Sobolev spaces, theory and applications

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Introduction

These are the notes that I prepared for the participants of the Summer School in Mathematics in Jyväskylä, August, 1998. I thank Pekka Koskela for his kind invitation.

This is the second summer course that I delivere in Finland. Last August I delivered a similar course entitled *Sobolev spaces and calculus of variations* in Helsinki. The subject was similar, so it was not posible to avoid overlapping. However, the overlapping is little. I estimate it as 25%. While preparing the notes I used partially the notes that I prepared for the previous course. Moreover Lectures 9 and 10 are based on the text of my joint work with Pekka Koskela [33].

The notes probably will not cover all the material presented during the course and at the some time not all the material written here will be presented during the School. This is however, not so bad: if some of the results presented on lectures will go beyond the notes, then there will be some reasons to listen the course and at the same time if some of the results will be explained in more details in notes, then it might be worth to look at them.

The notes were prepared in hurry and so there are many bugs and they are not complete. Some of the sections and theorems are unfinished.

At the end of the notes I enclosed some references together with comments. This section was also prepared in hurry and so probably many of the authors who contributed to the subject were not mentioned. I would like to to appologise for that. Actually the number of relared papers is so huge that it would be not possible to mention all the names.

I kindly welcome all the comments concerning the notes. I think the best way is to send comments by e-mail: `hajlasz@mimuw.edu.pl`

Lecture 1

Dirichlet problem. The classical *Dirichlet problem* reads as follows. Given an open

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domain $\Omega \subset \mathbb{R}^n$ and $g \in C^0(\partial\Omega)$. Find $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ such that

$$\begin{cases} \Delta u &= 0 \quad \text{in } \Omega\,, \\ u|_{\partial\Omega} &= g\,. \end{cases}$$

This problem arose in XIX'th century physics. We will explain the physical context of the equation, but we will not be very rigorous. It is just to give an intuition for the principles of the calculus of variations that will be developed in the sequel.

Let $\Omega \subset \mathbb{R}^3$ be a vacuum region and let $E : \Omega \to \mathbb{R}^3$ be an electric field. Given two points $x, y \in \Omega$, the integral

$$\int_x^y E \cdot ds \tag{1}$$

does not depend on the choice of the curve that joins x with y inside Ω (provided Ω is simply connected). By the definition (1) equals to

$$\int_a^b E(\gamma(t)) \cdot \dot{\gamma}(t)\, dt,$$

where γ is a parametrization of the given curve that joins x with y. Here and in the sequel $A \cdot B$ denotes the scalar product. We will also denote the scalar product by $\langle A, B\rangle$.

Fix $x_0 \in \Omega$ and define the potential u as follows

$$u(x) = -\int_{x_0}^x E \cdot ds.$$

Potential u is a scalar function defined up to a constant (since we can change the base point x_0). We have

$$E = -\nabla u\,.$$

It is well known that the electric field is divergence free $\operatorname{div} E = 0$ and hence

$$\Delta u = \operatorname{div} \nabla u = -\operatorname{div} E = 0.$$

Thus potential u is a harmonic function inside Ω. Assume that the vacuum Ω is bounded by a surface $\partial\Omega$. Moreover assume that the surface contains an electrical charge that induces potential g on $\partial\Omega$. Electrical charge on the boundary induces an electric field in Ω and hence the induced potential u in Ω has the following properties: $\Delta u = 0$ in Ω, $u|_{\partial\Omega} = g$. Thus u is a solution to the Dirichlet problem stated at the begining.

The energy of the electric field (up to a constant factor) is given by the formula

$$\text{Energy} = \int_\Omega |E|^2 = \int_\Omega |\nabla u|^2\,.$$

It is a general principle in physics that all the sistems approach the configuration with the minimal energy. Thus given potential g on the boundary $\partial\Omega$ one may expect that induced potential u in Ω has the property that it minimizes the *Dirichlet integral*

$$I(u) = \int_\Omega |\nabla u|^2$$

among all the functions $u \in C^2(\overline{\Omega})$ such that $u|_{\partial\Omega} = g$. As we will see it is true. Now we put all the physics aside and till the end of lectures we will be concerned with the rigorous mathematics.

Theorem 1 (Dirichlet principle) *Let $\Omega \subset \mathbb{R}^n$ be an arbitrary open and bounded set and let $u \in C^2(\Omega)$. Then the following statements are equivalent:*

1. $\Delta u = 0$ *in* Ω*,*

2. *u is a critical point of the functional I in the sense that*

$$\frac{d}{dt} I(u + t\varphi)|_{t=0} = 0 \quad \text{for all} \quad \varphi \in C_0^\infty(\Omega)\,.$$

If in addition $u \in C^2(\overline{\Omega})$, and $u|_{\partial\Omega} = g$, then we have one more equivalent condition:

3. *u minimizes I in the sense that $I(u) \le I(w)$ for all $w \in C^2(\overline{\Omega})$ with $w|_{\partial\Omega} = g$.*

Remark. The assumption $u \in C^2(\overline{\Omega})$ is certainly to strong, but we do not care about the minimal conditions.

Proof of Theorem 1. To prove the equivalence between 1. and 2. observe that for any $u \in C^2(\Omega)$ and $\varphi \in C_0^\infty(\Omega)$ we have

$$\frac{d}{dt} \int_\Omega |\nabla(u + t\varphi)|^2|_{t=0} = 2\int_\Omega \langle \nabla u, \nabla \varphi \rangle = -2\int_\Omega \Delta u\, \varphi\,.$$

The last equality follows from integration by parts. Now the implication 1. $\Rightarrow$ 2. is obvious. The implication 2. $\Rightarrow$ 1. follows from the following important lemma.

Lemma 2 *If $f \in L^1_{\mathrm{loc}}(\Omega)$ satisfies $\int_\Omega f\varphi = 0$ for any $\varphi \in C_0^\infty(\Omega)$, then $f = 0$ a.e.*

Proof. Suppose that $f \not\equiv 0$. We can assume f is positive on a set of positive measure (otherwise we replace f by $-f$). Then there is a compact set $K \subset \Omega$, $|K| > 0$ and $\varepsilon > 0$ such that $f \ge \varepsilon$ on K.

Let G_i be a sequence of open sets such that $K \subset G_i \subset\subset \Omega$, $|G_i \setminus K| \to 0$ ad $i \to \infty$. Now take $\varphi_i \in C_0^\infty(G_i)$ with $0 \le \varphi \le 1$, $\varphi_i|_K \equiv 1$. Then

$$0 = \int_\Omega f\varphi_i \ge \varepsilon|K| - \int_{G_i\setminus K} |f| \to \varepsilon|K|\,,$$

as $i \to \infty$, which is a contradiction. The proof is complete. □

We are left with the proof of the equivalence with 3. under the given additional regularity assumptions. For $u = w$ on $\partial\Omega$ we have

$$\begin{aligned} \int_\Omega |\nabla w|^2 &= \int_\Omega |\nabla(w-u)+u|^2 \\ &= \int_\Omega |\nabla(w-u)|^2 + \int_\Omega |\nabla u|^2 + 2\int_\Omega \langle \nabla(w-u), \nabla u\rangle \\ &= \int_\Omega |\nabla(w-u)|^2 + \int_\Omega |\nabla u|^2 - 2\int_\Omega (w-u)\Delta u. \end{aligned} \tag{2}$$

The last equality follows from the integration by parts and the fact that $w - u = 0$ on $\partial\Omega$.

1. $\Rightarrow$ 3. $\Delta u = 0$ and hence the last summand in (2) equals zero, so $\int_\Omega |\nabla w|^2 > \int_\Omega |\nabla u|^2$ unless $w = u$.

3. $\Rightarrow$ 2. Take $w = u + t\varphi$. Then $\int_\Omega |\nabla(u+t\varphi)|^2 \geq \int_\Omega |\nabla u|^2$ for all t. Hence I attains the minimum at $t = 0$. This implies 2.

Direct method in the calculus of variations. Riemann concluded that the Dirichlet problem was solvable, reasoning that I is nonnegative and so must attain a minimum value. Choosing a function u with $I(u) = \min I$ solves the problem.

Of course this "proof" of the existence of the solution is not correct. The function I is defined on an infinite dimensional object: the space of functions and there is no reason why the minimum of I should be attained.

The first rigorous proof of the existence of the solution of the Dirichlet problem was obtained by a different method. Later, however, Hilbert showed that it was possible to solve Dirichlet problem using Riemann's strategy. This was the begining of the so called direct method in the calculus of variations. We describe the method in a very general setting.

Till the end of the lecture we assume that $I : X \to \mathbb{R}$ is a function (called functional) defined on a Banach space X equipped with a norm $\|\cdot\|$. We want to emphasize that despite the name "functional" we do not assume I is linear. Actually, in all the interesting instances I will not be linear. We look for a condition which will guarantee the existence of $\bar{u}$ such that

$$I(\bar{u}) = \inf_{u\in X} I(u). \tag{3}$$

Function $\bar{u}$ as in (3) is called a *minimizer* of I and the problem of finding a minimizer is called a *variational problem.*

We say that I is *sequentially weakly lower semicontinuous* (swlsc) if for every sequence $u_n \rightharpoonup u$ weakly convergent in X, $I(u) \leq \liminf_{n\to\infty} I(u_n)$.

Recall that the weak convergence $u_n \rightharpoonup u$ in X means that for every linear continuous functional $e \in X^*$ there is $\langle e, u_n\rangle \to \langle e, u\rangle$.

We say that the functional I is *coercive* if $\|u_n\| \to \infty$ implies $I(u_n) \to \infty$.

Reflexive spaces play a special role, particularly because of the following result.

Theorem 3 *Every bounded sequence in a reflexive space contains a weakly convergent subsequence.* □

The following result is a basic result for the direct method in the calculus of variations.

Theorem 4 *If X is a reflexive Banach space and $I : X \to \mathbb{R}$ is swlsc and coercive then there exists $\bar{u} \in X$ such that $I(\bar{u}) = \inf_{u \in X} I(u)$.*

Proof. Let u_n be a sequence such that $I(u_n) \to \inf_X I$. Such a sequence will always be called *minimizing sequence.*

Because of the coercivity, the sequence u_n is bounded in X. Since the space is reflexive, we can substract a subsequence $u_{n_k} \rightharpoonup \bar{u}$ weakly convergent to some $\bar{u} \in X$. Then

$$I(\bar{u}) \leq \liminf_{k \to \infty} I(u_{n_k}) = \inf_{u \in X} I(u),$$

and hence the theorem follows. □

In general, the most difficult condition to deal with is the swlsc condition. Note that it does not follow from the continuity of I, which would be much easier to check.

An important class of functionals for which it is relatively easy to verify the swlsc condition is the class of convex functionals. Recall that the functional $I : X \to \mathbb{R}$ is *convex* if $I(tu + (1-t)v) \leq tI(u) + (1-t)I(v)$ whenever $t \in [0,1]$ and $u, v \in X$. We say that I is *strictly convex* if $I(tu + (1-t)v) < tI(u) + (1-t)I(v)$ whenever $t \in (0,1)$ and $u \neq v$.

We say that I is *lower semicontinuous* if the convergence in norm $u_n \to u$ implies $I(u) \leq \liminf_{n \to \infty} I(u_n)$.

Theorem 5 *If X is a Banach space and $I : X \to \mathbb{R}$ is convex and lower semicontinuous, then I is swlsc.*

Proof. In the proof we need Mazur's lemma which states that for a weakly convergent sequence $u_n \rightharpoonup u$ in X a sequence of convex combinations of u_n converges to u in the norm (we do not assume that the space is reflexive). Let us state the lemma precisely.

Lemma 6 (Mazur's lemma) *Let X be a Banach space and let $u_n \rightharpoonup u$ be a sequence weakly convergent in X. Then $v_n \to u$ in the norm for some sequence v_n of the form*

$$v_n = \sum_{k=n}^{N(n)} a_k^n u_k$$

where $a_k^n \geq 0$, $\sum_{k=n}^{N(n)} a_k^n = 1$. □

To prove the theorem we have to prove that $u_n \rightharpoonup u$ implies $I(u) \leq \liminf_{n\to\infty} I(u_n)$. We can assume that $I(u_n)$ has a limit $\lim_{n\to\infty} I(u_n) = g$. Let v_n be a sequence as in Mazur's lemma. Then by lower semicontinuity and convexity we have

$$I(u) \leq \liminf_{n\to\infty} I(v_n) \leq \liminf_{n\to\infty} \sum_{k=n}^{N(n)} a_k^n I(u_k) = g.$$

This completes the proof of the theorem. □

Corollary 7 *If* $I : X \to \mathbb{R}$ *is a convex, lower semicontinuous and coercive functional defined an a reflexive Banach space, then* I *attains minimum on* X *i.e. there exists* $\bar{u} \in X$ *such that* $I(\bar{u}) = \inf_X I(u)$. *If in addition the functional is strictly convex, then the minimum is unique.* □

As we will see, in many cases it is very easy to verify assumptions of the above corollary. Such an abstract approach to the existence of minimizers of variational problems was proposed by Mazur and Schauder in 1936 on the International Congress of Mathematics in Oslo.

Origin of the Sobolev spaces. Now we show how to apply the above direct method to the Dirichlet problem stated at the begining of the lecture. As we know, the equivalent problem is to find a minimizer of the functional $I(u) = \int_\Omega |\nabla u|^2$ in the class of functions with given restriction to the boundary.

Let us try to apply Corollary 7. Forst of all the functional I is defined on the space

$$C_g^2(\Omega) = w + C_b^2(\overline{\Omega}) = \{w + u :\ u \in C_b^2(\overline{\Omega})\}$$

where $w \in C^2(\overline{\Omega})$ is any functionn such that $w|_{\partial\Omega} = g$ and $C_b^2(\overline{\Omega})$ is a subspace of $C^2(\overline{\Omega})$ consisting of functions vanishing at the boundary.

The space $C_b^2(\overline{\Omega})$ is not even a linear! To overcome this difficulty we make the following trick. Define $J : C_b^2(\overline{\Omega}) \to \mathbb{R}$ by the formula $J(u) = I(u + w)$. Now the equivalent problem is to find a minimizer of J in the Banach space $C_b^2(\overline{\Omega})$.

