leq C(\Gamma^7 + \Gamma^{11/2}) \leq C\Gamma^7, \\ \|DP_1\| &\leq C\Gamma^{11/2}, & \|P_1\|_{L^\infty(\overline{\mathbb{B}^2})} &\leq C\Gamma^{5/2}. \end{aligned}$$

The inverse derivative of P_1 is bounded by $C\Gamma^7$: compose P^{-1} , whose derivative is at most Γ by (3.4), with $\mathcal{J}$. Its domain is the whole plane, so this gives the inequality for every pair. For the ambient coordinates, take $\xi, \xi' \in U_{d_1}$, and put $w = \mathcal{T}^{-1}(\xi)$, $w' = \mathcal{T}^{-1}(\xi')$. Both lie in the convex slab U_{d_0} , where the forward derivative bound for $\mathcal{T}$ holds. Equations (5.17) and (3.4) give

$$|\xi - \xi'| \leq C\Gamma^{9/2} |w - w'| \leq C\Gamma^{11/2} |\Xi_1(\xi) - \Xi_1(\xi')|.$$

This is a bound for every pair, not just a bound for a local inverse.

Step 5: choose the shear coefficient.

At this step we distinguish explicitly the old and new plane variables. Define

$$\begin{aligned} B'_1(\xi, u) &= \partial_x \Xi_1^h(\xi, 0, u), & B'_2(\xi, u) &= \partial_y \Xi_1^h(\xi, 0, u), \\ m_1(\xi, u) &= \mu_1(\xi, 0, u) = m(\mathcal{J}(\xi, u)). \end{aligned}$$

Differentiation of (5.12) on the plane gives

$$B'_1 \circ \mathcal{L} = B_1/A, \quad B'_2 \circ \mathcal{L} = AB_2.$$

For the first identity, varying ξ changes x by $1/A$; for the second, varying $\bar{y}$ changes y by A at $\bar{y} = 0$. All other contributions in these two directions vanish on the plane. On $P_1(\overline{\mathbb{B}^2})$, therefore,

$$(5.19) \quad \frac{1}{2}\sqrt{m_1} \leq |B'_1| \leq 2\sqrt{m_1}.$$

The unwanted parallel coefficient is

$$q_0(\xi, u) = (A^2 c_{\parallel}) (\mathcal{J}(\xi, u)), \quad c_{\parallel} = \frac{\langle B_2, B_1 \rangle}{b_1^2}.$$

Here is its derivative estimate without differentiating Ξ_1 twice. For clarity, the quotient rule here is

$$Dc_{\parallel} = \frac{\langle DB_2, B_1 \rangle + \langle B_2, DB_1 \rangle}{b_1^2} - 2 \frac{\langle B_2, B_1 \rangle}{b_1^3} Db_1.$$

Consequently the original data give

$$|c_{\parallel}| \leq \frac{\Gamma}{b_1}, \quad |Dc_{\parallel}| \leq C \left(\frac{\Gamma}{b_1} + \frac{\Gamma^2}{b_1^2} \right).$$

Since $A \leq C(1 + b_1/\sqrt{m})$, $b_1 \geq m/\Gamma$, $b_1 \geq \Gamma^{-2}$, and $m \geq \Gamma^{-1}$, we get

$$A^2 |c_{\parallel}| \leq C\Gamma^3, \quad A^2 |Dc_{\parallel}| \leq C\Gamma^6.$$

For example, $(1+b_1^2/m)\Gamma/b_1 = \Gamma/b_1 + \Gamma b_1/m \leq 2\Gamma^3$, and $(1+b_1^2/m)\Gamma^2/b_1^2 = \Gamma^2/b_1^2 + \Gamma^2/m \leq 2\Gamma^6$. The cross terms are bounded by these using $2ab \leq a^2 + b^2$. Together with (5.7), this proves

$$|D(A^2 c_{\parallel})| \leq 2 \frac{|DA|}{A} A^2 |c_{\parallel}| + A^2 |Dc_{\parallel}| \leq C\Gamma^6.$$

Composing with $\mathcal{J}$ yields $|q_0| \leq C\Gamma^3$ and $|Dq_0| \leq C\Gamma^{12}$.

Cut off q_0 outside a square of radius $C\Gamma^{5/2}$, with the cutoff one on a smaller square containing $P_1(\mathbb{B}^2)$. Its first derivative is $O(\Gamma^{-5/2})$, so these bounds persist. Mollify at scale $c_4\Gamma^{-12}$ and call the result q . The absolute-error version of Lemma 4.1, with c_4 small enough, gives

$$(5.20) \quad |q - q_0| \leq 1/8 \text{ on } P_1(\mathbb{B}^2), \quad \|q\| \leq C\Gamma^3, \quad \|D^j q\| \leq C_j \Gamma^{12j} \quad (1 \leq j \leq 3).$$

Thus Lemma 5.2 applies with $M = C\Gamma^3$ and $K = C'\Gamma^{12}$. Enlarge C' so that $K \geq M$ and the resulting $d = c/K$ satisfies $2d < d_1$.

Step 6: compose the shear and finish the estimate.

Let Θ, κ be supplied by that lemma, and put

$$\tilde{\Xi} = \Xi_1 \circ \Theta : U_d \rightarrow \mathbb{R}^3, \quad \tilde{P} = P_1, \quad \tilde{\mu} = (\mu_1 \circ \Theta)\kappa.$$

The inclusion $\Theta(U_d) \subset U_{2d} \subset U_{d_1}$ makes these definitions valid. The contact identity follows by composition. Since Θ fixes the plane, the represented map is still F . The two derivative bounds for Θ are $C\Gamma^3$ and $C\Gamma^{15}$. Consequently

$$\|D\tilde{\Xi}\| \leq C\Gamma^{10}, \quad \|D^2\tilde{\Xi}\| \leq C(\Gamma^{16}\Gamma^6 + \Gamma^7\Gamma^{15}) \leq C\Gamma^{22}.$$

Also $\tilde{\mu} \geq c/\Gamma$, $\|\tilde{\mu}\| \leq C\Gamma$, and $\|D\tilde{\mu}\| \leq C(\Gamma^7\Gamma^3 + \Gamma) \leq C\Gamma^{10}$. The reciprocal width is $C\Gamma^{12}$. For $\xi, v \in U_d$, the global inverse bound for Θ and the pair bound from Step 4 give

$$\begin{aligned} |\xi - v| &\leq C\Gamma^3 |\Theta(\xi) - \Theta(v)| \\ &\leq C\Gamma^{3+11/2} |\tilde{\Xi}(\xi) - \tilde{\Xi}(v)|. \end{aligned}$$

Thus the ambient inverse inequality in (3.4) holds with constant $C\Gamma^{17/2}$. The bounds for $D\tilde{P}$, $\tilde{P}|_{\mathbb{B}^2}$, and the inverse of $\tilde{P}$ remain those obtained for P_1 in Step 4. Set $\tilde{\delta} = d$. The width, contact factor, derivative estimates, and both inverse inequalities therefore satisfy Definition 3.1 with a common bound $C\Gamma^{22}$.

Finally the new horizontal columns on the plane are B'_1 and $B'_2 - qB'_1$. Their determinant is still m_1 . By definition of q_0 , the perpendicular component of the second column has length $m_1/|B'_1| \leq 2\sqrt{m_1}$, and its parallel component has length $|q - q_0||B'_1| \leq \sqrt{m_1}/4$. Together with (5.19) this gives a sum of lengths at most $(2 + 2 + 1/4)\sqrt{m_1} < 8\sqrt{m_1}$. On the new plane $\tilde{\mu}(\xi, 0, u) = m_1(\xi, u)$, so this is exactly (3.5). The normalization proposition follows. $\square$

6. THE EMBEDDED GRAPH AND ITS FLATTENING

6.1. Periodic functions with the required cancellation. The next profiles are chosen so that a certain scalar function ψ is strictly increasing. This will prove that the corrected plane remains an embedded graph, even though its derivatives may be large. Positivity

of the second profile will also keep the graph inside one half-space, where its contact coordinates have an explicit inverse. The quadratic cancellation follows the corrugation mechanism of [11, Lemma 3.3 and Proposition 3.5]; the positive profiles below enforce these additional graph properties.

Lemma 6.1 (Periodic profiles). *There exist a constant $c_0 > 0$, a smooth 1-periodic function $\eta : \mathbb{R} \rightarrow \mathbb{R}$, and, for every $0 < \theta \leq 1/4$, smooth 1-periodic functions $\zeta_\theta, \chi_\theta : \mathbb{R} \rightarrow \mathbb{R}$, with the following properties. At every $u \in \mathbb{R}$,*

$$(6.1) \quad \begin{aligned} c_0\theta \leq \zeta_\theta(u) \leq 2, \quad \chi'_\theta(u) &= \zeta_\theta(u)\eta'(u) - 1, \\ \zeta_\theta(u)\eta'(u) &\geq -\theta/8. \end{aligned}$$

Moreover, $\eta(0) = \chi_\theta(0) = 0$. For every integer $j \geq 0$ there is a constant $C_j < \infty$, independent of θ , such that

$$\|\eta\|_{C^j(\mathbb{R})} + \|\zeta_\theta\|_{C^j(\mathbb{R})} + \|\chi_\theta\|_{C^j(\mathbb{R})} \leq C_j.$$

In particular, η is chosen once and does not change with θ .

Proof. The derivative η' will have a positive part and a negative part. We make ζ_θ large on the positive part and small on the negative part, so the weighted derivative has positive average and only a small negative part. Choose a nonnegative smooth bump ϱ , supported in $(-1/16, 1/16)$, with integral one; it can be constructed by the formula given after Lemma 4.1. Periodize its translates to $1/4$ and $3/4$ to obtain a and b . Their supports on the circle $\mathbb{R}/\mathbb{Z}$ are disjoint, and their integrals on $[0, 1]$ are one. Choose a smooth periodic ρ , with $0 \leq \rho \leq 1$, equal to one near the support of a and zero near that of b . Such a function is obtained using the same smooth transition function on the two intervening intervals of the circle. Set

$$E(u) = \int_0^u (a(v) - b(v)) dv, \quad \eta(u) = E(u).$$

Since the integral of $a - b$ over a period is zero, E and η are periodic, and $\eta' = a - b$. Choose $0 < c_0 \leq 1$ so that $c_0\|b\|_\infty \leq 1/8$, and set $\zeta_\theta = \rho + c_0\theta$. The integral needed for a periodic primitive is

$$\int_0^1 \zeta_\theta \eta' = \int_0^1 \rho a - \int_0^1 \rho b + c_0\theta \left(\int_0^1 a - \int_0^1 b \right) = 1 - 0 + 0 = 1.$$

Thus $E_\theta(u) = \int_0^u (\zeta_\theta \eta' - 1)$ is periodic, and define $\chi_\theta(u) = E_\theta(u)$. This gives the claimed differential identity and fixes its constant. The product $\zeta_\theta \eta'$ is nonnegative on $\text{supp } a$, equals $-c_0\theta b$ on $\text{supp } b$, and is zero elsewhere. This proves (6.1). To verify the uniform norms, note that η is fixed and $\zeta_\theta = \rho + c_0\theta$, with $0 < \theta \leq 1/4$. All derivatives of these functions are uniformly bounded. Derivatives of χ_θ of positive order are derivatives of $\zeta_\theta \eta' - 1$ of one lower order, so are uniformly bounded by the product rule. Its value is bounded by the $L^1(0, 1)$ norm of that function, using the explicit primitive. This proves every assertion of the lemma. $\square$

6.2. The corrected plane and its explicit flattening. The next formula is the entire oscillatory construction. The remaining work will be to express this corrected plane in the old coordinates and to verify that another correction can follow it.

Lemma 6.2 (Graph and contact coordinates). *Fix the profiles from Lemma 6.1. Let $0 < \theta \leq 1/4$ and $0 < \varepsilon \leq 1$, and define*

$$L = \frac{1 - \theta}{\varepsilon^2}, \quad \psi(u) = u + \varepsilon^2 \chi_\theta(Lu).$$

The number L need not be an integer. There are constants $c_ > 0$ and $C \geq 1$, independent of θ, ε , for which the following statements hold.*

(i) The corrected plane. *The function $\psi : \mathbb{R} \rightarrow \mathbb{R}$ is a smooth increasing diffeomorphism, with $\psi' \geq 7\theta/8$. The map*

$$(6.2) \quad \mathcal{G}(s, u) = (s + \varepsilon \eta(Lu), \varepsilon \zeta_\theta(Lu), \psi(u))$$

is a smooth embedding of $\mathbb{R}^2$ into $\mathbb{R}^3$. Its image is the graph

$$\{(x, \varepsilon \zeta_\theta(L\psi^{-1}(t)), t) : x, t \in \mathbb{R}\}$$

over the vertical plane $\{y = 0\}$, and its contact defect is

$$(6.3) \quad \mathcal{G}^* \lambda = \theta du - \varepsilon \zeta_\theta(Lu) ds.$$

(ii) Ambient and planar coordinates for this graph. *Put $d = c_* \varepsilon \theta$. There exist smooth diffeomorphisms*

$$\Phi : U_d \longrightarrow \Phi(U_d), \quad B : \mathbb{R}^2 \longrightarrow \mathbb{R}^2,$$

and a smooth positive function κ on U_d satisfying

$$\Phi \circ \iota \circ B = \mathcal{G} \quad \text{on } \mathbb{R}^2, \quad \Phi^* \lambda = \kappa \lambda \quad \text{on } U_d, \quad \det DB = 1.$$

The image $\Phi(U_d)$ lies in the convex slab

$$(6.4) \quad V = \mathbb{R} \times (c_0 \theta \varepsilon / 2, 4\varepsilon) \times \mathbb{R}.$$

(iii) Bounds for these coordinate maps. *On the entire slab U_d ,*

$$(6.5) \quad \begin{aligned} \|D\Phi\|_{L^\infty(U_d)} &\leq C(\varepsilon^{-1} + \theta^{-1}), & \|D^2\Phi\|_{L^\infty(U_d)} &\leq C\varepsilon^{-1}\theta^{-2}, \\ c_0\theta/2 &\leq \kappa \leq 4, & \|D\kappa\|_{L^\infty(U_d)} &\leq C\varepsilon^{-1}\theta^{-1}. \end{aligned}$$

On the entire parameter plane,

$$(6.6) \quad \begin{aligned} \|DB\|_{L^\infty(\mathbb{R}^2)} + \|DB^{-1}\|_{L^\infty(\mathbb{R}^2)} &\leq C/\varepsilon, \\ |B(s, u)| &\leq C(|s| + |u|)/\varepsilon \quad ((s, u) \in \mathbb{R}^2). \end{aligned}$$

(iv) Extension of the inverse. *The inverse Φ^{-1} extends to a smooth diffeomorphism K from the half-space $\{y > 0\}$ onto its image. Thus $K|_{\Phi(U_d)} = \Phi^{-1}$. It satisfies*

$$|K(p) - K(q)| \leq C\varepsilon^{-1}\theta^{-2}|p - q| \quad (p, q \in V).$$

This estimate holds on the whole slab V , including pairs whose joining segment leaves $\Phi(U_d)$.

Proof. Step 1: compute the defect and prove the graph is embedded.

The central-coordinate correction cancels the order-one oscillation in the contact form. Suppress the subscript θ on ζ, χ . From $\chi' = \zeta\eta' - 1$,

$$\psi'(u) = \theta + (1 - \theta)\zeta(Lu)\eta'(Lu) \geq 7\theta/8.$$

Indeed $\zeta\eta' \geq -\theta/8$ gives the stronger lower bound $\theta - (1 - \theta)\theta/8$. Also ψ' is bounded above by a fixed constant. Thus ψ is strictly increasing and onto, since $\psi(u) - u$ is bounded. It has a smooth inverse with derivative $1/\psi'$ at the corresponding point.

Equality of two images under $\mathcal{G}$ first gives equality of u from the central coordinate, then equality of s from the x -coordinate. At an image point (x, y, t) the inverse is

$$u = \psi^{-1}(t), \quad s = x - \varepsilon\eta(Lu).$$

The derivative columns are independent: the s column is $(1, 0, 0)$, while the u column has positive central component $\psi'(u)$. This proves smooth embeddedness. It also identifies the image as

$$\left\{ (x, \varepsilon\zeta(L\psi^{-1}(t)), t) : x, t \in \mathbb{R} \right\},$$

an ordinary graph over the vertical (x, t) -plane. Direct differentiation gives

$$\mathcal{G}^*\lambda = \psi'(u) du - \varepsilon\zeta(Lu)(ds + \varepsilon L\eta'(Lu) du) = \theta du - \varepsilon\zeta(Lu) ds.$$

Step 2: choose coordinates from the phase parameter.