The functional J is convex and continuous. Unfortunatelly neither the space $C_b^2(\overline{\Omega})$ is reflexive nor the functional J is coercive. If $n \geq 2$, then one can construct a sequence of $C_b^2(\overline{\Omega})$ functions with the supremum norm (and hence the C^2 norm) tending to infinity, but with the L^2 norm of the gradient tending to zero. This proves that the functional is not coercive. It is also easy to construct a relevant example when $n = 1$. We leave details to the reader.

The problem is caused by the fact that there is no way yo bound the C^2 norm of a function by the L^2 norm of its gradient[2] Hence we should change the norm in the space in a way that everything will be governed by $\|u\|_2$.

Observefirst that $u \mapsto \|\nabla u\|_2$ is not a norm on C^2 as it anihilates alll the constant functions, so we shall add a term to prevent this phenomenon. This suggents the norm $\|u\|_{1,2} = \|u\|_2 + \|\nabla u\|_2$. Then however the space C^2 equipped with the norm $\|\cdot\|_{1,2}$ is not complete. Hence we have to take a completion of the C^2 (or C^1) functions with respect to this norm. All that motivates the following definition.

Let $\Omega \subset \mathbb{R}^n$ be an open set and $1 \leq p < \infty$. Sobolev space $W^{1,p}(\Omega)$ is defined as a closure of the set of $C^1(\Omega)$ functions in the norm

$$\|u\|_{1,p} = \|u\|_p + \|\nabla u\|_p \,.$$

Of course we take into account only those C^1 functions for which the norm is finite.

Now we extend I by continuity to $W^{1,2}(\Omega)$.

$W^{1,2}(\Omega)$ is a Hilbert space with the scalar product

$$\langle u, v\rangle = \int_\Omega uv + \int_\Omega \nabla u \cdot \nabla v.$$

It is still not a correct setting for the Dirichlet problem since we seek a minimizer among the functions with fixed restriction (called *trace*) to the boundary. Thus we need more definitions.

Let $W_0^{1,p}(\Omega) \subset W^{1,p}(\Omega)$ be defined as the closure of the subset $C_0^\infty(\Omega)$ in the Sobolev norm. Rougly speaking $W_0^{1,p}(\Omega)$ is a subspace of $W^{1,p}(\Omega)$ consisting of functions which vanish on the boundary.

Fix $w \in W^{1,p}(\Omega)$ and define $W_w^{1,p}(\Omega) = w + W_0^{1,p}(\Omega)$. Thus $W_w^{1,p}(\Omega)$ consints of all those functions in the Sobolev space that, in some sense, have the same trace at the boundary as w. Note that $W_w^{1,p}(\Omega)$ is not linear but an affine subspace of $W^{1,p}(\Omega)$. The elements of the Sobolev space need not be continuous, so it does not make sense to take a restriction to the boundary. Thus we should understand that elements of $W_w^{1,p}(\Omega)$ have the same trace on the boundary as w only in a very rough sense.

Now we can formulate the variational problem of finding minimizer of $I(u) = \int_\Omega |\nabla u|^2$ with given trace on the boundary in the setting of Sobolev spaces.

Let $\Omega \subset \mathbb{R}^n$ be open and bounded and let $w \in W^{1,2}(\Omega)$. Find $\bar{u} \in W_w^{1,2}(\Omega)$ such that

$$\int_\Omega |\nabla \bar{u}|^2 = \inf_{u \in W_w^{1,2}(\Omega)} \int_\Omega |\nabla u|^2. \tag{4}$$

[2] This suggests the question of finding norms which can be estimated by the L^2 or, more generally, by the L^p norm of the gradient. This will lead us to so called Sobolev inequalities.

In a moment we will see that it easily follows from Corollary 7 that the above problem has a solution. Does the minimizer of (4) solve the Laplace equation in any reasonable sense as it was in the case of the classical Dirichlet principle? If yes, then does it have anything to do with the classical solution to the Dirichlet problem as stated at the begining?

The method of solving variational problems (more general than the one described above) consists very often of two main steps. First we prove the existence of the solution in a Sobolev space. This space is very large. To large. Then using the theory of Sobolev spaces one can prove that this solution is in fact more regular. For example later we will prove that the Sobolev minimizer of (4) is C^∞ smooth, which will imply that $\bar{u}$ is actually the classical harmonic function.

Theorem 8 *Let $\Omega \subset \mathbb{R}^n$ be open and bounded and let $w \in W^{1,2}(\Omega)$. Then there exists unique $\bar{u} \in W^{1,2}_w(\Omega)$ which minimizes the Dirichlet integral in the sense of (4).*

Proof. In order to have functional defined on a Banach space set $J(u) = I(u + w)$ for $u \in W^{1,2}_0(\Omega)$. Now the equivalent problem is to prove the existence of the unique minimizer of the functional J on $W^{1,2}_0(\Omega)$.

The functional J is strictly convex and continuous. It remains to prove that J is coercive. We need the following lemma for $p = 2$.

Lemma 9 (Poincaré) *Let $\Omega \subset \mathbb{R}^n$ be open and bounded. For $u \in W^{1,p}_0(\Omega)$, $1 \le p < \infty$, we have*

$$\left(\int_\Omega |u|^p \, dx\right)^{1/p} \le C(p, \Omega) \left(\int_\Omega |\nabla u|^p \, dx\right)^{1/p} .$$

Proof. Assume first that $u \in C^\infty_0(\Omega)$. The general case follows by the approximation argument. Let $M > 0$ be such that $\Omega \subset [-M, M]^n$. Then for every $x \in [-M, M]^n$

$$u(x) = \int_{-M}^{x_1} D_1 u(t, x_2, \ldots, x_n) \, dt \le \int_{-M}^{M} |D_1 u| \, dt.$$

By Hölder inequality,

$$|u(x)|^p \le 2^{p-1} M^{p-1} \int_{-M}^{M} |D_1 u|^p \, dt,$$

and the assertion follows by integration with respect to x. The proof is complete. □

The space $W^{1,2}_0(\Omega)$ is equipped with a norm $\|u\|_{1,2} = \|u\|_2 + \|\nabla u\|_2$. The Poincaré inequality for $p = 2$ states that $\|\nabla u\|_2$ is an equivalent norm on the space $W^{1,2}_0(\Omega)$. Hence for a sequence $u_k \in W^{1,2}_0(\Omega)$, $\|u_k\|_{1,2} \to \infty$ if and only if $\|\nabla u_k\|_2 \to \infty$.

Now the inequality $\|\nabla(u + w)\|_2 \ge \|\nabla u\|_2 - \|\nabla w\|_2$ implies that if $\|u\|_{1,2} \to \infty$ in $W^{1,2}_0(\Omega)$, then $J(u) \to \infty$ which means the functional J is coercive. The proof of Theorem 8 is complete. □

Remark. Observe that $I(u) = \int_\Omega |\nabla u|^2$ is strictly convex and continuous on any of the spaces $W^{1,p}(\Omega)$ for $p \geq 2$. As we will see all the spaces $W^{1,p}$, $1 < p < \infty$ are reflexive. However I is not coercive when $p > 2$.

Lecture 2

Sobolev spaces. For the sake of further applications (including the proof that the minimizer obtained in Theorem 8 is C^∞ smooth), we need develope the theory of Sobolev spaces. We will see that the scope of applications of Sobolev spaces is very wide and it goes far beyond the calculus of variations and differential equations.

Let $\Omega \subset \mathbb{R}^n$ be an open set, $u, v \in L^1_{\mathrm{loc}}(\Omega)$ and let α be a multiindex. We say that $D^\alpha u = v$ in the *weak sense* if for every $\varphi \in C_0^\infty(\Omega)$

$$\int_\Omega v\varphi = (-1)^{|\alpha|} \int_\Omega u D^\alpha \varphi.$$

If $u \in C^\infty$, then the weak and classical derivatives are equal to each other due to the integration by parts. More generally, if $P = \sum_{|\alpha| \leq m} a_\alpha D^\alpha$ is a differential operator with sufficently regular coefficients, and $u, v \in L^1_{\mathrm{loc}}$, then

$$Pu = v \qquad \text{in } \Omega$$

in the *weak sense* if

$$\int_\Omega v\varphi = \int_\Omega u \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha (a_\alpha \varphi),$$

for all $\varphi \in C_0^\infty(\Omega)$. The coefficients have to be sufficently regular, in order to know that $\sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha (a_\alpha \varphi)$ is a bounded function with the compact support, so we can integrate it against $u \in L^1_{\mathrm{loc}}$.

In particular $\Delta u = 0$ is the weak sense means $\int_\Omega u \, \Delta\varphi = 0$ for all $\varphi \in C_0^\infty(\Omega)$.

It follows from Lemma 2 that Pu is unique when understood in the weak sense (provided it exists).

During the lectures all the derivatives and the differential operators will be understood in the weak sense.

Let $1 \leq p \leq \infty$ and let m be an integer. *Sobolev space* $W^{m,p}(\Omega)$ is the set of all functions $u \in L^p(\Omega)$ such that the partial derivatives of order less than or equal to m exist in the weak sense and belong to $L^p(\Omega)$. The space is equipped with a norm

$$\|u\|_{m,p} = \sum_{|\alpha| \leq m} \|D^\alpha u\|_p.$$

The definition is of a different nature than the definition of the Sobolev space given in the previous lecture. We will see later that the two definitions are equivalent.

Theorem 10 *$W^{m,p}(\Omega)$ is a Banach space.*

Proof. If $\{u_k\}$ is a Cauchy sequence in $W^{m,p}(\Omega)$, then for every $|\alpha| \le m$, $D^\alpha u_k$ converges in $L^p(\Omega)$ to some $u_\alpha \in L^p(\Omega)$ (we will write u instead of u_0). Since

$$\int_\Omega u D^\alpha \varphi \leftarrow \int_\Omega u_k D^\alpha \varphi = (-1)^{|\alpha|} \int_\Omega D^\alpha u_k \varphi \to (-1)^{|\alpha|} \int_\Omega u_\alpha \varphi,$$

we conclude that $u_\alpha = D^\alpha u$ and that u_k converges to u in the norm of $W^{m,p}$. $\square$

Exercise. *Prove that if* $u \in W^{m,p}(\Omega)$ *and* $\varphi \in C^\infty(\Omega)$ *has bounded derivatives* $D^\alpha \varphi$ *for* $|\alpha| \le m$*, then* $u\varphi \in W^{m,p}(\Omega)$ *and* $D^\alpha(u\varphi)$ *can be computed from the Leibniz formula.*

If $u \in W^{m,p}(\mathbb{R}^n)$, $1 \le p < \infty$ and φ_ε is a standard mollifier kernel, then it follows readily from the properties of the convolution and from the definition of weak derivative that $C^\infty \ni u * \varphi_\varepsilon \stackrel{\varepsilon \to 0}{\longrightarrow} u$ in $W^{m,p}(\mathbb{R}^n)$. If η_R is a cutoff function, i.e. $\eta_R(x) = \eta(x/R)$, $\eta \in C_0^\infty(B(0,2))$, $\eta|_{B(0,1)} \equiv 1$, $0 \le \eta \le 1$, then one easily checks that $\eta_R u \stackrel{R \to \infty}{\longrightarrow} u$ in $W^{m,p}(\mathbb{R}^n)$. As we have just noticed, $\eta_R u$ can be approximated, via convolution, by smooth functions. Since $(\eta_R u) * \varphi_\varepsilon$ has a compact support we arrive to the following result.

Proposition 11 $C_0^\infty(\mathbb{R}^n)$ *is dense in* $W^{m,p}(\mathbb{R}^n)$ *for all* $1 \le p < \infty$ *and* $m = 0, 1, 2, \ldots$
$\square$

For $1 \le p < \infty$, by $W_0^{m,p}(\Omega)$ we denote the closure of $C_0^\infty(\Omega)$ in the $W^{m,p}(\Omega)$ norm, so Proposition 11 can be reformulated now as follows: $W^{m,p}(\mathbb{R}^n) = W_0^{m,p}(\mathbb{R}^n)$.

Very often, we will write L^p, $W^{m,p}$,... in place of $L^p(\mathbb{R}^n)$, $W^{m,p}(\mathbb{R}^n)$,... If in the definition of $W^{m,p}(\Omega)$ the space $L^p(\Omega)$ is replaced by $L^p_{\mathrm{loc}}(\Omega)$, then we obtain the space $W^{m,p}_{\mathrm{loc}}(\Omega)$. In the following result we prove that smooth functions form a dense subset of the Sobolev space. This implies that equivalently one can define Sobolev space as a closure of the set of smooth functions in the Sobolev norm. This establishes the equivalence with the definition employed in the previous lecture.

Theorem 12 (Meyers–Serrin) *If* $u \in W^{m,p}_{\mathrm{loc}}(\Omega)$*, where* $1 \le p < \infty$*, then to every* $\varepsilon > 0$ *there exists* $v \in C^\infty(\Omega)$ *such that*

1. $u - v \in W_0^{m,p}(\Omega)$,
2. $\|u - v\|_{m,p} < \varepsilon$.

Proof. Let $\{B_k\}_{k=1}^\infty$ be a locally finite covering of Ω by balls with subordinated partition of unity $\{\varphi_k\}_{k=1}^\infty$ such that the family $\{2B_k\}_{k=1}^\infty$ also forms a locally finite covering of Ω (by $2B_k$ we denote a ball with the same center as B_k and twice the radius).

Let $\varepsilon > 0$ be taken arbitrarily. Each of the functions $u\varphi_k$ (which belongs to $W_0^{m,p}(\Omega)$) can be approximated by a smooth function with compact support contained in $2B_k$ (standard approximation by convolution). Hence there exists $v_k \in C_0^\infty(2B_k)$ such that

$$\|u\varphi_k - v_k\|_{m,p} \le \varepsilon/2^k.$$

Now the series $\sum(u\varphi_k - v_k)$ converges in $W_0^{m,p}(\Omega)$, but we also have the pointwise convergence $\sum(u\varphi_k - v_k) = u - v$, where $v = \sum v_k \in C^\infty(\Omega)$ and hence

$$\|u - v\|_{m,p} \le \sum_{k=1}^{\infty} \|u\varphi_k - v_k\|_{m,p} \le \varepsilon.$$

□

Remark. In the previous lecture we have defined the Sobolev space as the closure of $C^1(\Omega)$ functions in the norm of $W^{1,p}(\Omega)$. It was however not obvious whether given function in $W^{1,p}(\Omega)$ has the unique gradient. Namely if we could find a sequence $u_k \in C^1(\Omega)$ such that $u_k \to 0$ in L^p and $\nabla u_k \to v$ in L^p, $v \not\equiv 0$, then $(0, v)$ would belong to $W^{1,p}(\Omega)$ and so v would be the gradient of 0! Fortunately this will not happend: since the Sobolev space can be defined in terms of weak derivatives we see that any function from the Sobolev space has the unique gradient — the weak gradinent.

One can ask whetcher $C^\infty(\overline{\Omega})$ is a dense subset of $W^{m,p}(\Omega)$. In general it is not. For example if Ω is a two dimensional disc with a radius removed, then $C^\infty(\overline{\Omega})$ is not dense in $W^{1,p}(\Omega)$ for any p (Why?). However one can prove the following

Theorem 13 *If Ω is a bounded domain whose boundary is locally a graph of a continuous function, then $C^\infty(\overline{\Omega})$ is dense in $W^{m,p}(\Omega)$ for any $1 \le p < \infty$ and any m,*

We will not prove it. □

The following result is a counterpart of the Dirichlet principle in the setting of Sobolev spaces.

Theorem 14 *u is a minimizer of the Dirichlet integral (4) if and only if u is a weak solution to the Dirichlet problem: $\Delta u = 0$, $u \in W_w^{1,2}(\Omega)$.*

Proof. $\Rightarrow$. Let u be a minimizer of (4). Then $u + t\varphi \in W_w^{1,2}(\Omega)$ for any $\varphi \in C_0^\infty(\Omega)$ and hence

$$0 = \frac{d}{dt}|_{t=0} I(u + t\varphi) = 2\int_\Omega \langle \nabla u, \nabla\varphi\rangle = -2\int_\Omega u\Delta\varphi\,,$$

which means $\Delta u = 0$ in the weak sense.

$\Leftarrow$. Since by Theorem 8 the minimizer exists we get that one of the weak solutions is the minimizer. Now it remains to prove that the solution is unique. Assume that $u_1, u_2 \in W_w^{1,2}(\Omega)$ are weak solutions to $\Delta u = 0$. Then

$$\int_\Omega \nabla u_1 \cdot \nabla\varphi = 0, \qquad \int_\Omega \nabla u_2 \cdot \nabla\varphi = 0\,,$$

for all $\varphi \in C_0^\infty(\Omega)$, and so

$$\int_\Omega (\nabla u_1 - \nabla u_2)\cdot\nabla\varphi = 0 \qquad \forall\varphi\in C_0^\infty(\Omega)\,.$$

By the approximation argument it is true also for all $\varphi \in W_0^{1,2}(\Omega)$. Taking $\varphi = u_1 - u_2$ w conclude

$$\int_\Omega |\nabla(u_1 - u_2)|^2 = 0\,.$$

Hence by Poincaré inequality (Lemma 9), $u_1 = u_2$. □

For the sake of simplicity we will be concerned with Sobolev spaces $W^{m,p}(\Omega)$ for $m = 1$ only. However, we want to point out that most of the results have their counterparts for higher order derivatives.

It was essential for the direct method that the functional was defined on a reflexive space. In this regard, the following result is very important.

Theorem 15 *If $1 < p < \infty$, then the space $W^{1,p}(\Omega)$ is reflexive.*

Proof. Closed subspace of a reflexive space is reflexive. Thus it suffices to find an isomorphism between $W^{1,p}(\Omega)$ and a closed subspace of $L^p(\Omega, \mathbb{R}^{n+1})$. The isomorphism is given by the mapping $\Phi(u) = (u, \nabla u)$. The proof is complete. □

ACL characterization. The definition of the Sobolev space is quite abstract and it is not obvious how to verify whether given function $u \in L^p(\Omega)$ belongs to $W^{1,p}(\Omega)$. Below we provide a characterization of the Sobolev spaces which goes back to Nikodym. The characterization is very convenient when checking whether given function belongs to the Sobolev space.

First we need recall the definition of the absolutely continuous function.

We say that a continuous function u defined on an interval $[a,b]$ is *absolutely continuous* if to every $\varepsilon > 0$ there is $\delta > 0$ such that the following implication holds: If $I_1,\ldots,I_k$ are pairwise disjoint segments contained in $[a,b]$ with $\sum_{i=1}^k |I_i| < \delta$, then $\sum_{i=1}^k |u(I_i)| < \varepsilon$.

We will denote the class of absolutely continuous functions on $[a,b]$ by $AC[a,b]$.

It is easy to see that the function $u(x) = c + \int_a^x h(t)\,dt$, where $h \in L^1(a,b)$ and c is a constant is absolutely continuous. As it follows from the following lemma these are the only absolutely continuous functions.

Lemma 16 *If $u \in AC[a,b]$, then u' exists a.e., $u' \in L^1(a,b)$ and $u(x) = u(a) + \int_a^x u'(t)\,dt$ for all $x \in [a,b]$.* □

We skip the proof. Also we will not prove the following integration by parts formula.

Lemma 17 *If $u, v \in AC[a,b]$, then the formula for integration by parts holds*

$$\int_a^b u(x)v'(x)\,dx = uv|_a^b - \int_a^b u'(x)v(x)\,dx.$$

□

If $U \subset \mathbb{R}$ is an open set, then we say that $u \in AC(U)$ if $u \in AC[a,b]$, whenever $[a,b] \subset U$.

Let $\Omega \subset \mathbb{R}^n$ be an open set. We say that $u \in ACL(\Omega)$ (absolutely continuous on lines) if the function u is Borel measurable and absolutely continuous on almost all lines paralell to coordinate axes. Since absolutely continous functions are differentiable a.e., $u \in ACL(\Omega)$ has partial derivatives a.e. and hence the classical gradient ∇u is defined a.e. Now we say that $u \in ACL^p(\Omega)$ if $u \in L^p(\Omega) \cap ACL(\Omega)$ and $|\nabla u| \in L^p(\Omega)$. The following result goes back to Nikodym.

Theorem 18 (ACL characterization) $W^{1,p}(\Omega) = ACL^p(\Omega)$, $1 \leq p \leq \infty$. □

Since maybe it is not evident how to understand the theorem we shall comment it now. The theorem asserts that each $ACL^p(\Omega)$ function belongs to $W^{1,p}(\Omega)$ and that the classical partial derivatives (which exist a.e. for elements of $ACL^p(\Omega)$) are equal to weak partial derivatives. On the other hand every element $u \in W^{1,p}(\Omega)$ can be alternated on a set of measure zero in a way that the resulting function belongs to $ACL^p(\Omega)$.

The proof of the inclusion $ACL^p(\Omega) \subset W^{1,p}(\Omega)$ is easy. It follows from the fact that integration by parts holds for the absolutely continuous functions, from the Fubini theorem, and from the definition of the weak derivative. The opposite implication is more involved and we will not prove it.

Example. The radial projection mapping

$$u_0(x) = \frac{x}{|x|} : B^n(0,1) \to S^{n-1}(0,1) \subset \mathbb{R}^n,$$

is discontinuous at $x = 0$. The coordinate functions $x_i/|x|$ of u_0 are absolutely continuous on almost all lines. Moreover

$$\frac{\partial}{\partial x_j}\left(\frac{x_i}{|x|}\right) = \frac{\delta_{ij}|x| - x_i x_j/|x|}{|x|^2} \in L^p(B^n(0,1)),$$

for all $1 \leq p < n$. Hence by the ACL characterization $u_0 \in W^{1,p}(B^n, \mathbb{R}^n)$ for all $1 \leq p < n$. Here δ_{ij} is the Kronecker symbol i.e., $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. □

The following four results are direct consequences on the ACL characterization.

Corollary 19 *Functions in the space* $W^{1,\infty}(\Omega)$ *are locally Lipschitz continuous. If in addition* Ω *is a bounded Lipschitz domain (i.e.* $\partial\Omega$ *is locally a graph of a Lipschitz function), then* $W^{1,\infty}(\Omega) = \mathrm{Lip}\,(\Omega)$. □

Corollary 20 *If* $u \in W^{1,p}(\Omega)$, *where* Ω *is connected and* $\nabla u = 0$ *a.e., then* u *is constant.* □

Corollary 21 *If* $u \in W^{1,p}(\Omega)$ *is constant in a measurable set* $E \subset \Omega$, *then* $\nabla u = 0$ *a.e. in* E. □

Corollary 22 *Let* $u \in W^{1,p}(\Omega)$, $1 \le p \le \infty$. *Then* $u_\pm \in W^{1,p}(\Omega)$, *where* $u_+ = \max\{u, 0\}$, $u_- = \min\{u, 0\}$ *and*

$$\nabla u_+(x) = \begin{cases} \nabla u(x) & \text{if} \quad u(x) > 0\,, \\ 0 & \text{if} \quad u(x) \le 0\,, \end{cases}$$

almost everywhere. Similar formula holds for ∇u_-. □

Another characterization of the Sobolev space. In the sequel we will need the following elegant characterization of the Sobolev space.

Theorem 23 *Let* $1 < p < \infty$ *and suppose that* $\Omega \subset \mathbb{R}^n$ *is an open set. Then* $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ *if and only if* $u \in L^p_{\mathrm{loc}}(\Omega)$ *and for every* $\Omega' \subset\subset \Omega$ *there is a constant* $C_{\Omega'}$ *such that*

$$\|u(\cdot + h) - u(\cdot)\|_{L^p(\Omega')} \le C_{\Omega'}|h|$$

provided $|h| < \min\{\frac{1}{2}\mathrm{dist}\,(\Omega', \partial\Omega), 1\}$.

Proof. 1. ⇒ 2. Assume first that $u \in C^\infty(\Omega)$. Then we have

$$\frac{u(x+h) - u(x)}{|h|} = \frac{1}{|h|}\int_0^{|h|} \nabla u\left(x + t\frac{h}{|h|}\right) \cdot \frac{h}{|h|}\,dt.$$

Hence applying Hölder's inequality and integrating over Ω' we get

$$|h|^{-p}\int_{\Omega'} |u(x+h) - u(x)|^p\,dx \le \frac{1}{|h|}\int_0^{|h|}\int_{\Omega'}\left|\nabla u\left(x + t\frac{h}{|h|}\right)\right|^p dx\,dt \le \int_V |\nabla u(x)|^p\,dx,$$

for some V with $\Omega' \subset\subset V \subset\subset \Omega$ that does not depend on h. Now by Meyers–Serrin's theorem the inequality holds for any $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$.

2. ⇒ 1. Dentote by e_k, one of the coordinate directions. Let $h_i \to 0$. Then the sequence $(u(x + h_ie_k) - u(x))/h_i$ is bounded in $L^p(\Omega')$. By the reflexivity of $L^p(\Omega')$ we can substract a weakly convergent subsequence to some $u_k \in L^p(\Omega')$. It remais to prove

that u_k coincides with the distributional derivative $\partial u/\partial x_k$. To this end note that for every $\varphi \in C_0^\infty(\Omega')$

$$\begin{aligned}
\int_{\Omega'} u_k \varphi \, dx &= \lim_{i\to\infty} \int_{\Omega'} \left(\frac{u(x+h_i e_k) - u(x)}{h_i} \right) \varphi(x) \, dx \\
&= \lim_{i\to\infty} \int_{\Omega'} u(x) \left(\frac{\varphi(x-h_i e_k) - \varphi(x)}{h_i} \right) dx \\
&= -\int_{\Omega'} u \frac{\partial \varphi}{\partial x_k} \, dx.
\end{aligned}$$

□

As we have seen, the difference quotient is weakly convergent in L^p to the distributional partial deriative. In fact one can prove the strong convergence in L^p. Namely we have.

Proposition 24 *If $u \in W^{1,p}_{\rm loc}(\Omega)$, $1 < p < \infty$, then*

$$\frac{u(x+he_k) - u(x)}{h} \xrightarrow{h\to 0} \frac{\partial u}{\partial x_k}$$

in $L^p(\Omega')$ for every $\Omega' \subset\subset \Omega$.

We leave the proof as an exercise. □

Poincaré inequality and Riesz potentials. The lemma below provides a very powerful integral estimate for Sobolev functions.

Lemma 25 *Let $B \subset \mathbb{R}^n$ be a ball. Then for every $u \in W^{1,p}(B)$, $1 \le p \le \infty$,*

$$|u(x) - u_B| \le C(n) \int_B \frac{|\nabla u(z)|}{|x-z|^{n-1}} \, dz \ \ a.e. \tag{5}$$

Proof. First we prove the inequality for $u \in C^\infty(B)$. Fix $x \in B$. For $y \in B$, $y \ne x$ set

$$y = x + t\frac{y-x}{|y-x|} = x + tz, \qquad z \in S^{n-1}$$

and let $\delta(z) = \max\{t > 0 : \, x + tz \in B\}$. We have

$$|u(x) - u(y)| \le \int_0^{|y-x|} \left| \nabla u(x + s\frac{y-x}{|y-x|}) \right| ds \le \int_0^{\delta(z)} |\nabla u(x+sz)| \, ds.$$

Denoting by dz the surface measure on S^{n-1} we get

$$|u(x) - u_B| \le |B|^{-1} \int_B |u(x) - u(y)| \, dy$$

(polar coordinates)

$$
\begin{aligned}
&= |B|^{-1}\int_{S^{n-1}}\int_0^{\delta(z)} t^{n-1}|u(x)-u(x+tz)|\,dt\,dz \\
&\leq |B|^{-1}\int_{S^{n-1}}\int_0^{\delta(z)} t^{n-1}\int_0^{\delta(z)}|\nabla u(x+sz)|\,ds\,dt\,dz \\
&\leq |B|^{-1}\int_{S^{n-1}}\int_0^{2r} t^{n-1}\,dt\int_0^{\delta(z)}|\nabla u(x+sz)|\,ds\,dz \\
&= C(n)\int_{S^{n-1}}\int_0^{\delta(z)}\frac{|\nabla u(x+sz)|}{s^{n-1}}s^{n-1}\,ds\,dz \\
&\quad \text{(polar coordinates)} \\
&= C(n)\int_B \frac{|\nabla u(y)|}{|x-y|^{n-1}}\,dy.
\end{aligned}
$$

The case of general $u \in W^{1,p}(B)$ follows by approximating it by C^∞ smooth functions. to this end we have to know that if $u_k \to u$ in $W^{1,p}$, then after substracting a subsequence, $I_1^B|\nabla u_k| \to I_1^B|\nabla u|$ a.e. where $I_1^B g(x) = \int_B g(z)|x-z|^{1-n}\,dz$. This will follow from Lemma 27 below.