We make the new horizontal coordinate equal to u/ε . This avoids repeated derivatives of ψ^{-1} in the forward coordinate map. Write

$$\nu(u) = \psi'(u), \quad H(u) = \int_0^u \frac{\theta}{\zeta(Lv)} dv.$$

The letter ν denotes a scalar function, and primes on it denote u derivatives. The primitive H is suggested by the contact form on the graph:

$$\frac{\mathcal{G}^*\lambda}{\zeta(Lu)} = \frac{\theta}{\zeta(Lu)} du - \varepsilon ds = d(H(u) - \varepsilon s).$$

Thus $H(u) - \varepsilon s$ is a natural new central coordinate on the surface. Together with the horizontal coordinate u/ε , it will give an explicit flattening. Here $\theta/2 \leq H' \leq 1/c_0$ and $|H(u)| \leq C|u|$.

For an ambient input (x, y, t) , set $u = \varepsilon x$ and define

$$\begin{aligned} \Phi^x(x, y, t) &= \varepsilon\eta(Lu) + \frac{H(u) - t}{\varepsilon}, \\ (6.7) \quad \Phi^y(x, y, t) &= \frac{\varepsilon\zeta(Lu)}{1 - (y/\varepsilon)\zeta(Lu)/\nu(u)}, \\ \Phi^t(x, y, t) &= \psi(u). \end{aligned}$$

The superscripts specify the output coordinates of Φ ; its input coordinate t is distinct from the phase parameter u . For $|y| < d = c_*\varepsilon\theta$, choose for example $c_* = 1/100$. Then

$$\left| \frac{y\zeta(Lu)}{\varepsilon\nu(u)} \right| \leq \frac{16c_*}{7} < \frac{1}{4}.$$

The denominator in (6.7) lies between $3/4$ and $5/4$. In particular, putting

$$(6.8) \quad \kappa = \frac{\zeta(Lu)}{1 - (y/\varepsilon)\zeta(Lu)/\nu(u)} = \Phi^y/\varepsilon,$$

we have $c_0\theta/2 \leq \kappa \leq 4$ and $\Phi(U_d) \subset V$.

We check the contact identity explicitly. The relation $\nu = \theta + (1 - \theta)\zeta\eta'$ implies

$$d\Phi^t = \varepsilon\nu dx, \quad d\Phi^x = \frac{\nu}{\zeta} dx - \frac{dt}{\varepsilon}.$$

Consequently,

$$\Phi^*\lambda = d\Phi^t - \Phi^y d\Phi^x = \kappa dt + \varepsilon\nu(1 - \kappa/\zeta) dx = \kappa(dt - y dx).$$

The last equality follows from $\kappa(1 - y\zeta/(\varepsilon\nu)) = \zeta$.

Define the planar change by

$$(6.9) \quad B(s, u) = (u/\varepsilon, H(u) - \varepsilon s).$$

Since $\iota(s, u) = (s, 0, u)$, substitution in (6.7) gives $\Phi \circ \iota \circ B = \mathcal{G}$. At a target (ξ, v) , the inverse is $u = \varepsilon\xi$ and $s = (H(\varepsilon\xi) - v)/\varepsilon$. Thus B is a global smooth diffeomorphism. Its derivative and inverse derivative at corresponding points are

$$DB = \begin{pmatrix} 0 & \varepsilon^{-1} \\ -\varepsilon & H' \end{pmatrix}, \quad DB^{-1} = \begin{pmatrix} H' & -\varepsilon^{-1} \\ \varepsilon & 0 \end{pmatrix}.$$

In particular, $\det DB = 1$ and the derivative bounds in (6.6) follow from $|H'| \leq 1/c_0$. The value bound follows from $|H(u)| \leq C|u|$ and $\varepsilon \leq 1$. This change has no additional inverse power of θ .

Step 3: give the inverse and the estimate for every pair.

For any point (x, y, t) with $y > 0$, put $u = \psi^{-1}(t)$ and set

$$(6.10) \quad K(x, y, t) = \left(\frac{u}{\varepsilon}, \frac{\varepsilon\nu(u)}{\zeta(Lu)} - \frac{\varepsilon^2\nu(u)}{y}, H(u) + \varepsilon^2\eta(Lu) - \varepsilon x \right).$$

To verify the inverse, denote the recovered input of Φ by $(\bar{x}, \bar{y}, \bar{t})$. The last equation in (6.7) determines $u = \psi^{-1}(t)$, hence $\bar{x} = u/\varepsilon$. The first gives $\bar{t} = H(u) + \varepsilon^2\eta(Lu) - \varepsilon x$. The middle equation is equivalent to

$$1 - \frac{\bar{y}\zeta(Lu)}{\varepsilon\nu(u)} = \frac{\varepsilon\zeta(Lu)}{y},$$

which gives the middle component of K . Conversely, substitution of these variables into Φ returns (x, y, t) . These substitutions also give $K \circ \Phi = \text{id}$ wherever the denominator in (6.7) is positive. More explicitly, K maps $\{y > 0\}$ onto

$$\left\{ (x, y, t) : y < \frac{\varepsilon\nu(\varepsilon x)}{\zeta(L\varepsilon x)} \right\}.$$

The formula for Φ is smooth on this set and has positive denominator. This proves the global extension and smooth invertibility on U_d .

There is a useful cancellation in differentiating the central component of K :

$$dK^t = \frac{dt}{\zeta(Lu)} - \varepsilon dx, \quad dK^x = \frac{dt}{\varepsilon\nu(u)}.$$

The first identity uses $H' + \varepsilon^2 L\eta' = \nu/\zeta$ and $du = dt/\nu$. These formulas also verify the inverse contact identity:

$$\begin{aligned} K^* \lambda &= dK^t - K^y dK^x \\ &= \frac{dt}{\zeta} - \varepsilon dx - \left(\frac{\varepsilon\nu}{\zeta} - \frac{\varepsilon^2\nu}{y} \right) \frac{dt}{\varepsilon\nu} \\ &= \frac{\varepsilon}{y} (dt - y dx). \end{aligned}$$

For the derivative bounds on V , use $y \geq c_0\theta\varepsilon/2$, $\nu \geq 7\theta/8$, $|\nu'| \leq C\varepsilon^{-2}$, and $|\zeta'| \leq C$. The remaining nonzero derivatives of the middle component are

$$\begin{aligned} \partial_y K^y &= \frac{\varepsilon^2\nu(u)}{y^2}, \\ \partial_t K^y &= \varepsilon \frac{\nu'(u)}{\nu(u)} \left(\frac{1}{\zeta(Lu)} - \frac{\varepsilon}{y} \right) - \frac{\varepsilon L\zeta'(Lu)}{\zeta(Lu)^2}. \end{aligned}$$

Their bounds are $C\theta^{-2}$ and $C\varepsilon^{-1}\theta^{-2}$, respectively. All first derivatives of K^x, K^t have the latter bound as well. The slab V is convex, so integrating DK along any segment in V gives the stated Lipschitz inequality for every pair. The segment need not lie in the smaller set $\Phi(U_d)$.

Step 4: estimate two derivatives of the forward coordinates.

We retain the factor ε in front of Φ^y when differentiating. Write $\zeta_*(x) = \zeta(L\varepsilon x)$, $\nu_*(x) = \nu(\varepsilon x)$, and $q = \zeta_*/\nu_*$. Both ζ_*, ν_* have first and second derivative bounds $C\varepsilon^{-1}, C\varepsilon^{-2}$. Their values are bounded, and $\nu_* \geq 7\theta/8$. With primes in the next display denoting x derivatives, the quotient rule gives

$$\begin{aligned} q' &= \frac{\zeta'_*}{\nu_*} - \frac{\zeta_*\nu'_*}{\nu_*^2}, \\ q'' &= \frac{\zeta''_*}{\nu_*} - \frac{2\zeta'_*\nu'_* + \zeta_*\nu''_*}{\nu_*^2} + \frac{2\zeta_*(\nu'_*)^2}{\nu_*^3}. \end{aligned}$$

Using the lower bound term by term yields

$$|q| \leq C\theta^{-1}, \quad |q'| \leq C\varepsilon^{-1}\theta^{-2}, \quad |q''| \leq C\varepsilon^{-2}\theta^{-3}.$$

Put $v = (y/\varepsilon)q(x)$, so $\kappa = \zeta_*(1-v)^{-1}$. The small range of y on U_d gives

$$|v| \leq 1/4, \quad |Dv| \leq C\varepsilon^{-1}\theta^{-1}, \quad |D^2v| \leq C\varepsilon^{-2}\theta^{-2}.$$

Indeed $v_y = q/\varepsilon$, $v_x = (y/\varepsilon)q'$, $v_{xy} = q'/\varepsilon$, $v_{xx} = (y/\varepsilon)q''$, and $v_{yy} = 0$. The reciprocal and product rules give

$$|D\kappa| \leq C\varepsilon^{-1}\theta^{-1}, \quad |D^2\kappa| \leq C\varepsilon^{-2}\theta^{-2}.$$

The second estimate is used here to bound $\Phi^y = \varepsilon\kappa$; it is not retained as contact-factor data in the iteration.