To get further estimates we introduce Riesz potentials. The Riesz potential is an integral operator I_α, $0<\alpha<n$, defined by the formula

$$I_\alpha g(x) = \int_{\mathbb{R}^n}\frac{g(z)}{|x-z|^{n-\alpha}}\,dz.$$

If $\Omega \subset \mathbb{R}^n$, then we set

$$I_\alpha^\Omega g(x) = \int_\Omega \frac{g(z)}{|x-z|^{n-\alpha}}\,dz.$$

We start with an elementary, but very useful observation which will be frequently employed in the sequel.

Lemma 26 *If $E \subset \mathbb{R}^n$ is a measurable set of finite measure, then*

$$\int_E \frac{dz}{|x-z|^{n-1}} \leq C(n)|E|^{1/n},$$

for all $x \in \mathbb{R}^n$.

Proof. Let $B = B(x,r)$ be a ball with $|B| = |E|$. Then it easily followins that

$$\int_E \frac{dz}{|x-z|^{n-1}} \leq \int_B \frac{dz}{|x-z|^{n-1}} = C(n)r = C'(n)|E|^{1/n}.$$

The proof is complete. □

Lemma 27 *If $|\Omega| < \infty$, then for $1 \leq p < \infty$ we have*

$$\|I_1^\Omega g\|_{L^p(\Omega)} \leq C(n,p)|\Omega|^{1/n}\|g\|_{L^p(\Omega)}.$$

Proof. It followins from the provious lemma that

$$\int_\Omega \frac{dz}{|x-z|^{n-1}} \leq C(n)|\Omega|^{1/n}\,.$$

Now if $p>1$, then Hölder's inequality with respect to the measure $|x-z|^{1-n}dz$ implies

$$\begin{aligned}\int_\Omega \frac{|g(z)|}{|x-z|^{n-1}}\,dz &\leq \left(\int_\Omega \frac{|g(z)|^p}{|x-z|^{n-1}}\,dz\right)^{1/p}\left(\int_\Omega \frac{dz}{|x-z|^{n-1}}\,dz\right)^{1-1/p}\\ &\leq C|\Omega|^{\frac{p-1}{np}}\left(\int_\Omega \frac{|g(z)|^p}{|x-z|^{n-1}}\,dz\right)^{1/p}.\end{aligned}$$

If $p=1$, then the above inequality is obvious. Now we can conclude the proof using Fubini's theorem.

$$\begin{aligned}\int_\Omega |I_1^\Omega g(x)|^p\,dx &\leq C|\Omega|^{\frac{p-1}{n}}\int_\Omega\int_\Omega \frac{|g(z)|^p}{|x-z|^{n-1}}\,dz\,dx\\ &\leq C|\Omega|^{\frac{p-1}{n}}|\Omega|^{\frac{1}{n}}\int_\Omega |g(z)|^p\,dz.\end{aligned}$$

This completes the proof of Lemma 27 and hence that for Lemma 25. □

Remark. The proof of Lemma 25 easily extends to the case of an arbitrary bounded, convex domain.

As a direct consequence of Lemma 25 and Lemma 27 we obtain

Corollary 28 *If $u\in W^{1,p}(B)$, where B is a ball of radius r, and $1\leq p<\infty$, then*

$$\left(⨍_B |u-u_B|^p\,dx\right)^{1/p} \leq C(n,p)r\left(⨍_B |\nabla u|^p\,dx\right)^{1/p}.$$

Lecture 3

Bi-Lipschitz change of variables. We say that the mapping $T:\Omega\to\mathbb{R}^n$, $\Omega\subset\mathbb{R}^n$ is *bi-Lipschitz* if there is a constant $C\geq 1$ such that

$$C^{-1}|x-y|\leq |T(x)-T(y)|\leq C|x-y|\,,$$

for all $x,y\in\Omega$. Obviously bi-Lipschitz mapping is a homeomorphism.

An important property of Lipschitz functions is that they are differentiable a.e. In particular bi-Lipschitz mappings are differentiable a.e. Namely we have

Theorem 29 (Rademacher) *If u is a Lipschitz function defined in an open subset of $\mathbb{R}^n$, then u is differentiable a.e.*

The following result is a generalization of the classical change of variables formula to the case of bi-Lipschitz transformations.

Theorem 30 *Let* $T : \Omega \to \mathbb{R}^n$, $\Omega \subset \mathbb{R}^n$, *be a bi-Lipschitz homeomorphism and let* $f : T(\Omega) \to \mathbb{R}$ *be measurable. Then*

$$\int_\Omega (f \circ T)|J_T| = \int_{T(\Omega)} f\,,$$

in the sense that if one of the integrals exists then the second one exists and the integrals are equal one to another.

We will not prove it. □

As a consequence we obtain that the Sobolev space $W^{1,p}$ is invariant under the bi-Lipschitz change of variables. Namely we have

Theorem 31 *Let* $T : \Omega_1 \to \Omega_2$, *be a bi-Lipschitz homeomorphism between domains* $\Omega_2, \Omega_2 \subset \mathbb{R}^n$. *Then* $u \in W^{1,p}(\Omega_2)$, $1 \le p \le \infty$ *if and only if* $v = u \circ T \in W^{1,p}(\Omega_1)$, *and*

$$Dv(x) = Du(T(x)) \cdot DT(x) \tag{6}$$

for almost all $x \in \Omega_1$. *Moreover the transformation* $T^* : W^{1,p}(\Omega_2) \to W^{1,p}(\Omega_1)$ *given by* $T^*u = u \circ T$ *is an isomorphism of Sobolev spaces.*

Proof. Assume first that u is locally Lipschitz. Then (6) is obvious.

Since T is Lipschitz, we have $|DT| \le C$, so the chain rule (6) implies

$$|Dv(x)|^p \le C|Du(T(x))|^p.$$

The fact that T is bi-Lipschitz implies $|J_T| > C$ and hence

$$|Dv(x)|^p \le C|Du(T(x))|^p|J_T(x)|$$

Now applying the change of variables formula we conclude

$$\int_{\Omega_1} |Dv|^p \le C\int_{\Omega_1} |Du(T(x))|^p|J_T(x)| = \int_{\Omega_2} |Du|^p.$$

By a similar argument $\int_{\Omega_1} |v|^p \le C\int_{\Omega_2} |u|^p$, and hence $\|T^*u\|_{W^{1,p}(\Omega_1)} \le C\|u\|_{W^{1,p}(\Omega_2)}$. We proved the inequality when u is locally Lipschitz. By the density argument (Theorem 12 and Corollary 19) it is true for any $u \in W^{1,p}(\Omega_2)$.

Applying the above argument to T^{-1} we conclude that T^* is an isomorphism of Sobolev spaces. Finally the density argument proves (6) for any $u \in W^{1,p}(\Omega_2)$. □

The change of variables transformation shows that one can define Sobolev spaces on manifolds. We will come back to this question later on.

Sobolev embedding theorem. By the definition the function $u \in W^{1,p}(\Omega)$ is only L^p integrable. However as we will see, we can say much more about the regularity of u. We start with the case $1 \leq p < n$. Later we will see that in the case $p > n$ the function is Hölder continuous.

Theorem 32 (Sobolev embedding theorem) *Let $1 \leq p < n$ and $p^* = np/(n-p)$. Then for $u \in W^{1,p}(\mathbb{R}^n)$ we have*

$$\left(\int_{\mathbb{R}^n} |u(x)|^{p^*}\, dx\right)^{1/p^*} \leq C(n,p) \left(\int_{\mathbb{R}^n} |\nabla u(x)|^p\, dx\right)^{1/p}. \tag{7}$$

Proof. Step 1. $p = 1$. This is the crucial step in the proof. As we will see later, the general case in which $1 \leq p < n$ easily follows from the case in which $p = 1$.

By the density argument we can assume that $u \in C_0^\infty(\mathbb{R}^n)$. We have

$$|u(x)| \leq \int_{-\infty}^{x_1} |D_1 u(t_1, x_2, \ldots, x_n)|\, dt_1 \leq \int_{-\infty}^{\infty} |D_1 u(t_1, x_2, \ldots, x_n)|\, dt_1.$$

Here by D_i we denote the partial derivative with respect to i-th coordinate. Analogous inequalities hold with x_1 replaced by $x_2, \ldots, x_n$. Hence

$$|u(x)|^{\frac{n}{n-1}} \leq \prod_{i=1}^{n} \left(\int_{-\infty}^{\infty} |D_i u|\, dt_i\right)^{\frac{1}{n-1}}.$$

Now we integrate both sides with respect to $x_1 \in \mathbb{R}$. Note that exactly one integral in the product on the right hand side does not depend on x_1. Applying Hölder's inequality to the remaining $n-1$ integrals yields

$$\int_{-\infty}^{\infty} |u(x)|^{\frac{n}{n-1}}\, dx_1 \leq \left(\int_{-\infty}^{\infty} |D_1 u|\, dt_1\right)^{\frac{1}{n-1}} \prod_{i=2}^{n} \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |D_i u|\, dt_i\, dx_1\right)^{\frac{1}{n-1}}.$$

Next, we integrate both sides with respect to $x_2 \in \mathbb{R}$ and apply Hölder's inequality in a similar way as above. This leads to an inequality which we then integrate with respect to $x_3 \in \mathbb{R}$ etc. In the end, we obtain the inequality

$$\int_{\mathbb{R}^n} |u(x)|^{\frac{n}{n-1}}\, dx \leq \prod_{i=1}^{n} \left(\int_{\mathbb{R}^n} |D_i u|\, dx\right)^{\frac{1}{n-1}},$$

which readily implies (7) when $p = 1$.

Step 2. General case. Let $u \in C_0^\infty(\mathbb{R}^n)$. Define a nonnegative function f of class C^1 by

$$f^{\frac{n}{n-1}} = |u|^{\frac{np}{n-p}}.$$

Applying (7) with $p = 1$ to f yields

$$\left(\int |u|^{\frac{np}{n-p}}\right)^{\frac{n-1}{n}} = \left(\int f^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}} \leq C \int |\nabla f|.$$

Since

$$|\nabla f| = \frac{p(n-1)}{n-p} |u|^{\frac{n(p-1)}{n-p}} |\nabla u|,$$

the theorem easily follows by Hölder's inequality (with suitable exponents) applied to $\int |\nabla f|$. The proof is complete. □

Extension operator. Any bounded linear operator $E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$ such that $Eu|_\Omega = u$ for $u \in W^{1,p}(\Omega)$ is called an *extension operator.*

We say that $\Omega \subset \mathbb{R}^n$ is a *Lipschitz domain* if Ω is bounded and the boundary of Ω is localy a graph of a Lipschitz function.

Theorem 33 *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $1 \leq p \leq \infty$. Then there exists an extension operator $E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$.*

Proof. Let $\mathbb{R}^n_+ = \{(x_1, x') : x_1 > 0\}$ be a halfspace. Then we define an extension operator $E : W^{1,p}(\mathbb{R}^n_+) \to W^{1,p}(\mathbb{R}^n)$ be reflection i.e. $Eu = u$ on $\mathbb{R}^n_+$ and $Eu(x_1, x') = u(-x_1, x')$ for $x_1 < 0$. By the ACL characterization of the Sobolev space $\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^n_+)}$ for all $u \in W^{1,p}(\mathbb{R}^n_+)$.

Now if Ω is a bounded Lipschitz domain then we use a partition of unity to localize u near the boundary, next we flat small parts of the boundary using bi-Lipschitz homeomorphisms and we extend the localizied pieces of function across the flat boundaries using the reflection described above. Finally we come back using the inverse bi-Lipschitz homeomorphisms. This is only a sketch of the proof. We leave details to the reader. □

We will use the above extension operator to prove a general Sobolev inequality.

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and let $E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$ be an extension operator. Then invoking Theorem 32 we get for $1 \leq p < n$

$$\begin{aligned} \left(\int_\Omega |u|^{p^*}\, dx\right)^{1/p^*} &\leq \left(\int_{\mathbb{R}^n} |Eu|^{p^*}\, dx\right)^{1/p^*} \leq C\left(\int_{\mathbb{R}^n} |\nabla(Eu)|^p\, dx\right)^{1/p} \\ &\leq C\left(\left(\int_\Omega |\nabla u|^p\, dx\right)^{1/p} + \left(\int_\Omega |u|^p\, dx\right)^{1/p}\right). \end{aligned}$$

Hence we proved the following result

Proposition 34 *If $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain and $1 \leq p < n$, then $W^{1,p}(\Omega) \subset L^{p^*}(\Omega)$. Moreover*

$$\|u\|_{L^{p^*}(\Omega)} \leq C(\|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)}).$$

Compactness. One of the most important results in the theory of Sobolev spaces is the following theorem.

Theorem 35 (Rellich–Kondrachov) *Let Ω be a bounded Lipschitz domain. The the embedding*

$$W^{1,p}(\Omega) \subset L^q(\Omega)$$

is compact, if $q < p^*$ and $1 \le p < n$ or $q < \infty$ and $n \le p < \infty$.

We will not prove the theorem. □

We remark that the embedding $W^{1,p}(\Omega) \subset L^{p^*}(\Omega)$, $1 \le p < n$ is *not* compact.

As an application of the theorem we prove the following

Theorem 36 (Sobolev–Poincaré inequality) *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $1 \le p < n$. Then for every $u \in W^{1,p}(\Omega)$*

$$\left(\int_\Omega |u - u_\Omega|^{p^*} \, dx \right)^{1/p^*} \le C(\Omega, p) \left(\int_\Omega |\nabla u|^p \, dx \right)^{1/p},$$

where $p^* = np/(n-p)$.

Proof. Applying Proposition 34 to $u - u_\Omega$ we see that it remains to prove that

$$\|u\|_{L^p(\Omega)} \le C \|\nabla u\|_{L^p(\Omega)},$$

for all $u \in W^{1,p}(\Omega)$ with $\int_\Omega u = 0$.

Suppose this is not true. Then there is a sequence $u_k \in W^{1,p}(\Omega)$, such that $\int_\Omega u_k \, dx = 0$ and

$$\int_\Omega |u_k|^p \, dx \ge k \int_\Omega |\nabla u_k|^p \, dx. \tag{8}$$

Multiplying u_k by a suitable constant we may assume in addition that $\int_\Omega |u_k|^p = 1$. Since the embedding $W^{1,p}(\Omega) \subset L^p(\Omega)$ is compact we may substract a subsequence u_{k_i} such that $u_{k_i} \to u$ in $L^p(\Omega)$. Hence $\int_\Omega |u|^p = 1$ ad $\int_\Omega u = 0$. Inequality (8) implies that $\nabla u_{k_i} \to 0$ in $L^p(\Omega)$. Hence u_{k_i} is a Cauchy sequence in $W^{1,p}(\Omega)$ and thus $u \in W^{1,p}(\Omega)$, $\nabla u = 0$ a.e. which means u is constant. This is a contradiction because $\int_\Omega u = 0$ and $\|u\|_p = 1$. □

Almost the same argument implies the following result.

Proposition 37 *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and let $E \subset \Omega$, $|E| > 0$, $1 \le p < n$. Then*

$$\left(\int_\Omega |u|^{p^*} \, dx \right)^{1/p^*} \le C(\Omega, E, p) \left(\int_\Omega |\nabla u|^p \, dx \right)^{1/p}$$

for all $u \in W^{1,p}(\Omega)$ with $u|_E \equiv 0$.

The argument employed in the proof of Theorem 36 establishes also the following

Theorem 38 *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $1 \leq p < \infty$. Then*

$$\left(\int_\Omega |u - u_\Omega|^p\, dx\right)^{1/p} \leq C(n,p) \left(\int_\Omega |\nabla u|^p\, dx\right)^{1/p}$$

for all $u \in W^{1,p}(\Omega)$.

Later we will need the following special case.

Corollary 39 *Let $1 \leq p < \infty$, and $n \geq 2$. Then for any $u \in W^{1,p}(B(2r) \setminus B(r))$ we have*

$$\left(\int_{B(2r)\setminus B(r)} |u - u_{B(2r)\setminus B(r)}|^p\, dx\right)^{1/p} \leq C(n,p) r \left(\int_{B(2r)\setminus B(r)} |\nabla u|^p\, dx\right)^{1/p}.$$

Proof. The assumption $n \geq 2$ is to guarantee that the annulus $B(2r)\setminus B(r)$ is connected. It remains to prove that the constant in the inequality is proportional to the radius of the ball. This easily follows from the scaling argument: If the radius of the ball is one, then the inequality holds with some constant $C = C(n,p)$. If the radius is arbitrary, then by a linear change of variables we can reduce it to the case in which $r = 1$. □

Hölder continuity. If $p \geq n$ and Ω is a bounded domain with the Lipschitz boundary, then $u \in W^{1,p}(\Omega)$ belongs to Sobolev spaces for all exponents less than n. Hence the Sobolev embedding theorem implies that u is integrable with aby exponent less than ∞. As we will see a much better result holds. Namely we will prove that if $p > n$, then the Sobolev functions are Hölder continuous.

Theorem 40 *If $u \in W^{1,p}(B)$, $n < p < \infty$, then $u \in C^{0,1-n/p}(B)$. Moreover*

$$|u(x) - u(y)| \leq C(n,p)|x-y|^{1-n/p} \left(\int_B |\nabla u(z)|^p\, dz\right)^{1/p}.$$

Proof. Let $E : W^{1,p}(B) \to W^{1,p}(2B)$ be an extension operator such that $\int_{2B} |\nabla(Eu)|^p \leq C \int_B |\nabla u|^p$ (show that such an operator exists). Denote $\tilde{u} = Eu$ for the simplicity of notation. Then for any ball $\widetilde{B} \subset 2B$ of radius r we have

$$\begin{aligned} |\tilde{u}(x) - \tilde{u}_{\widetilde{B}}| &\leq C(n) \int_{\widetilde{B}} \frac{|\nabla \tilde{u}(z)|}{|x-z|^{n-1}}\, dz \\ &\leq C(n) \left(\int_{\widetilde{B}} |\nabla \tilde{u}(z)|^p\, dz\right)^{1/p} \left(\int_{\widetilde{B}} \frac{dz}{|x-z|^{(n-1)p/(p-1)}}\right)^{1-1/p} \\ &\leq C(n,p) r^{1-\frac{n}{p}} \left(\int_{\widetilde{B}} |\nabla \tilde{u}(z)|^p\, dz\right)^{1/p} \\ &\leq C'(n,p) r^{1-n/p} \left(\int_B |\nabla u(z)|^p\, dz\right)^{1/p}. \end{aligned}$$

Now let $x, y \in B$ and let $\widetilde{B} \subset 2B$ be a ball such that $x, y \in \widetilde{B}$, $\operatorname{diam} \widetilde{B} \approx |x-y|$. The inequality

$$|u(x) - u(y)| \leq |u(x) - u_{\widetilde{B}}| + |u(y) - u_{\widetilde{B}}|$$

together with the previous inequlity yields the result. □

This result and the extension theorem imply the folowing corollary.

Corollary 41 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with the Lipschitz boundary. Then for $n < p < \infty$ $W^{1,p}(\Omega) \subset C^{0,1-n/p}(\Omega)$.*

Corollary 42 *Let $1 \leq p < \infty$. If $u \in W^{k,p}(\Omega)$ for all $k = 1, 2, \ldots$, then $u \in C^\infty(\Omega)$.*

Proof. If $p > n$, then $W^{1,p} \subset C^{0,\alpha}$, so $W^{k,p} \subset C^{k-1,\alpha}$ and the claim follows. Let $p < n$. Take k such that $kp < n$, but $(k+1)p > n$. Then $W^{1,p} \subset L^{np/(n-p)}$, so by induction

$$W^{k+1,p} \subset W^{k,np/(n-p)} \subset W^{k-1,np/(n-2p)} \subset \ldots \subset W^{1,np/(n-kp)} \subset C^{0,\alpha},$$

because $np/(n-kp) > n$. Hence $W^{m,p} \subset C^{m-k-1,\alpha}$ and then the claim follows. □

What does happen if $u \in W^{1,n}(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is bounded Lipschiz domain? As we have already mentioned u is integrable with any exponent strictly less than ∞. However u need not be bounded. Moreover one can construct $u \in W^{1,n}(\mathbb{R}^n)$ which is essentially discontinuous everywhere.

Exercise. *Show that $\log|\log|x||$ belongs to the Sobolev space $W^{1,n}$ in a neighborhood of 0. Use this function to constract $u \in W^{1,n}(\mathbb{R}^n)$ such that its essential supremum on every ball is $+\infty$ and its essential infimum on every ball is $-\infty$.*

More information about the integrability of $u \in W^{1,n}(\Omega)$ is provided by the following result.

Theorem 43 (Trudinger) *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Then there exist constants C_1, C_2 depending on n, p and Ω only such that*

$$⨍_\Omega \exp\left(\frac{|u - u_\Omega|}{C_1 \|\nabla u\|_{L^n(\Omega)}}\right)^{\frac{n}{n-1}} \leq C_2 \,.$$

for any $u \in W^{1,n}(\Omega)$.

We will not provide a proof. □

Traces. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. We want to restrict the function $u \in W^{1,p}(\Omega)$ to the boundary of Ω. If $p > n$, then the function u is Hölder continuous and such a restriction makes sense. However if $p \leq n$, then the function can be essentially discontinuous everywhere and the restriction makes no sense when understood in the usual way.

Thus in the case $p \leq n$ we want to describe the trace in the following way. Find a function space defined on the boundary $X(\partial\Omega)$ with a norm $\|\cdot\|_X$ such that the operator of restriction $Tu = u|_{\partial\Omega}$ defined for $u \in C^\infty(\overline{\Omega})$ is continuous in the sense that it satisfies the estimate $\|Tu\|_X \leq C\|u\|_{W^{1,p}(\Omega)}$. We will assume that Ω is a bounded Lipschitz domain. Observe that in this case the space $C^\infty(\overline{\Omega})$ is dense in $W^{1,p}(\Omega)$ (Theorem 13) and hence T extends in a unique way to an operator defined on $W^{1,p}(\Omega)$.

Theorem 44 *Let Ω be a bounded Lipschitz domain, and $1 \leq p < n$. Then there exists a unique bounded operator*

$$\mathrm{Tr} : W^{1,p}(\Omega) \to L^{p(n-1)/(n-p)}(\partial\Omega),$$

such that $\mathrm{Tr}(u) = u|_{\partial\Omega}$*, for all* $u \in C^\infty(\overline{\Omega})$.

Proof. Using the partition of unity and flatting the boundary argument, it suffices to assume that $\Omega = Q^n = Q^{n-1} \times [0,1]$, $u \in C^\infty(\overline{\Omega})$, $\operatorname{supp} u \subset Q^{n-1} \times [0, 1/2)$, and prove the estimate

$$\|u\|_{L^q(Q^{n-1})} \leq C\Big(\int_\Omega |\nabla u|^p \, dx\Big)^{1/p},$$

where $q = p(n-1)/(n-p)$, and $Q^{n-1} = Q^{n-1} \times \{0\}$.

In Q^n we use coordinates (x', t), where $x' \in Q^{n-1}$, $t \in [0,1]$. Let $w = |u|^q$. We have

$$w(x', 0) = -\int_0^1 \frac{\partial w}{\partial t}(x', t)\, dt,$$

and hence

$$|u(x',0)|^q \leq q \int_0^1 |u(x',t)|^{q-1} |\frac{\partial u}{\partial t}(x',t)|\, dt.$$

Now we integrate both sides with respect to $x' \in Q^{n-1}$. If $p = 1$, then $q = 1$ and the theorem follows. If $p > 1$, we use Hölder's inequality which yields

$$\begin{aligned}\int_{Q^{n-1}} |u(x',0)|^{\frac{p(n-1)}{n-p}}\, dx' \quad &\leq \quad \frac{p(n-1)}{n-p}\Big(\int_{Q^{n-1}}\int_0^1 |u(x',t)|^{\frac{np}{n-p}}, dt\, dx'\Big)^{1-1/p} \\ &\qquad \times \Big(\int_{Q^{n-1}}\int_0^1 \Big|\frac{\partial u}{\partial t}(x',t)\Big|^p dt\, dx'\Big)^{1/p}.\end{aligned}$$

Now we use Sobolev embedding theorem (Proposition 37) to estimate the first integral on the right-hand side and the theorem follows. □

Lecture 4

This lecture will be devoted to study of the pointwise and geometric properties of Sobolev functions. In the second part of the lecture we will show some applications to conformal mappings.

Lebesgue points. One of the most important results of the theory of the Lebesgue integral is the following Lebesgue differentiation theorem.

Theorem 45 (Lebesgue) *If $u \in L^1_{\rm loc}(\mathbb{R}^n)$, then*

$$u(x) = \lim_{r\to 0} \fint_{B(x,r)} u(z)\, dz \ \ a.e. \tag{9}$$

Before we proceed to the proof we have to recall some properties of the maximal function. The *Hardy–Littlewood maximal function* of $g \in L^1_{\rm loc}(\mathbb{R}^n)$ is defined by the formula.

$$Mg(x) = \sup_{r>0} \fint_{B(x,r)} |g(z)|\, dz.$$

The following theorem is due to Hardy and Littlewood.

Theorem 46 (Hardy–Littlewood) *If $g \in L^1(\mathbb{R}^n)$, then*

$$|\{x \in \mathbb{R}^n :\ Mg(x) > t\}| \leq C(n)t^{-1} \int_{\mathbb{R}^n} |g(z)|\, dz. \tag{10}$$

If $g \in L^p(\mathbb{R}^n)$, $1 < p \leq \infty$, then

$$\|Mg\|_{L^p(\mathbb{R}^n)} \leq C(n,p)\|g\|_{L^p(\mathbb{R}^n)}\, . \tag{11}$$

If $u \in L^1(\mathbb{R}^n)$, then Chebyschev's inequality implies

$$|\{|u| > t\}|t \leq \int_{\mathbb{R}^n} |u|.$$

Hence (10) is weaker than (11) when $p = 1$. Actually inequality (11) fails for $p = 1$.

Inequality (10) is called weak type estimate for the maximal function.

Sketch of the proof of Theorem 46. We will prove inequality (10) only. The main new idea employed in the proof is an application of the following version of the Vitali covering lemma.

Theorem 47 (5r-covering lemma.) *Let $\mathcal{B}$ be a family of balls in a metric space (X,d) with* $\sup\{\operatorname{diam} B : B \in \mathcal{B}\} < \infty$. *Then there is a pairwise disjoint subcollection $\mathcal{B}' \subset \mathcal{B}$ such that*

$$\bigcup_{B\in\mathcal{B}} B \subset \bigcup_{B\in\mathcal{B}'} 5B$$

If (X,d) is separable, then $\mathcal{B}'$ is countable and we can represent $\mathcal{B}'$ as a sequence $\mathcal{B}' = \{B_i\}_{i=1}^\infty$, and so

$$\bigcup_{B\in\mathcal{B}} B \subset \bigcup_{i=1}^\infty 5B_i.$$

We will not prove it. □

Let $g \in L^1(\mathbb{R}^n)$. Fix $t>0$. Then for every $x \in \{Mg > t\}$ there is $r_x > 0$ such that

$$\int_{B(x,r_x)} |g| > t|B(x,r_x)|\,.$$

The family of balls $\{B(x,r_x)\}_{x\in\{Mg>t\}}$ forms a covering of the set $\{Mg>t\}$. Since the radii of the balls are uniformly bounded we can apply the $5r$-covering lemma. Thus we find a sequence $\{B(x_i,r_{x_i})\}_{i=1}^\infty$ of pairwise disjoint balls such that $\{Mg > t\} \subset \bigcup_{i=1}^\infty B(x_i, 5r_{x_i})$. Hence

$$\begin{aligned} |\{Mg>t\}| &\leq \sum_{i=1}^\infty |B(x_i,5r_{x_i})| \\ &= C\sum_{i=1}^\infty |B(x_i,r_{x_i})| \\ &\leq \frac{C}{t}\sum_{i=1}^\infty \int_{B(x_i,r_{x_i})} |g| \\ &\leq \frac{C}{t}\int_{\mathbb{R}^n} |g|\,. \end{aligned}$$

The proof of (10) is complete. □

Proof of Theorem 45. Define

$$\Phi(u,x) = \limsup_{r\to 0} ⨍_{B(x,r)} u(z)\,dz - \liminf_{r\to 0} ⨍_{B(x,r)} u(z)\,dz.$$

Then $\Phi(u,x) \leq 2Mu(x)$. Moreover $\Phi(u-h,x) = \Phi(u,x)$, whenever h is continuous. Choosing h with $\|u-h\|_{L^1} < \varepsilon$ yields

$$|\{\Phi(u,x)>t\}| = |\{\Phi(u-h,x)>t\}| \leq |\{M(u-h,x)>t/2\}| \leq \frac{C\varepsilon}{t}.$$

Since $\varepsilon > 0$ was choosen arbitrarily we conclude that $\Phi(u,x) = 0$ a.e. which means, the limit on the right hand side of (9) exists a.e. Now using similar arguments as above it is not difficult to prove that the limit equals u a.e. We leave details to the reader. □

Elements of the space $L^1_{\rm loc}$ are equivalence classes of functions which are equal except a set of measure zero. It will be however convenient to choose a particular representative in each class.