The central component satisfies $|D\Phi^t| \leq C\varepsilon$ and $|D^2\Phi^t| \leq C$. For the first component the only nonzero first derivatives are

$$\partial_t \Phi^x = -\varepsilon^{-1}, \quad \partial_x \Phi^x = \frac{\theta}{\zeta(L\varepsilon x)} + (1-\theta)\eta'(L\varepsilon x).$$

The latter is bounded by a fixed constant. Its only nonzero second derivative is

$$\partial_{xx} \Phi^x = L\varepsilon \left(-\frac{\theta\zeta'(L\varepsilon x)}{\zeta(L\varepsilon x)^2} + (1-\theta)\eta''(L\varepsilon x) \right),$$

which is bounded by $C\varepsilon^{-1}\theta^{-1}$. Together with $|D\Phi^y| \leq C\theta^{-1}$ and $|D^2\Phi^y| \leq C\varepsilon^{-1}\theta^{-2}$, these bounds prove (6.5). $\square$

6.3. A short estimate for the oscillatory amplitudes. The normalization gives an amplitude bounded by $C\sqrt{m}$, while the contact defect gives $|DU| \leq r/m$. To combine these bounds when m varies, we use its minimum on the segment joining two arguments. No derivative of m is needed.

Lemma 6.3 (An oscillation with variable amplitude). *Let $C_0 \geq 1$, $\Gamma \geq 2$, $0 < r \leq 1$, $0 < \varepsilon \leq 1$, and $0 < L \leq \varepsilon^{-2}$. Let $m : \overline{\mathbb{B}^2} \rightarrow (0, \infty)$ be continuous, and let $U : \overline{\mathbb{B}^2} \rightarrow \mathbb{R}$ and $v : \overline{\mathbb{B}^2} \rightarrow \mathbb{R}^N$ be restrictions of smooth functions defined on a neighborhood of $\overline{\mathbb{B}^2}$. Assume that, at every $a \in \overline{\mathbb{B}^2}$,*

$$\begin{aligned} m(a) &\leq \Gamma, & m(a)|DU(a)| &\leq r, \\ |v(a)| &\leq C_0\sqrt{m(a)}, & |Dv(a)| &\leq C_0\Gamma^2. \end{aligned}$$

Let $\zeta \in C^1(\mathbb{R})$ satisfy $\|\zeta\|_{L^\infty(\mathbb{R})} + \|\zeta'\|_{L^\infty(\mathbb{R})} \leq C_0$. Then the function $W(a) = \varepsilon v(a)\zeta(LU(a))$ satisfies

$$\|W\|_{C^{1/2}(\overline{\mathbb{B}^2})} \leq C(\sqrt{r} + \varepsilon\Gamma^2),$$

where C depends only on C_0 . No derivative bound for m is assumed.

Proof. Put $W = \varepsilon v\zeta(LU)$ and take distinct $a, b \in \overline{\mathbb{B}^2}$. Choose $c \in [a, b]$ where m is smallest, and set $m_* = m(c) > 0$ and $d = |a - b|$. Convexity of the disk and integration along the segment give

$$|U(a) - U(b)| \leq rd/m_*.$$

All these segments lie in the original disk $\overline{\mathbb{B}^2}$. In the application, no convexity of its image $P(\overline{\mathbb{B}^2})$ is required. Freeze the amplitude at the minimizing point c . The exact decomposition is

$$\begin{aligned} W(a) - W(b) &= \varepsilon(v(a) - v(c))\zeta(LU(a)) \\ &\quad + \varepsilon v(c)(\zeta(LU(a)) - \zeta(LU(b))) \\ &\quad + \varepsilon(v(c) - v(b))\zeta(LU(b)). \end{aligned}$$

The first and last terms together are at most $C\varepsilon\Gamma^2d$, because $|a - c| + |c - b| = d$. The middle term is at most

$$C\varepsilon\sqrt{m_*} \min\{1, Lrd/m_*\} \leq C\varepsilon\sqrt{L}\sqrt{rd} \leq C\sqrt{rd},$$

where we used $\min\{1, \tau\} \leq \sqrt{\tau}$ for $\tau \geq 0$. Dividing by $\sqrt{d}$ and using $d \leq 2$ proves the seminorm bound. Finally $\|W\|_\infty \leq C\varepsilon\sqrt{\Gamma} \leq C\varepsilon\Gamma^2$. $\square$

7. PROOF OF THE MAIN CORRECTION PROPOSITION

The defect controls the derivative of the phase. It should therefore appear not only in the half-Hölder estimate, but also in the first derivative estimate. Retaining this small factor improves interpolation. We first compute the exact defect and Taylor expansion, and only then estimate their terms. The final part checks the coordinate hypotheses needed to repeat the proposition.

Proof of Proposition 3.2. Step 1: construct the embedded correction.

Take $\mathcal{G}$ from Lemma 6.2 and define

$$G = \Xi \circ \mathcal{G} \circ P.$$

Since $\delta \geq \Gamma^{-1}$ and $\varepsilon \leq c\Gamma^{-2}$, choosing $c \leq 1/8$ ensures $4\varepsilon < \delta$. The graph lies inside U_δ , so this is a composition of an embedding and diffeomorphisms on their correct domains. In particular G is a smooth embedding of the entire plane. The distance between $\mathcal{G}(P(a))$ and $\iota(P(a))$ is at most $C\varepsilon$. The segment between these points lies in U_δ . The mean value estimate for Ξ proves the global displacement bound.

Step 2: compute the new defect.

Put $m = \mu(\iota(P))$ and $m_+ = \mu(\mathcal{G}(P))$. The graph identity in Lemma 6.2 gives

$$(7.1) \quad F^*\lambda = m dU, \quad G^*\lambda = m_+ \left(\theta dU - \varepsilon \zeta_\theta(LU) dS \right).$$

To see precisely which terms need estimating, write this as

$$(7.2) \quad \begin{aligned} G^*\lambda &= \theta F^*\lambda + E_{\text{fac}} + E_{\text{tr}}, \\ E_{\text{fac}} &= \theta(m_+ - m) dU, \\ E_{\text{tr}} &= -\varepsilon m_+ \zeta_\theta(LU) dS. \end{aligned}$$

The first error records variation of the contact factor when moving off the plane; the second comes from the transverse oscillation. There is no further oscillatory term of order one: it was canceled exactly by the central coordinate of $\mathcal{G}$.

Since $\|D\mu\| \leq \Gamma$ and the displacement in the old coordinates is at most $C\varepsilon$, we have $|m_+ - m| \leq C\Gamma\varepsilon$. Combining this with $m|DU| \leq r$ and $m \geq \Gamma^{-1}$ gives

$$\|E_{\text{fac}}\|_\infty \leq C\theta\Gamma\varepsilon \frac{r}{m} \leq C\theta\varepsilon\Gamma^2 r.$$

Choose c in $\varepsilon \leq c\Gamma^{-2}$ so that the last bound is at most θr . Also $m_+ \leq \Gamma$, $|DS| \leq \Gamma$, and $\zeta_\theta \leq 2$, so $\|E_{\text{tr}}\|_\infty \leq 2\varepsilon\Gamma^2$. Together with $\|\theta F^*\lambda\| \leq \theta r$, this proves the defect estimate in (3.9).

Step 3: separate the leading oscillations.

The displacement is small even though its derivative may be large. Taylor expansion in the displacement therefore separates the two principal oscillations from terms with an extra factor of ε . Write $p = \iota(P)$ and

$$w = (\varepsilon\eta(LU), \varepsilon\zeta_\theta(LU), \varepsilon^2\chi_\theta(LU)).$$

The full columns $a_1 = \partial_x \Xi(p)$ and $a_2 = \partial_y \Xi(p)$ satisfy

$$(7.3) \quad |a_i| \leq C\sqrt{m}, \quad |Da_i| \leq C\Gamma^2.$$

Indeed the contact identity at $y = 0$ says $a_i^t = F^y a_i^x$. Therefore

$$|a_i|^2 = |a_i^y|^2 + (1 + |F^y|^2)|a_i^x|^2 \leq 2|(a_i^x, a_i^y)|^2.$$

The normalization controls the last expression by Cm . For the derivative, the chain rule gives $Da_i = D(\partial_j \Xi)(p) Dp$ for $j = x$ or y ; its norm is at most $\Gamma\Gamma = \Gamma^2$. This is why the global bound on the second coordinate of the surface is among the hypotheses.