It follows from the Lebesgue theorem that the function $\tilde{u}$ defined *everywhere* by the formula

$$\tilde{u}(x) = \limsup_{r\to 0} \fint_{B(x,r)} u(z)\, dz \tag{12}$$

satisfies $\tilde{u} = u$ a.e. and hence $\tilde{u}$ is a Borel measurable representative of $u \in L^1_{\mathrm{loc}}$.

We say that $x \in \Omega$ is a p-Lebesgue point of $u \in L^1_{\mathrm{loc}}$ if

$$\left(\fint_{B(x,r)} |u(x) - \tilde{u}(x)|^p\, dz\right)^{1/p} \to 0 \qquad \text{as } r \to 0.$$

We will also say *Lebesgue point* instead of 1-Lebesgue point.

Roughly speaking the p-Lebesgue points of the function are those points where the function behaves nicely.

Theorem 48 *Given $u \in L^p_{\mathrm{loc}}$, $1 \le p < \infty$, then almost all points are the p-Lebesgue points of u.*

Proof. By the Lebesgue theorem and the fact that the set of rational numbers is countable we conclude that for almost all x and all $c \in \mathbb{Q}$ the averages $(\fint_{B(x,\varepsilon)} |u - c|^p)^{1/p}$ converge to $|u(x) - c|$ as $\varepsilon \to 0$. Then by the density it is true for aby $c \in \mathbb{R}$ and in particular it is true for $c = \tilde{u}(x)$. The proof is complete. $\square$

As an application of the above result we will prove the following result.

Theorem 49 (Calderón) *If $u \in W^{1,p}(\Omega)$, where $\Omega \subset \mathbb{R}^n$, $n < p \le \infty$, then u is differentiable a.e.*

Proof. We can assume that $p < \infty$. Fix $x_0 \in \Omega$ and set $v(x) = u(x) - \nabla u(x_0)(x - x_0)$. Obviously $v \in W^{1,p}(\Omega)$. If x is sufficiently close to x_0, then $B = B(x_0, 2|x - x_0|) \subset \Omega$ and hence Theorem 40 yields

$$|u(x) - u(x_0) - \nabla u(x_0)(x - x_0)| = |v(x) - v(x_0)| \le C|x - x_0| \left(\fint_B |\nabla u(z) - \nabla u(x_0)|^p\, dz\right)^{1/p}.$$

This implies that u is differentiable at x_0 whenever x_0 is a p-Lebesgue point of u. The proof is compliete. $\square$

Observe that by the ACL characterization Lipschitz functions locally belong to $W^{1,\infty}$ and hence as a corollary we get the classical Rademacher theorem.

Corollary 50 (Rademacher) *Lipschitz functions are differentiable a.e.* $\square$

There is also another neat generalization of Rademacher's theorem due to Stepanov.

Theorem 51 (Stepanov) *Let u be a function defined in an open set $\Omega \subset \mathbb{R}^n$. Then u is differentiable a.e. if and only if*

$$\limsup_{y \to x} \frac{|u(y) - u(x)|}{|y - x|} < \infty \quad a.e.$$

$\square$

Actually Stepanov's theorem can be rather easily deduced form the Rademacher theorem.

Now we turn our attantion to study of the Lebesgue pointes of Sobolev functions.

As we will see in the case in which u belongs to the Sobolev space not only the measure of the set of points which are not Lebesgue points is zero but also its Hausdorff dimension is small. Actually it is not very surprising: we can take a trace of a Sobolev function on a $(n-1)$-dimensional set. This suggests that the dimension of the set of points which are not Lebesgue points should be at most $(n-1)$. Moreover if $p > n$, then the Sobolev function is continuous and hence all points are the Lebesgue points. This also suggests that as $p < n$ approaches n the Hausdorff dimension of the set of non-Lebesgue points should go to zero. The above two facts suggest $n - p$ as a natural candidate for the Hausdorff dimension.

First we need recall the definition of the Hausdorff measure and the Hausdorff dimension.

The volume of the unit ball in $\mathbb{R}^n$ equal $\omega_n = 2\pi^{n/2}/(n\Gamma(n/2))$. This formula allows one to define ω_n for any real number $n > 0$. For $n = 0$ we set $\omega_n = 1$.

Let $s \geq 0$. For any $E \subset \mathbb{R}^n$ and $0 < \delta < \infty$ we put

$$H^s_\delta(E) = \inf \frac{\omega_s}{2^s} \sum_i (\operatorname{diam} E_i)^s \,,$$

where the infimum is taken over all coverings $E \subset \bigcup_{i=1}^\infty E_i$ such that $\sup_i \operatorname{diam} E_i < \delta$. Now we define the *Haudorff meaure* as

$$H^s(E) = \lim_{\delta \to 0} H^s_\delta(E) = \sup_{\delta > 0} H^s_\delta(E) \,.$$

The existence of the limit follows from the fact that the function $\delta \mapsto H^s_\delta(E)$ is nondecreasing.

H^s is a Borel measure and H^0 is a counting measure. One can prove that for any $E \subset \mathbb{R}^n$, $H^n(E) = |E|$, where $|E|$ is the outher Lebegue measure. Thus the Haudorff measure is a natural generalization of the Lebesgue measure to the case in which the dimension of the measure is different than the dimension of the space.

Given a set $E \subset \mathbb{R}^n$ it is easy to see that there is a real number $0 \leq s \leq n$ such that $H^t(E) = 0$ for all $t > s$ and $H^t(E) = \infty$ for all $0 \leq t < s$. The number s is called the *Hausdorff dimension* of the set E.

There is another slightly more convenient — but equivalent — way to define the Hausdorff dimension.

For a set $E \subset \mathbb{R}^n$ and $s > 0$ we define the *Hausdorff content* as $H^s_\infty(A) = \inf \sum_i r_i^s$, where infimum is taken over all coverings of E by balls B_i with radii r_i.

The definition of H^s_∞ differs from the above definition of H^s_δ with $\delta = \infty$ by a constant factor and by a fact that coverings by arbitrary sets are replaced by coverings by balls.

Given a set $E \subset \mathbb{R}^n$, it is easy to see that there is a real number $0 \le s \le n$ such that $H^t_\infty(E) = 0$ for $t > s$ and $H^t_\infty(E) > 0$ for $0 \le t < s$. It is easy to see that s equals the Haudorff dimension of the set E.

The last statement follows from an elementary observation that $H^t(E) = 0$ if and only if $H^t_\infty(E) = 0$. Observe however that the properties of the content H^t_∞ are very different from the properties of the Hausdorff measure H^t. Indeed, H^t_∞ is not a Borel measure and it is finite on all bounded sets while H^t is not.

All the above definitions of the Haudorff measure, Hausdorff content and the Hausdorff dimension extend without any modification to the setting of metric spaces.

If $u \in W^{1,p}(\Omega)$, $p > n$, then the function is locally Hölder continuous and hence all the points are the Lebesgue points. Thus we may assume that $p \le n$.

The choice of the representative (12) is particularly convenient when we deal with Sobolev functions. Till the end of the lecture we will *always* assume that $u \in W^{1,p}(\Omega)$ coincides *everywhere* with the representative $\tilde{u}$ and we will omit the tilde sign. However one has to be very careful since, for example, it may happen that for some x, $\tilde{u}(x) \ne -(\widetilde{-u})(x)$.

Theorem 52 *Let $u \in W^{1,p}(\Omega)$, where $\Omega \subset \mathbb{R}^n$ and $1 \le p \le n$. Then the Hausdorff dimension of the set of points which are not Lebesgue points of the function u is less than or equal to $n - p$.*

Before we prove the theorem we need some auxiliary results. As we know the function $u \in W^{1,p}(B)$ satisfies inequality (5) a.e. If we however choose the representative (12) then the inequality holds everywhere! More precisely we have.

Theorem 53 *If $u \in W^{1,p}(B)$ coincides everywhere with the representative (12) then there is a constant $C = C(n,p)$ such that*

$$|u(x) - u_B| \le C \int_B \frac{|\nabla u(z)|}{|x - z|^{n-1}}\, dz \tag{13}$$

for all $x \in B$.

Remark. It does not follow from the proof that the constant C in inequality (13) equals the constant in inequalty (5).

Proof. Let $u_\varepsilon(x) = \fint_{B(x,\varepsilon)} u(z)\, dz$. Then by Lemma 25 we have

$$\begin{aligned} |u_\varepsilon(x) - u_B| &\leq \fint_{B(x,\varepsilon)} |u(z) - u_B|\, dz \\ &\leq C \fint_{B(x,\varepsilon)} \int_B \frac{|\nabla u(w)|}{|z-w|^{n-1}}\, dw\, dz \\ &= C \int_B \left(\fint_{B(x,\varepsilon)} \frac{dz}{|z-w|^{n-1}} \right) |\nabla u(w)|\, dw \\ &\leq C' \int_B \frac{|\nabla u(w)|}{|x-w|^{n-1}}\, dw. \end{aligned} \tag{14}$$

The last inequality follows form the following elementary lemma whose proof is left to the reder.

Lemma 54 *There is a constant $C = C(n)$ such that for any $\varepsilon > 0$ and any $x, y \in \mathbb{R}^n$ there is*

$$\fint_{B(x,\varepsilon)} \frac{dz}{|z-w|^{n-1}} \leq \frac{C}{|x-w|^{n-1}}\,.$$

□

Now the claim follows by passing to the limit in inequality (14) as $\varepsilon \to 0$. The proof is complete. □

Recall the notation

$$I_1^\Omega g(x) = \int_\Omega \frac{g(z)}{|x-z|^{n-1}}\, dz.$$

Corollary 55 *Let $u \in W^{1,p}(\Omega)$. Then the set of points which are not Lebesgue points for u is contained in the set*

$$\{x \in \Omega : \, I_1^\Omega |\nabla u|(x) = \infty\}\,.$$

Proof. Since

$$\fint_{B(x,\varepsilon)} |u(z) - u(x)|\, dz \leq \fint_{B(x,\varepsilon)} |u(z) - u_{B(x,\varepsilon)}|\, dz + |u(x) - u_{B(x,\varepsilon)}|,$$

it suffices to prove that the right hand side converges to zero as $\varepsilon \to 0$ when

$$I_1^\Omega |\nabla u|(x) < \infty. \tag{15}$$

Observe that (15) implies that $\int_{B(x,\varepsilon)} |\nabla u(z)||x-z|^{1-n}\, dz \to 0$ as $\varepsilon \to 0$. Hence

$$|u(x) - u_{B(x,\varepsilon)}| \leq C \int_{B(x,\varepsilon)} \frac{|\nabla u(z)|}{|x-z|^{n-1}}\, dz \to 0$$

as $\varepsilon \to 0$. Moreover applying Lemma 54 we get

$$\begin{aligned} ⨍_{B(x,\varepsilon)} |u(z) - u_{B(x,\varepsilon)}|\, dz &\leq C ⨍_{B(x,\varepsilon)} \int_{B(x,\varepsilon)} \frac{|\nabla u(w)|}{|z-w|^{n-1}}\, dw\, dz \\ &\leq C' \int_{B(x,\varepsilon)} \frac{|\nabla u(w)|}{|x-w|^{n-1}}\, dw \to 0 \end{aligned}$$

as $\varepsilon \to 0$. The above estimates easily imply the claim. The proof is complete. □

Proof of Theorem 52. Observe that the case of Theorem 52 in which $p = n$ follows from the case in which $1 \leq p < n$. Thus in what follows we may assume that $1 \leq p < n$.

Since the question of the estimate of the Hausdorff dimension is local in its nature we may assume that the domain Ω is a ball. Thus it suffices to prove that for any $g \in L^p(B)$, $1 \leq p < n$ and any $\varepsilon > 0$ there is

$$H_\infty^{n-p+\varepsilon}(\{I_1^B g = \infty\}) = 0\,.$$

We need the following modification of the Hardy–Littlewood maximal function.

Let $R > 0$, $\lambda \geq 0$. The fractional maximal function is defined as

$$M_R^\lambda g(x) = \sup_{0<r<R} r^\lambda ⨍_{B(x,r)} |g(z)|\, dz.$$

Obviously $M_\infty^0 g = Mg$ is the classical Hardy–Littlewood maximal function.

The next lemma provides an estimate of the Riesz potential in terms of the fractional maximal function.

Lemma 56 *If $0 \leq \lambda < 1$ then there exists a constant C such that*

$$\int_B \frac{|g(z)|}{|x-z|^{n-1}}\, dz \;\leq\; C r^{1-\lambda} M_{\operatorname{diam} B}^\lambda g(x)$$

for all $g \in L^1(B)$ and all $x \in B$. Here B is a ball of radius r.

Proof. We break the integral on the left hand side into the sum of the integrals over 'rings' $B \cap (B(x, \operatorname{diam} B/2^k) \setminus B(x, \operatorname{diam} B/2^{k+1}))$. In each 'ring' we have $|x-z|^{1-n} \approx (\operatorname{diam} B/2^k)^{1-n}$. Now we estimate the integral over the 'ring' by the integral over the ball $B(x, r/2^k)$ and the lemma follows easily. □

The following result is a variant of weak type estimate (10). Actually the proof follows the same argument and we leave details to the reader.

Lemma 57 *Let $g \in L^p(\mathbb{R}^n)$, $1 \le p < \infty$, $R > 0$ and $\lambda \ge 0$. Then*

$$H_\infty^{n-\lambda p}\left(\{M_R^\lambda g > t\}\right) \le Ct^{-p}\int_{\mathbb{R}^n} g^p.$$

□

In particular $H_\infty^{n-\lambda p}(\{M_R^\lambda g = \infty\}) = 0$. Hence Lemma 56 yields that for $g \in L^p(B)$ and $0 \le \lambda < 1$ we have

$$H_\infty^{n-\lambda p}(\{I_1^B g = \infty\}) \le H_\infty^{n-\lambda p}(\{M_{\operatorname{diam} B}^\lambda g = \infty\}) = 0.$$

Since $n - p + \varepsilon = n - \lambda p$ for some $0 \le \lambda < 1$ the claim follows. The proof is complete. □

Pointwise inequality. As a by product of the above results we obtain the following powerful inequality.

Theorem 58 *If $u \in W^{1,p}(R^n)$, $1 \le p < \infty$ and $0 \le \lambda < 1$ then*

$$|u(x) - u(y)| \le C|x-y|^{1-\lambda}\left(M_{2|x-y|}^\lambda|\nabla u|(x) + M_{2|x-y|}^\lambda|\nabla u|(y)\right)$$

for all $x, y \in \mathbb{R}^n$, $x \ne y$.

The above inequality will be often called *pointwise inequality*.

The inequality requires some explanations. It may happen that the left hand side of the inequality is of the indefinite form like $|\infty - \infty|$, then we adopt the convention $|\infty - \infty| = \infty$. In such a case the inequality is still valid since $\tilde{u}(z) = \pm\infty$ implies $M_R^\lambda|\nabla u|(z) = \infty$ for any $R > 0$. We assume that $x \ne y$ as it could lead to ambiguity when $u(x) = \infty$. Indeed, in such a case we would have ∞ on the left hand side while the right hand side would be of the form $0\cdot\infty$. Anyway the inequality is not interesting when $x = y$.

Proof of Theorem 58. For x, $y \in B$, $\operatorname{diam} B \approx |x-y|$ we have

$$\begin{aligned} |u(x) - u(y)| &\le |u(x) - u_B| + |u(y) - u_B| \le C\left(\int_B \frac{|\nabla u(z)|}{|x-z|^{n-1}}\,dz + \int_B \frac{|\nabla u(z)|}{|y-z|^{n-1}}\,dz\right) \\ &\le C|x-y|^{1-\lambda}\left(M_{2|x-y|}^\lambda|\nabla u|(x) + M_{2|x-y|}^\lambda|\nabla u|(y)\right). \end{aligned}$$

The last inequality follows from Lemma 56. The proof is complete. □

Now we will provide some applications of Theorem 58.

Obviously Theorem 58 generalizes also to $u \in W^{1,p}(\Omega)$, where Ω is a bounded domain with the *extension propoerty* which means there is a bounded linear extension operator $E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$. Namely we have

$$|u(x) - u(y)| \le C|x-y|^{1-\lambda}\left(M_{2\operatorname{diam}\Omega}^\lambda|\nabla(Eu)|(x) + M_{2\operatorname{diam}\Omega}^\lambda|\nabla(Eu)|(y)\right), \text{ a.e.} \quad (16)$$

Theorem 58 immediately implies the folloinwg important Morrey's lemma.

Corollary 59 (Morrey's lemma) *Let $u \in W^{1,p}(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is open and $1 \le p < \infty$. Suppose that for some constants $0 < \mu \le 1$, $M > 0$,*

$$\int_{B(x,R)} |\nabla u|^p \le M^p R^{n-p+p\mu},$$

holds whenever $B(x,R) \subset \Omega$. Then $u \in C^{0,\mu}_{\rm loc}(\Omega)$, and in each ball B such that $5B \subset \Omega$ the inequality

$$|u(x) - u(y)| \le CM|x-y|^\mu$$

holds for all $x, y \in B$ with a constant C depending on n, p and μ only.

Proof. Indeed, the hypothesis of the corollary implies that the suitable fractional maximal function with $\lambda = 1 - \mu$ is finite. Since the Morrey lemma is local in its nature we do not need any regularity upon the boundary of Ω. □

Assume for a moment that $u \in W^{1,p}(\mathbb{R}^n)$ has compact support with diameter less than 1/2. Than it easily follows from Theorem 58 that

$$|u(x) - u(y)| \le C|x-y|^{1-\lambda}\left(M_1^\lambda|\nabla u|(x) + M_1^\lambda|\nabla u|(y)\right), \tag{17}$$

for *all* $x, y \in \mathbb{R}^n$, $x \ne y$. Let $E_{\lambda,t} = \{x \in \mathbb{R}^n \,|\, M_1^\lambda|\nabla u|(x) \le t\}$. Obviously $u|_{E_{\lambda,t}}$ is $C^{0,1-\lambda}$-Hölder continuous with the constant $2tC$. Note, that we employed (17) *everywhere.*

Lemma 60 *Any Lipschitz function defined on an arbitrary subset of an arbitrary metric space can be extended to the entire space with the same Lipschitz constant.*

Proof. Let X be a metric space and $u : E \to \mathbb{R}$, $E \subset X$ a Lipschitz function with the Lipschitz constant L. Then we extend u to the entire space X by the formula

$$\overline{u}(x) = \inf_{a \in E}\{u(a) + L\rho(a,x)\},$$

where ρ denotes the metric. It is easy to see that $\overline{u}$ is Lipschitz with the same Lipschitz constant L. □

The result holds also for the extensions of $C^{0,\mu}$-Hölder continuous functions, because $C^{0,\mu}$ function is Lipschitz with respect to a new metric $d'(x,y) = d(x,y)^\mu$. This implies that there exists a $C^{0,1-\lambda}$ function $u_{\lambda,t}$ defined on the entire $\mathbb{R}^n$ such that $u_{\lambda,t}|_{E_{\lambda,t}} = u|_{E_{\lambda,t}}$.

As it follows from Lemma 57 $H^{n-\lambda p}_\infty(\mathbb{R}^n \setminus E_{\lambda,t}) \le Ct^{-p}\int_{\mathbb{R}^n}|\nabla u|^p$.

If $\lambda \to 1$ and $t \to \infty$, then $\mathbb{R}^n \setminus E_{\lambda,t}$ is a decreasing sequence of open sets and it follows from the estimate that the intersection of these sets has the Hausdorff dimension less than or equal to $n - p$.

Using the partition of unity the above argument applies to $u \in W^{1,p}_{\rm loc}(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is an arbitrary open set. This leads to the following theorem.

Theorem 61 *If $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ coincides everywhere with the representative (12), then there exists a sequence of compact sets $X_1 \subset X_2 \subset \ldots \subset X_k \subset \ldots \subset \Omega$ and a sequence of Hölder continuous functions u_k on Ω such that $u_k|_{X_k} = u|_{X_k}$ and $\Omega \setminus \bigcup_k X_k$ has the Hausdorff dimension less than or equal to $n - p$.*

Both of Theorems 52 and 61 show that it is reasonable to talk about values of $u \in W^{1,p}(\Omega)$ except the set of dimension $n-p$. Hence if $p > 1$, we can define a trace of the Sobolev function on a $(n-1)$-dimensional submanifold of Ω just as a restriction. If we want to define the trace on the boundary of Ω (provided it is sufficiently regular), we first extend the Sobolev function to $W^{1,p}(\mathbb{R}^n)$ and then we take a restriction. As we have seen earlier, in the case in which $p = 1$ one can also define a trace, however the above approach does not cover that case.

In general, the Hölder exponent of the functions u_k in Theorem 61 has to go to 0 as $k \to \infty$. If we want to have an additional condition that all the functions u_k have a fixed Hölder exponent $1 - \lambda$, then by the same argument as above we get $H^{n-\lambda p}_\infty(\Omega \setminus \bigcup_k X_k) = 0$, which is slightly more than to say that the Hausdorff dimension is less than or equal to $n-\lambda p$. In particular we get the classical Sobolev imbedding into the Hölder continuous functions when $p > n$. Indeed, take $\lambda = n/p$. Then $n - \lambda p = 0$ and hence $u \in C^{0,1-n/p}_{\mathrm{loc}}(\Omega)$.

On the other hand if $\lambda = 0$ and $1 \le p < \infty$, we get that the Sobolev function coincides with a Lipschitz continuous function outside a set of an arbitrary small measure. Lipschitz functions belong also to the Sobolev spaces $W^{1,p}$ for all p. Careful estimates of the $W^{1,p}$ norm of these Lipschitz functions shows that they approximate our Sobolev function in the Sobolev norm. Hence we obtain

Theorem 62 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with the Lipschitz boundary and $u \in W^{1,p}(\Omega)$, $1 \le p \le \infty$. Then to every $\varepsilon > 0$ there is a Lipschitz function $w \in \mathrm{Lip}\,(\Omega)$ such that*

1. $|\{x \in \Omega :\, u(x) \neq w(x)\}| < \varepsilon$,
2. $\|u - w\|_{W^{1,p}(\Omega)} < \varepsilon$.

As a next application of the pointwise inequality we give a characterization the Sobolev space that does not involve the notion of the derivative.

If $1 < p \le \infty$, then by the Hardy–Littlewood theorem the maximal operator is bounded in L^p. This in connection with (16) for $\lambda = 0$, implies that to every $u \in W^{1,p}(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain with sufficiently regular boundary, there exists $g \in L^p(\Omega)$, $g \ge 0$, such that

$$|u(x) - u(y)| \le |x - y|(g(x) + g(y)) \text{ a.e.} \tag{18}$$

On the other hand if $u \in L^p$ $1 < p \leq \infty$ satisfies (18) with $0 \leq g \in L^p(\Omega)$, then it follows from Theorem 23 that $u \in W^{1,p}(\Omega)$. Hence we have proved the following result.

Theorem 63 *Let Ω be a bounded domain with the extension property and $1 < p \leq \infty$. Then $u \in W^{1,p}(\Omega)$ if and only if there exists $0 \leq g \in L^p(\Omega)$, such that (18) holds. Moreover*

$$\|\nabla u\|_{L^p(\Omega)} \approx \inf_g \|g\|_{L^p(\Omega)},$$

where the infimum is taken over all functions g which satisfy (18).

In the case $p = \infty$ we recover a classical result which states that $W^{1,p}(\Omega) = \mathrm{Lip}\,(\Omega)$. Hence it is natural to call the above theorem a Lipschitz type characterization of Sobolev functions.

We will see later that the above characterization is very usefull when trying to generalize the notion of the Sobolev space to the setting of metric spaces. Indeed, if (X, d, μ) is a metric space equipped with a Borel measure, then we can define Sobolev space for $1 < p \leq \infty$ as the set of all L^p functions for which there exists $0 \leq g \in L^p(X)$ such that $|u(x) - u(y)| \leq d(x, y)(g(x) + g(y))$ a.e. We will come back to this construction in the last lecture.

Conformal mappings and Sobolev spaces. Theory of Sobolev spaces has many applications in the areas different than calculus of variations or PDE. In this last section we describe some of the basic applications to the theory of conformal mappings.

Let $\Omega \subset \mathbb{C}$ be bounded and simply connected domain. Let $\psi : D \to \Omega$ be a Riemann mapping, where D is the unit disc $D = B^2(1)$.

It follows from the Cauchy–Riemann equations that $|\psi'|^2 = J_\psi$, where J_ψ denotes the Jacobian. Hence

$$\int_D |\psi'|^2 = \int_D J_\psi = |\psi(D)| = |\Omega| < \infty.$$

Thus $\psi \in W^{1,2}(D)$. As we will see this simply observation has many important consequences. At the first glance it is surprising. We know that ψ is smooth, but Sobolev functions are measurable only, so how does the information that the Riemann mapping belongs to the Sobolev space can be employed?

Let $u : \mathbb{R}^n \to \mathbb{R}^n$ be Borel mapping. We say that u has *Lusin's property* (N) if it maps sets of Lebesgue measure zero into sets of measure zero. $|E| = 0 \Rightarrow |f(E)| = 0$. Here $|E|$ as usual denotes the Lebesgue measure.

Theorem 64 *Any mapping $u \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^n)$, where $p > n$ has the Lusin property.*

Here $u \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^n)$ means that the coordinate functions of u belong to the Sobolev space $W^{1,p}(\mathbb{R}^n)$.

Before we prove the theorem we show that the theorem fails for $p = n$.

Let Ω be a bounded Jordan domain whose boundary has positive two dimensional Lebesgue measure and let $\psi : D \to \Omega$ be a Riemann mapping. Since $\psi \in W^{1,2}$ we can extend it to a mapping $E\psi \in W^{1,2}(\mathbb{R}^2, \mathbb{R}^2)$. Note that ψ is continuous up to the boundary of D, so $E\psi$ is continuous on the entire $\mathbb{R}^2$. Now $E\psi(\partial D) = \partial\Omega$ and $\partial\Omega$ has positive measure, so $E\psi$ does not satisfy Lusin's property. It is possible to extend the example to any dimension $n \geq 2$. However in dimensions greater than 2 the construction has to be different: we cannot use the Riemann mapping.

Proof of Theorem 64. Let $|E| = 0$. Given $\varepsilon > 0$ take an open set Ω such that $E \subset \Omega$ and $|\Omega| < \varepsilon$ Let $\Omega = \bigcup_i Q_i$ be a decomposition of Ω into a family of cubes with pairwise disjoint interiors.

By the Sobolev inequality

$$|u(x) - u(y)| \leq C|x-y|^{1-n/p} \left(\int_{Q_i} |\nabla u|^p \right)^{1/p},$$

for all $x, y \in Q_i$ and hence

$$\operatorname{diam} u(Q_i) \leq C(\operatorname{diam} Q_i)^{1-n/p} \left(\int_{Q_i} |\nabla u|^p \right)^{1/p}.$$

Since the set $u(E)$ is covered by $\bigcup_i u(Q_i)$ we obtain

$$\begin{aligned} |u(E)| &\leq C \sum_i (\operatorname{diam} u(Q_i))^n \leq C \sum_i (\operatorname{diam} Q_i)^{n\left(\frac{p-n}{p}\right)} \left(\int_{Q_i} |\nabla u|^p \right)^{n/p} \\ &\leq C \left(\sum_i (\operatorname{diam} Q_i)^n \right)^{(p-n)/p} \left(\sum_i \int_{Q_i} |\nabla u|^p \right)^{n/p} \leq C\varepsilon^{\frac{p-n}{p}} \left(\int_{\mathbb{R}^n} |\nabla u|^p \right)^{n/p} \to 0 \end{aligned}$$

as $\varepsilon \to 0$. We employed here the Hölder inequality. This completes the proof. □

Corollary 65 *Let $\Omega \subset \mathbb{C}$ be a bounded Jordan domain whose boundary has positive two dimensional Lebesgue measure and let $\psi : D \to \Omega$ be a Riemann mapping. Then*

$$\int_D |\psi'|^p = \infty,$$

for any $p > 2$.