Taylor's formula, obtained by integrating twice in the scalar variable along $p + \tau w$, is

$$(7.4) \quad G - F = \varepsilon\eta(LU)a_1 + \varepsilon\zeta_\theta(LU)a_2 + R_0 + R_1,$$

where

$$\begin{aligned} R_0 &= \varepsilon^2 \chi_\theta(LU) \partial_t \Xi(p), \\ R_1 &= \int_0^1 (1 - \tau) D^2 \Xi(p + \tau w)[w, w] d\tau. \end{aligned}$$

The term R_0 is the smaller correction in the central coordinate; R_1 is the nonlinear Taylor remainder. The leading two terms have the balanced amplitudes a_1, a_2 , which is exactly where the normalization enters the proof. Every point of this integral belongs to U_δ . Its middle coordinate is between zero and $\varepsilon\zeta_\theta(LU)$.

Step 4: keep the small factor in the phase derivative.

Using only $|DU| \leq \Gamma$ here would throw away the improvement obtained by making the contact defect small. We keep r every time the phase is differentiated. The exact inequality $m|DU| \leq r$ and $m \geq \Gamma^{-1}$ give

$$(7.5) \quad |w| \leq C\varepsilon, \quad |Dp| \leq \Gamma, \quad |Dw| \leq C\varepsilon L|DU| \leq Cr\Gamma/\varepsilon.$$

For a unit parameter direction v , differentiating the first remainder gives

$$\begin{aligned} DR_0[v] &= \varepsilon^2 L\chi'_\theta(LU) DU[v] \partial_t \Xi(p) \\ &\quad + \varepsilon^2 \chi_\theta(LU) D(\partial_t \Xi)(p)[Dp[v]]. \end{aligned}$$

Therefore $\|DR_0\| \leq C\Gamma^2(r + \varepsilon^2)$. For the second remainder, use [Lemma 4.2](#) instead of differentiating its integral formula. Equation (7.5) gives

$$|DR_1| \leq C\Gamma\varepsilon(2\Gamma + r\Gamma/\varepsilon) \leq C\Gamma^2(\varepsilon + r).$$

Also $\|R_0\| + \|R_1\| \leq C\Gamma\varepsilon^2$. Hence

$$(7.6) \quad \|R_0 + R_1\|_\infty \leq C\Gamma\varepsilon^2, \quad \|D(R_0 + R_1)\|_\infty \leq C\Gamma^2(\varepsilon + r).$$

This is slightly weaker in ε than a bound using $D^3\Xi$, but it will give the same correction estimates. Only $D^2\Xi$ and DP have been used, with no second derivative of P .

For each leading term in (7.4), differentiation of the amplitude costs at most $C\varepsilon\Gamma^2$. Differentiation of the profile costs at most

$$C\varepsilon L\sqrt{m}|DU| \leq \frac{Cr}{\varepsilon\sqrt{m}} \leq Cr\sqrt{\Gamma}/\varepsilon.$$

We conclude that

$$\|D(G - F)\| \leq C \left(\varepsilon \Gamma^2 + \frac{r\sqrt{\Gamma}}{\varepsilon} + \Gamma^2 r \right).$$

Under $\varepsilon \Gamma^2 \leq \sqrt{r}$ we have $r \geq \varepsilon^2 \Gamma^4$, and hence

$$\frac{\varepsilon \Gamma^2}{r\sqrt{\Gamma}/\varepsilon} = \frac{\varepsilon^2 \Gamma^{3/2}}{r} \leq \Gamma^{-5/2} \leq 1.$$

For the last term the ratio is $\Gamma^2 r / (r\sqrt{\Gamma}/\varepsilon) = \varepsilon \Gamma^{3/2} \leq 1$, by $\varepsilon \leq c \Gamma^{-2}$ after taking $c \leq 1$. This proves (3.10).

Step 5: the Hölder estimate.

Apply Lemma 6.3 to the leading terms of (7.4), using (7.3) and $m|DU| \leq r$. The remainders satisfy, even without the extra smallness assumption,

$$[R_0 + R_1]_{1/2} \leq (2\|R_0 + R_1\|_\infty \|D(R_0 + R_1)\|_\infty)^{1/2} \leq C\varepsilon \Gamma^{3/2} \leq C\varepsilon \Gamma^2,$$

since $r, \varepsilon \leq 1$. This proves the half-Hölder bound in (3.9). Under $\varepsilon \Gamma^2 \leq \sqrt{r}$, its seminorm is at most $C\sqrt{r}$. Interpolation (4.3) with $\gamma = 1/2$ and (3.10) gives

$$[G - F]_\alpha \leq C_\alpha (\sqrt{r})^{2-2\alpha} (r\sqrt{\Gamma}/\varepsilon)^{2\alpha-1} = C_\alpha r^\alpha (\sqrt{\Gamma}/\varepsilon)^{2\alpha-1}.$$

Adding the global supremum bound proves (3.11).

Step 6: prepare the next coordinates.

Let Φ, B, κ be given by Lemma 6.2, with $d = c_* \varepsilon \theta$, and put

$$\Xi_1 = \Xi \circ \Phi : U_d \rightarrow \mathbb{R}^3, \quad P_1 = B \circ P, \quad \mu_1 = (\mu \circ \Phi) \kappa.$$

The image of Φ is inside the old slab since $4\varepsilon < \delta$. The data (d, Ξ_1, P_1, μ_1) represent G on all of $\mathbb{R}^2$, and $\Xi_1^* \lambda = \mu_1 \lambda$ on U_d . Set $W = \varepsilon^{-2} + \varepsilon^{-1} \theta^{-2}$. The two derivative orders give

$$\begin{aligned} (7.7) \quad & \|D\Xi_1\| \leq C\Gamma(\varepsilon^{-1} + \theta^{-1}), \\ & \|D^2\Xi_1\| \leq C\Gamma((\varepsilon^{-1} + \theta^{-1})^2 + \varepsilon^{-1}\theta^{-2}) \leq C\Gamma W, \\ & c_0 \theta / (2\Gamma) \leq \mu_1 \leq 4\Gamma, \\ & \|D\mu_1\| \leq C\Gamma(\varepsilon^{-1} + \theta^{-1} + \varepsilon^{-1}\theta^{-1}) \leq C\Gamma W. \end{aligned}$$

For example, the last line is the product rule applied to $(\mu \circ \Phi) \kappa$. We used $\varepsilon, \theta \leq 1$ to absorb θ^{-2} and all first-order terms into W . Equation (6.6) also gives

$$\|DP_1\| + \|DP_1^{-1}\| + \|P_1\|_{L^\infty(\mathbb{B}^2)} \leq C\Gamma/\varepsilon.$$

Here $\|DP^{-1}\| \leq \Gamma$ follows from the original pair inequality by taking a difference quotient. Since P_1^{-1} is defined on all of $\mathbb{R}^2$, its derivative bound implies the corresponding inequality for every pair.

For the ambient pair bound, take $u, u' \in U_d$ and put $v = \Phi(u)$, $v' = \Phi(u')$. These belong to the convex slab V in (6.4). Its inverse estimate gives

$$\begin{aligned} |u - u'| &= |K(v) - K(v')| \leq C\varepsilon^{-1} \theta^{-2} |v - v'| \\ &\leq C\Gamma \varepsilon^{-1} \theta^{-2} |\Xi_1(u) - \Xi_1(u')|. \end{aligned}$$

Finally $d^{-1} = c_*^{-1}\varepsilon^{-1}\theta^{-1} \leq CW$ and $\mu_1^{-1} \leq C\Gamma/\theta \leq CTW$. Together with the preceding derivative and inverse estimates, these bounds verify all four conditions of [Definition 3.1](#) for (d, Ξ_1, P_1, μ_1) , with a common bound CTW . Applying [Proposition 5.1](#) gives normalized data of size $C\Gamma^{22}W^{22}$ without changing G . This proves (3.7). If $\varepsilon \leq \theta^2$, then $W \leq 2\varepsilon^{-2}$, proving (3.8) as well. $\square$

8. PARAMETERS, INDUCTION, AND CONVERGENCE

We now choose numerical sequences before constructing any map. The displacement must decrease faster than finite-stage separation, the defect must decrease to zero, and the Hölder increments must be summable. Each requirement is checked separately.

8.1. Why these parameters are chosen. Suppose $\varepsilon_k = \Gamma_k^{-p}$ and $\theta_k = (4\Gamma_k^a)^{-1}$. When $p > 2a$, a sufficiently large initial Γ_0 ensures $\varepsilon_k \leq \theta_k^2$. The cost (3.8) is then $C\Gamma_k^{22+44p}$, so we can choose $b = 22 + 44p$. This is the benefit of the new phase coordinates.

The displacement bound is $C\Gamma_k^{1-p}$, whereas separation is at least $\Gamma_k^{-2}|u - v|$. Thus $p > 3$ makes the remaining displacement smaller than separation for each fixed distinct pair. We take $p = 4$. The defect error is then $C\Gamma_k^{-2}$. We take $a = 99/50 < 2$, leaving a small power to absorb r_k^{-1} in the defect estimate. These choices also give $p - 2a = 1/25 > 0$. All constants in these comparisons are checked below.

Now $b = 22 + 44 \cdot 4 = 198$. A convenient upper bound for r_k has power $\sigma = a/b = 1/100$. The Hölder summability condition is

$$\sigma\alpha > \frac{9}{2}(2\alpha - 1).$$

Choose the following rational parameters:

$$(8.1) \quad p = 4, \quad a = \frac{99}{50}, \quad b = 198, \quad \sigma = \frac{1}{100}, \quad \alpha = \frac{901}{1800}.$$

For constants $C_0 \geq 2$ and $\Gamma_0 \geq C_0$, define the sequences recursively, for every integer $k \geq 0$, by

$$(8.2) \quad \begin{aligned} \Gamma_{k+1} &= C_0\Gamma_k^b, & r_0 &= 1, & r_{k+1} &= r_k\Gamma_k^{-a}, \\ \varepsilon_k &= \Gamma_k^{-4}, & \theta_k &= (4\Gamma_k^a)^{-1}. \end{aligned}$$

The numerical lemma below uses only $\Gamma_0 \geq C_0 \geq 2$. The additional choices needed for the geometric construction are stated explicitly in [Proposition 8.2](#).

8.2. Bounds for the numerical sequences.

Lemma 8.1 (Bounds for the parameter sequences). *Fix $C_0 \geq 2$ and $\Gamma_0 \geq C_0$, and define Γ_k, r_k by (8.2), with a, b, σ as in (8.1). For every integer $k \geq 0$,*

$$(8.3) \quad \Gamma_0^{1/50}\Gamma_k^{-1/50} \leq r_k \leq \Gamma_0^\sigma\Gamma_k^{-\sigma}.$$

Moreover, $\Gamma_k \rightarrow \infty$ and $r_k \rightarrow 0$. For every $s > 0$ and every integer $k \geq 0$,

$$\sum_{j=k}^{\infty} \Gamma_j^{-s} \leq \frac{\Gamma_k^{-s}}{1 - 2^{-s}} < \infty.$$

In particular, $\sum_{j=k}^{\infty} \Gamma_j^{-3} \leq 2\Gamma_k^{-3}$.

Proof. Multiplying the recurrence for r by the appropriate power of Γ gives

$$r_{k+1}\Gamma_{k+1}^{1/50} = r_k\Gamma_k^{1/50}C_0^{1/50}\Gamma_k^{(b-1)/50-a}.$$

The exponent is $49/25 > 0$, so this product is nondecreasing. For the other direction,

$$r_{k+1}\Gamma_{k+1}^{\sigma} = r_k\Gamma_k^{\sigma}C_0^{\sigma}\Gamma_k^{\sigma(b-1)-a} = r_k\Gamma_k^{\sigma}(C_0/\Gamma_k)^{\sigma}.$$

Here $\sigma(b-1)-a = -\sigma$, and $\Gamma_k \geq \Gamma_0 \geq C_0$. This product is nonincreasing, proving (8.3). Also $\Gamma_{k+1} \geq 2\Gamma_k$ and in fact $\Gamma_k \geq \Gamma_0^{b^k}$. The first inequality gives $\Gamma_j^{-s} \leq \Gamma_k^{-s}2^{-s(j-k)}$ for $j \geq k$. Summing the geometric series proves the stated estimate for the tails. For $s = 3$ the factor is $(1 - 2^{-3})^{-1} = 8/7 < 2$, which gives the stated tail bound. The upper bound in (8.3) also proves $r_k \rightarrow 0$. $\square$

8.3. Construction of the maps and the limit.

Proposition 8.2 (The sequence of embeddings and its limit). *Let c, C be the constants in Proposition 3.2, and choose $C_1 \geq \max\{1, C\}$ and $C_0 \geq \max\{2, C_1\}$. Choose $\Gamma_0 \geq C_0$ satisfying*

$$(8.4) \quad \Gamma_0^{-2} \leq c, \quad \Gamma_0^{1/25} \geq 16, \quad \Gamma_0^{1/50} \geq 2C_1, \quad 2C_1\Gamma_0^{-3} \leq 1.$$

Define $\Gamma_k, r_k, \varepsilon_k, \theta_k$ by (8.2) and set $\alpha = 901/1800$, as in (8.1).

There exist smooth embeddings $F_k : \mathbb{R}^2 \rightarrow \mathbb{R}^3$, for $k \geq 0$, and normalized coordinate data $(\delta_k, \Xi_k, P_k, \mu_k)$ of size at most Γ_k , such that

$$F_k = \Xi_k \circ \iota \circ P_k \quad \text{on } \mathbb{R}^2, \quad F_0 = \iota.$$

At every stage,

$$\sup_{a \in \mathbb{R}^2} |F_k^y(a)| \leq 1, \quad \|F_k^* \lambda\|_{L^\infty(\overline{\mathbb{B}^2})} \leq r_k.$$

The increments satisfy

$$\|F_{k+1} - F_k\|_{L^\infty(\mathbb{R}^2)} \leq C_1\Gamma_k^{-3}, \quad \sum_{k=0}^{\infty} \|F_{k+1} - F_k\|_{C^\alpha(\overline{\mathbb{B}^2})} < \infty.$$

Consequently, the restrictions $F_k|_{\overline{\mathbb{B}^2}}$ converge in the Euclidean C^α norm to an injective map $F : \overline{\mathbb{B}^2} \rightarrow \mathbb{R}^3$. This limit obeys

$$\|F - F_k\|_{L^\infty(\overline{\mathbb{B}^2})} \leq 2C_1\Gamma_k^{-3} \quad (k \geq 0).$$

Proof. The choices in the statement fix all constants before the induction. They are possible: taking Γ_0 at least $\max\{C_0, c^{-1/2}, 16^{25}, (2C_1)^{50}\}$ satisfies all four conditions in (8.4). The constant C_1 bounds both the coordinate constant in (3.8) and the defect and displacement constants in (3.9). We use these same constants at every stage.

Step 1: initialization.

Take $\Xi_0 = \text{id}$, $\mu_0 = 1$, $P_0 = \text{id}$, and $\delta_0 = 1$. Then $F_0(s, u) = (s, 0, u)$, $F_0^y = 0$, and $F_0^* \lambda = du$. The two horizontal columns have lengths one. All four conditions of Definition 3.1 hold for $\Gamma_0 \geq 2$, and their sum of lengths is $2 \leq 8\sqrt{\mu_0}$, which verifies normalization.

Step 2: the coordinate and defect inequalities close.

The first inequality in (8.4), together with $\Gamma_k \geq \Gamma_0$, ensures $\varepsilon_k \leq c\Gamma_k^{-2}$. The second gives the compatibility needed for the improved coordinate cost:

$$\frac{\varepsilon_k}{\theta_k^2} = 16\Gamma_k^{2a-4} = 16\Gamma_k^{-1/25} \leq 1.$$

The coordinate power is exactly $22 + 44p = 198 = b$, so the next normalized data have size at most Γ_{k+1} . For the defect estimate, the lower bound in (8.3) is used in the following direction:

$$\frac{r_k}{2\Gamma_k^{99/50}} \geq \frac{\Gamma_0^{1/50}}{2} \Gamma_k^{-(1/50+99/50)} \geq C_1 \Gamma_k^{-2} = C_1 \varepsilon_k \Gamma_k^2.$$

The last inequality is the third condition in (8.4). The other defect term is $2\theta_k r_k = r_k / (2\Gamma_k^{99/50})$. Their sum is r_{k+1} . The extra smallness condition for (3.11) also holds:

$$\varepsilon_k \Gamma_k^2 = \Gamma_k^{-2} \leq \Gamma_k^{-1/100} \leq \sqrt{r_k}.$$

Step 3: the coordinate F_k^y stays bounded globally.

At each step (3.9) gives

$$(8.5) \quad \|F_{k+1} - F_k\|_{L^\infty(\mathbb{R}^2)} \leq C_1 \Gamma_k^{-3}.$$

Suppose $F_0, \dots, F_k$ have been constructed and the global bound holds for F_k . The correction proposition gives F_{k+1} and (8.5). Since $F_0^y = 0$, summing only the already constructed increments yields

$$\sup_{\mathbb{R}^2} |F_{k+1}^y| \leq C_1 \sum_{j=0}^k \Gamma_j^{-3} \leq 2C_1 \Gamma_0^{-3} \leq 1.$$

The last inequality is the fourth condition in (8.4). This proves the global bound at the next stage. This establishes that the hypothesis used in constructing the next map remains true; there is no reliance on a bound available only on the disk. Steps 1–3 therefore give the entire sequence by induction.

Step 4: Hölder increments are summable.

The improved estimate (3.11) becomes

$$\|F_{k+1} - F_k\|_{C^\alpha(\overline{\mathbb{B}^2})} \leq C_\alpha \left(\Gamma_k^{-3} + r_k^\alpha \Gamma_k^{(9/2)(2\alpha-1)} \right).$$

By (8.3), the second term is bounded by $\Gamma_0^{\sigma\alpha} \Gamma_k^{-m_*}$, where the strictly positive margin is the exact rational number

$$(8.6) \quad m_* = \sigma\alpha - \frac{9}{2}(2\alpha - 1) = \frac{901}{180000} - \frac{1}{200} = \frac{1}{180000}.$$

The series of these norms converges by Lemma 8.1. Consequently the uniform limit F exists, and the seminorm of any tail is bounded by the sum of its seminorms. These tail sums tend to zero, proving convergence in C^α . In particular $\sup_k \|F_k\|_{C^\alpha(\overline{\mathbb{B}^2})} < \infty$.

Step 5: the limit is injective.

It remains to prevent two distinct points from merging in the limit. The global displacement estimate gives

$$\|F - F_k\|_{L^\infty(\overline{\mathbb{B}^2})} \leq 2C_1\Gamma_k^{-3}.$$

Fix distinct $u, v \in \overline{\mathbb{B}^2}$ and write $d = |u - v| > 0$. The finite-stage separation (3.6) yields

$$\begin{aligned} |F(u) - F(v)| &\geq |F_k(u) - F_k(v)| - 2\|F - F_k\|_\infty \\ &\geq \Gamma_k^{-2}d - 4C_1\Gamma_k^{-3} = \Gamma_k^{-2}(d - 4C_1\Gamma_k^{-1}). \end{aligned}$$

Choose k so large that $4C_1\Gamma_k^{-1} < d/2$. The last expression is positive, so $F(u) \neq F(v)$. $\square$

Remark 8.3 (The slightly larger open range). For the same sequence F_k and the same limit F from Proposition 8.2, the sum of the $C^\alpha(\overline{\mathbb{B}^2})$ increment norms is finite for every $1/2 < \alpha < 4925/9839$. Here α is allowed to vary; its fixed value $901/1800$ in Proposition 8.2 gave a convenient rational margin. We verify the larger range from the exact recurrence. Put

$$\tau = \frac{a}{b-1} = \frac{99}{9850}.$$

Since $\Gamma_{k+1} = C_0\Gamma_k^b$, direct induction gives the exact identity

$$(8.7) \quad r_k = \Gamma_0^\tau C_0^{k\tau} \Gamma_k^{-\tau}.$$

If

$$\tau\alpha > \frac{9}{2}(2\alpha - 1),$$

the Hölder increment is bounded by a negative power of Γ_k times $C_0^{k\tau\alpha}$. This is summable: taking logarithms gives a negative term of size b^k and a positive term of size k . To solve the inequality, move the terms containing α to one side:

$$\tau\alpha > \frac{9}{2}(2\alpha - 1) \iff (9 - \tau)\alpha < \frac{9}{2} \iff \alpha < \frac{9/2}{9 - 99/9850} = \frac{4925}{9839}.$$

The equality case is not obtained by this calculation. Injectivity and all other induction requirements are unchanged.

9. PASSAGE TO THE HEISENBERG METRIC

9.1. A dyadic integration estimate. The following is the elementary smooth-function case of Young's integration estimate [14]; compare [6, Corollary 3.16]. We prove exactly the version required for our approximations.

Lemma 9.1 (A dyadic bound for a signed area integral). *Let $1/2 < \alpha < 1$ and let $u, v \in C^1([0, 1]; \mathbb{R})$. Using the Euclidean Hölder seminorms on $[0, 1]$, one has*

$$(9.1) \quad \left| \int_0^1 (u(\tau) - u(0))v'(\tau) d\tau \right| \leq C_\alpha [u]_{\alpha; [0, 1]} [v]_{\alpha; [0, 1]}, \quad C_\alpha = \frac{2^{-2\alpha}}{1 - 2^{1-2\alpha}}.$$

The integral is the ordinary integral of a continuous function; the constant depends only on α .

Proof. A refinement of a Riemann sum produces a product of two small increments. Their Hölder bounds make the refinement errors summable when $2\alpha > 1$. On the partition $\tau_i = i2^{-j}$ put

$$R_j = \sum_{i=0}^{2^j-1} (u(\tau_i) - u(0))(v(\tau_{i+1}) - v(\tau_i)).$$

Then $R_0 = 0$. On a partition interval $[a, b]$ with midpoint m , subtracting the coarse summand from its two refined summands leaves $(u(m) - u(a))(v(b) - v(m))$. Hence

$$|R_{j+1} - R_j| \leq 2^j [u]_\alpha [v]_\alpha 2^{-2\alpha(j+1)}.$$

The geometric series in j sums to the constant in (9.1). Moreover R_j converges to the indicated integral: rewrite each $v(\tau_{i+1}) - v(\tau_i)$ as $\int_{\tau_i}^{\tau_{i+1}} v'$, and use uniform continuity of u to estimate the difference from the integral by $\sup_{|\tau-s| \leq 2^{-j}} |u(\tau) - u(s)| \int_0^1 |v'|$. This tends to zero. The denominator in C_α is positive precisely because $2\alpha > 1$. $\square$

The next proposition is the approximation version of the contact characterization in [6, Theorem 7.4]. We use smooth line integrals and then pass to endpoint values. This also follows the signed-area estimate in [11, Lemma 2.2].

Proposition 9.2 (From smooth contact defects to Heisenberg regularity). *Fix $1/2 < \alpha < 1$. Let $F_k : \mathbb{B}^2 \rightarrow \mathbb{R}^3$ be restrictions of smooth maps defined on neighborhoods of $\mathbb{B}^2$, and suppose that F_k converges uniformly in Euclidean coordinates to a map $F : \mathbb{B}^2 \rightarrow \mathbb{R}^3$. Assume that there are a finite constant Λ and numbers $r_k \geq 0$, with $r_k \rightarrow 0$, such that, for every k ,*

$$\|F_k\|_{C^\alpha(\mathbb{B}^2; \mathbb{R}^3)} \leq \Lambda, \quad \|F_k^* \lambda\|_{L^\infty(\mathbb{B}^2)} \leq r_k.$$

Then, for all $a, b \in \mathbb{B}^2$,

$$\begin{aligned} |F^t(b) - F^t(a) - F^y(a)(F^x(b) - F^x(a))| \\ \leq C_\alpha \Lambda^2 |a - b|^{2\alpha}, \end{aligned}$$

where C_α is the constant in Lemma 9.1. Consequently, for the Heisenberg metric in the matrix model (2.2),

$$d_{CC}(F(a), F(b)) \leq (2 + 4\sqrt{C_\alpha})\Lambda |a - b|^\alpha \quad (a, b \in \mathbb{B}^2).$$

If F is injective, this makes $F : \mathbb{B}^2 \rightarrow \mathbb{H}^1$ a Hölder embedding.

Proof. The horizontal group increment already has the required Hölder bound. Since the metric takes a square root of the central increment, we must prove that the latter has order $|a - b|^{2\alpha}$. Fix $a, b \in \mathbb{B}^2$. The segment $[a, b]$ lies in $\mathbb{B}^2$. Write $\Delta F_k^j = F_k^j(b) - F_k^j(a)$ for $j = x, y, t$. Integrating $F_k^* \lambda = dF_k^t - F_k^y dF_k^x$ along the segment gives

$$\Delta F_k^t = \int_{[a, b]} F_k^y dF_k^x + e_k, \quad |e_k| \leq r_k |a - b|.$$

The error bound is the pointwise bound on the defect integrated along a segment of length $|a - b|$.

The group difference formula (2.5) provides the cancellation. The central coordinate of $F_k(a)^{-1} \odot F_k(b)$ is

$$\begin{aligned} E_k &= \Delta F_k^t - F_k^y(a) \Delta F_k^x \\ &= \int_{[a,b]} (F_k^y - F_k^y(a)) dF_k^x + e_k. \end{aligned}$$

Indeed, $\Delta F_k^x = \int_{[a,b]} dF_k^x$, so the endpoint product subtracts the constant value $F_k^y(a)$ from the integrand. The resulting signed area integral involves two small horizontal increments.

Parametrize the segment by $a + \tau(b - a)$, $0 \leq \tau \leq 1$. The C^α seminorms of F_k^y, F_k^x along this parametrization are at most $\Lambda|a - b|^\alpha$. Lemma 9.1 gives

$$|E_k| \leq C_\alpha \Lambda^2 |a - b|^{2\alpha} + r_k |a - b|.$$

The defining expression for E_k uses only endpoint values. Uniform convergence and $r_k \rightarrow 0$ consequently imply that

$$E = F^t(b) - F^t(a) - F^y(a)(F^x(b) - F^x(a)) \quad \text{satisfies} \quad |E| \leq C_\alpha \Lambda^2 |a - b|^{2\alpha}.$$

Thus no limit has to be passed under an integral. The horizontal increments satisfy

$$|F^x(b) - F^x(a)| + |F^y(b) - F^y(a)| \leq 2\Lambda |a - b|^\alpha.$$

Left invariance and the upper bound in Lemma 2.1, applied to $F(a)^{-1} \odot F(b)$, yield

$$d_{CC}(F(a), F(b)) \leq (2 + 4\sqrt{C_\alpha})\Lambda |a - b|^\alpha.$$

This proves the asserted regularity for F itself.

If F is injective, its inverse on its image is continuous: a closed subset of the compact disk is compact, its image is compact, and a compact subset of a metric space is closed. The bijection onto the image is therefore a closed map, proving the embedding assertion. $\square$

Completion of the proof of Theorem 1.1. The limit F supplied by Proposition 8.2 is injective. Its smooth approximations have uniformly bounded C^α norms and contact defects tending to zero. Proposition 9.2 shows that this same map $F : \mathbb{B}^2 \rightarrow \mathbb{H}^1$ is the required Hölder embedding. Remark 8.3 proves the larger open range stated in the theorem. $\square$

10. WHAT REMAINS BEFORE THE EXPONENT $2/3$

The exponent in Theorem 1.1 is close to $1/2$. The estimates permit a precise explanation of this limitation. We first separate the simplifications proved above from further ideas which would need a new geometric construction.

10.1. What the isometric immersion arguments contribute. The proof uses three elementary simplifications. The remainder estimate (4.5) requires only $D^2\Xi$ and DP . Relative smoothing in (4.4) preserves the ratio $|DA|/A$, which is needed when differentiating the inverse stretch. Finally, the phase coordinate u/ε in (6.7) reduces the cost of the forward map. Together they give normalization at cost $C\Gamma^{22}$ and a complete correction at cost $C\Gamma^{22}\varepsilon^{-44}$ when $\varepsilon \leq \theta^2$.

The organization was guided by [3, Proposition 4.1 and Sections 6–8], and the derivative estimates by the elementary composition rules in [4, Proposition 2.1] and [1, Lemma A.1]. The formulas and inverse estimates needed for the contact problem were proved here. In particular, the algebraic error corrections of [3, Proposition 5.4] and [1, Lemma 2.2] have not been applied as lemmas: their geometric hypotheses concern a metric defect, not our one-form defect.

There is a simple reason why the conformal-coordinate device in [4, Proposition 2.3 and Section 3.4] cannot be transferred unchanged. If a contact diffeomorphism Θ preserves the plane and $\Theta \circ \iota = \iota \circ (S_*, U_*)$, restriction of its contact identity to the plane gives

$$dU_* = (\kappa \circ \iota) du.$$

Thus $\partial_s U_* = 0$. A general conformal change of the two plane variables need not satisfy this condition. Our triangular stretch and the shear respect this restriction.

Cao–Hirsch–Inauen [1, Proposition 2.4] use repeated integration by parts to reorganize oscillatory errors. A possible contact analogue starts from the following identity:

$$d(L^{-1}f H(LU)) = f h(LU) dU + L^{-1}H(LU) df, \quad H' = h,$$

where h is periodic with zero mean, so a periodic H exists. Thus subtracting $d(L^{-1}f H(LU))$ removes the term $f h(LU) dU$ and leaves $-L^{-1}H(LU) df$. The remaining term has an explicit factor L^{-1} ; to make it small, one must also control df . This alone does not remove the transverse error $-\varepsilon m_+ \zeta(LU) dS$ in (7.2). For that error, a new correction would have to preserve both embedding and quantitative contact coordinates for the next step. No such additional lemma is assumed here, and the numerical improvement above does not depend on the theorem in [1].

10.2. The inequality that further parameter tuning cannot fix. Consider power choices

$$\varepsilon_k = \Gamma_k^{-p}, \quad \theta_k = \frac{1}{4}\Gamma_k^{-a}, \quad \Gamma_{k+1} = C_0\Gamma_k^b, \quad r_{k+1} = r_k\Gamma_k^{-a}.$$

Using the general cost (3.7), the coordinate recurrence is closed by

$$(10.1) \quad b \geq 22 + 22 \max\{2p, p + 2a\}.$$

The asymptotic power of the defect relative to Γ_k is $\tau = a/(b-1)$, as in (8.7). At the level of powers, the defect error requires $p-2 \geq a+\tau$, and comparison of the available displacement and separation estimates uses $p > 3$. Strict inequalities leave room for fixed constants. The Hölder summability condition from (3.11) is

$$(10.2) \quad \tau\alpha > (p + \frac{1}{2})(2\alpha - 1).$$

At $\alpha = 2/3$ this would require $\tau > (p + 1/2)/2$. However (10.1), for $a, p > 0$, gives

$$\tau = \frac{a}{b-1} \leq \frac{a}{21 + 22(p+2a)} < \frac{1}{44},$$

whereas $(p + 1/2)/2 > 7/4$ for $p > 3$. Thus the estimates in this article do not close at $2/3$, or at exponents near it, by changing the numerical parameters. This is a limitation of these estimates, not an upper bound for all embedded constructions or for all possible estimates of our maps.

10.3. The remaining geometric issue. One scalar Γ still bounds several different quantities: the first and second derivatives of the ambient map, the contact factor and its first derivative, the plane parametrization, the slab width, and the constants protecting separation. Feeding all of them into one normalization bound loses information. A further advance would need separate recurrences for these quantities, or a correction whose coordinate and inverse bounds remain controlled at the oscillation scale.

This is also where the analogy with Isett is incomplete. His proof [8, Section 6] includes a gluing construction specific to the Euler equation, in addition to frequency and error estimates. We have used the discipline of retaining scales in the explicit estimates, but have not proved an embedded contact analogue of that gluing.

A proposed replacement must retain the three limiting properties in (3.12). A finite correction at a fixed cutoff does not provide an iteration if its smallness threshold is uncontrolled as the cutoffs change. A construction using characteristic flows would likewise need uniform bounds on their inverse derivatives. Most importantly, favorable Hölder estimates alone do not protect distinct points: one must compare finite-stage separation with the entire remaining displacement. The explicit pair estimates above perform that comparison for the present construction.

There is a direct obstruction to copying exact-solution gluing. If a smooth planar map satisfies $F^*\lambda = 0$ on an open patch, differentiating gives $dF^x \wedge dF^y = 0$. Since $dF^t = F^y dF^x$, its differential then has rank at most one. Such a patch cannot be a smooth immersed disk. Thus exact Euler intervals have no literal counterpart made of smooth contact disks.

Finally, obtaining every fixed $\alpha < 2/3$ and attaining $\alpha = 2/3$ are different goals. At the endpoint even a strict summability inequality may disappear. Neither goal is established by the present improvement.

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