Remark. The problem of finding the best exponent of the integrability of the gradient of the inverse Riemann mapping is very difficult and very deep. *Brenann's conjecture* states that for any simply connected domain $\Omega \subset \mathbb{C}$ and a Riemann mapping $\phi : \Omega \to D$ there is $\int_\Omega |\phi'|^p < \infty$ for every $p < 4$.

Proof of the corollary. Suppose that $\int_D |\psi'|^p < \infty$ for some $p > 2$. Then we can extend ψ to $E\psi \in W^{1,p}(\mathbb{R}^2, \mathbb{R}^2)$ and we arrive to a contradiction between the Lusin property for $E\psi$ and the fact that $|\psi(\partial D)| = |\partial\Omega| > 0$.

The next application will concern the estimates for a harmonic measure. Let $\Omega \subset \mathbb{C}$ be a bounded Jordan domain. Then for every $\varphi \in C(\partial\Omega)$ one can solve the following classical Dirichlet problem:

Find $u \in C^2(\Omega) \cap C(\overline{\Omega})$ such that

$$\begin{cases} \Delta u &= 0 \quad \text{in } \Omega \\ u|_{\partial\Omega} &= \varphi. \end{cases}$$

Given a boundary condition φ, denote the solution to the above problem by u_φ. Fix $z \in \Omega$. The mapping $\varphi \mapsto u_\varphi(z)$ is a continuous linear functional on the Banach space $C(\partial\Omega)$. This functional is also nonnegative in a sense that $u_\varphi(z) \geq 0$ when $\varphi \geq 0$. This is a direct consequence of the maximum principle. Thus there exists a nonnegative Radon measure ω_z supported on $\partial\Omega$ such that for every $\varphi \in C(\partial\Omega)$

$$u_\varphi(z) = \int_{\partial\Omega} \varphi \, d\omega_z \, .$$

The measure ω_z is called *harmonic measure.* It does not essentially depend on the choice of z in the sense that for any $z_1, z_2 \in \Omega$ the measures ω_{z_1} and ω_{z_2} are mutually absolutely continuous. For example the measure Ω_{z_1} is supported by a set E (i.e., $\omega_{z_1}(\mathbb{C} \setminus E) = 0$) if and only if the measure ω_{z_2} is supported by the set E.

For that reason very often when we talk about harmonic measures, we simply forget about the dependence on the point z.

Theorem 66 (Øksendal) *Harmonic measure is singular with respect to the two dimensional Lebesgue measure i.e. it is supported by a set of two dimensional measure zero.*

Nowdays there are much more sophisticated results saying that the harmonic measure is supported by a set of Hausdorff dimension 1. However we will not touch those deep results and we simply show how the theory of Sobolev spaces can be employed to provide a rather simple proof of the Øksendal theorem.

Before we prceed to the proof, we need know more about harmonic measures and Sobolev spaces.

Given a Jordan domain Ω one can solve the Dirichlet problem employing Riemann's mapping. Fix $z \in \Omega$ and let $\psi : D \to \Omega$ be a Riemann mapping such that $\psi(0) = z$. If $\varphi \in C(\partial\Omega)$, then the function $\varphi \circ \psi$ is contnuous on ∂D and hence we can find the harmonic function v in D which coincides with $\varphi \circ \psi$ on ∂D. Since harmonic functions are invariant under conformal mappings and ψ is continuous up to the boundary we conclude that $u = v \circ \psi^{-1}$ is harmonic in Ω such that $u|_{\partial\Omega} = \varphi$. Now

$$\int_{\partial\Omega} \varphi \, d\omega_z = u(z) = v(0) = \frac{1}{2\pi} \int_{\partial D} \varphi \circ \psi \, dH^1 \tag{19}$$

for every $\varphi \in C(\partial\Omega)$. The last equality follows from the mean value property for harmonic functions. Identity (19) implies that ω_z is the image of the measure $(2\pi)^{-1}H^1$ under the mapping ψ i.e.

$$\omega_z(A) = (2\pi)^{-1} H^1(\psi^{-1}(A)), \tag{20}$$

for every Borel set $A \subset \partial\Omega$.

We want to prove that there exists $F \subset \partial\Omega$ such that $H^2(F) = 0$ and $\omega_z(\partial\Omega \backslash F) = 0$. Because of (20) the equivalent problem is to find $E \subset \partial D$ such that $H^1(E) = 0$ and $H^2(\psi(\partial D \setminus E)) = 0$. Indeed, then the set $F = \psi(\partial D \setminus E)$ will have desired properties.

Thus Øksendal's theorem follows from the following more general result.

Theorem 67 *Let $G \subset \mathbb{R}^n$ be a bounded domain with the smooth boundary and let $u \in W^{1,n}(G, \mathbb{R}^n)$ be continuous up to the boundary. Then there exists $E \subset \partial G$ such that $H^{n-1}(E) = 0$ and $H^n(u(\partial G \setminus E)) = 0$.*

In the last theorem the condition $H^{n-1}(E) = 0$ can be replaced by a much stronger condition that the Hausdorff dimension of the set E is zero. However we will not go into details.

We will need the following lemma which is yet another variant of weak type estimates for fractional maximal functions. Its proof is almost the same as that for Lemma 57. We leave details to the reader.

Lemma 68 *Let G be as above and $1 \leq p < \infty$. Then there exsts a constant $C > 0$ such that*

$$H^{n-1}(\{x \in \partial G : \, M_1^{1/p} g(x) > t\}) \leq \frac{C}{t^p} \int_{\mathbb{R}^n} |g(z)|^p \, dz$$

for all $g \in L^p(\mathbb{R}^n)$.

We will also need the following

Lemma 69 *Let G be as above, $1 \leq p < \infty$ and $g \in L^p(\mathbb{R}^n)$. Then to every $\varepsilon > 0$ there exists $E \subset \partial G$ with $H^{n-1}(E) < \varepsilon$ such that*

$$\sup_{x \in \partial G \setminus E} M_r^{1/p} g(x) \to 0 \qquad \text{as } r \to 0.$$

Proof. Let $k \in C_0^\infty(\mathbb{R}^n)$ be such that $\|k-g\|_{L^p}^p < \varepsilon^{p+1}$. Let $h = k - g$. Obviously there exists $R < 1$ with $\sup_{x\in\mathbb{R}^n} M_R^{1/p} k < \varepsilon$ (because $k \in C_0^\infty$). Now since $M_R^{1/p} g \leq M_1^{1/p} h + M_R^{1/p} k$, then

$$H^{n-1}(\{M_R^{1/p} g > 2\varepsilon\}) \leq H^{n-1}(\{M_1^{1/p} h > \varepsilon\}) \leq \frac{C}{\varepsilon^p} \int_{\mathbb{R}^n} |h|^p < C\varepsilon.$$

Let R_i, be such that $H^{n-1}(\{M_{R_i}^{1/p} g > \varepsilon/(C2^{i-1})\}) < \varepsilon/2^i$. Now it suffices to put $E = \bigcup_i \{M_{R_i}^{1/p} g > \varepsilon/(C2^{i-1})\}$.

Proof of Theorem 67. Applying an extension operator we may assume that $u \in W^{1,n}(\mathbb{R}^n, \mathbb{R}^n)$. We prove that theorem holds with $E = \bigcap_i E_i$, where E_i is as in Lemma 69 with $g = |\nabla u|$, $p = n$ and $\varepsilon = 1/i$. It suffices to prove that for every i, $H^n(u(\partial G \backslash E_i)) = 0$. Let $\partial G\backslash E_i \subset \bigcup_j B(x_j, r_j)$, $x_j \in \partial G\backslash E_i$, $\sum r_j^{n-1} < CH^{n-1}(\partial G\backslash E_i)$, $\sup_j r_j < \varepsilon/4 < 1/4$. It follows from Theorem 58 that u is $(1-1/n)$-Hölder continuous on $\partial G \setminus E_i$, so $u(B(x_j, r_j) \cap (\partial G \setminus E_i) \subset B(u(x_j), s_j)$ where

$$s_j < C(\sup_{\partial G\backslash E_i} M_\varepsilon^{1/n} |\nabla u|) r_j^{1-1/n}.$$

Hence $H^n(u(\partial G \setminus E_i)) = 0$, because according to Lemma 69

$$\sum_j s_j^n \leq C(\sup_{\partial G\backslash E_i} M_\varepsilon^{1/n} |\nabla u|)^n \sum_j r_j^{n-1} \to 0,$$

as $\varepsilon \to 0$. The proof is complete. □

Lecture 5

Quasiconformal mappings. Today we will present an application of the theory of Sobolev spaces to some geometric problems of homeomorphisms in $\mathbb{R}^n$. Namely we will discuss the theory of quasiconformal mappings.

Given a linear mapping $A : \mathbb{R}^n \to \mathbb{R}^n$ we will use two different norms of A. Namely the Hilbert-Smith norm $|A| = (\sum_{i,j} a_{i,j}^2)^{1/2}$ and the operator norm $\|A\| = \sup_{\xi=1} |A\xi|$.

For a given continuous mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ we will use the notation $L_f(x, r) = \sup_{|y-x|=r} |f(y) - f(x)|$ and $l_f(x, r) = \inf_{|y-x|=r} |f(y) - f(x)|$.

We say that a homeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ is *quasiconformal* if there is a constant $H \geq 1$ such that

$$H(x, f) = \limsup_{r\to 0} \frac{L_f(x, r)}{l_f(x, r)} \leq H \qquad \text{for all } x \in \mathbb{R}^n. \tag{21}$$

If in addition to (21) $H(x, f) \leq K$ a.e. we say that f is K-quasiconformal.

We also define another class of homeomorphisms.

We say that a homeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ is *quasisymmetric* if there is a constant $H \geq 1$ such that

$$\frac{L_f(x,r)}{l_f(x,r)} \leq H \tag{22}$$

for all $x \in \mathbb{R}^n$ and all $r > 0$.

Homeomorphisms which are quasiconformal of quasisymmetric will be called quasiconformal or eqasisymmetric mappings.

Obviously the class of quasiconformal mappings contains that of quasisymetric mappings. Although condition (21) seems much weaker than (22) we will show that both classes of quasiconformal and quasisymmetric mappings coincide. The an tool in the proof is the theory of Sobolev spaces.

First we prove some regularity results for uasiconformal mappings.

If f is differentiable at a point x, then the condition $H(x,f) \leq K$ is equivalent to

$$\max_{|\xi|=1} |Df(x)\xi| \leq K \min_{|\xi|=1} |Df(x)\xi|. \tag{23}$$

The first deep result that we will prove is the following

Theorem 70 *If $f : \Omega \to \mathbb{R}^n$ is K-quasiconformal, then*

1. *f is differentiable a.e.,*
2. *$f \in W^{1,n}_{\rm loc}(\Omega, \mathbb{R}^n)$,*
3. *$\max_{|\xi|=1} |Df(x)\xi| \leq K \min_{|\xi|=1} |Df(x)\xi|$ a.e.*

First we need recall some facts form the measure theory.

A measure μ on $\mathbb{R}^n$ is called *Radon measure* if it is a Borel measure and $\mu(K) < \infty$ for all compact sets $K \subset \mathbb{R}^n$.

The following theorem due to Besicovitch generalizes the theorem of Lebesgue.

Theorem 71 (Besicovitch) *If μ and ν are two Radon measures on $\mathbb{R}^n$, then the limit*

$$\frac{d\mu}{d\nu} \overset{def}{=} \lim_{r\to 0} \frac{\mu(B(x,r))}{\nu(B(x,r))}$$

exists ν a.e. and $d\mu/d\nu \in L^1_{\rm loc}(\mathbb{R}^n, \nu)$.

We will not prove it. □

Later we will need the theorem in the special case when $\nu = H^n$ is the Lebesgue measure in $\mathbb{R}^n$.

If $f : \Omega \to \mathbb{R}^n$ is a homeomorphism, then f maps Borel sets onto Borel sets and hence we can define a Radon measure as follows

$$\mu_f(A) = |f(A)|.$$

Lemma 72 *If a homeomorphism $f : \Omega \to \mathbb{R}^n$, $\Omega \subset \mathbb{R}^n$ is differentiable a.e., then the Jacobian does not change the sign i.e. $J_f \geq 0$ a.e. or $J_f \leq 0$ a.e. Moreover in all the points where f is differentiable we have*

$$|J_f(x)| = \lim_{r\to 0} \frac{|f(B(x,r))|}{|B(x,r)|}, \tag{24}$$

and hence $|J_f| = d\mu_f/dH^n \in L^1_{\rm loc}$.

Proof. The proof that J_f does not change the sign follows form an argument based on the topological degree and we skip it. The equality (24) is simpler: If f is differentiable at $x \in \Omega$, then for small r, $f(B(x,r))$ almost coincides with the ellipsoid $Df(x)(B(x,r))$ and hence $|f(B(x,r)| = |Df(x)(B(x,r))| + o(r^n) = |J_f(x)||B(x,r)| + o(r^n)$. Actually the proof of this estimate for the volume of $f(B(x,r))$ follows form the definition of the differential and from the Brouwer theorem. □

Lemma 73 *Let $f : \Omega \to \mathbb{R}^n$ be a homeomorphism such that*

$$H(x,f) = \limsup_{r\to 0} \frac{L_f(x,r)}{l_f(x,r)} < \infty \quad a.e.$$

Then f is differentiable a.e.

The assumption about f is weaker than that for the quasiconformal mapping: we do not assume neither that $H(x,f)$ is finite everywhere, nor that $H(x,f)$ is bounded.

Proof. According to the Stepanov theorem (Theorem 51) it suffices to prove that

$$|D^+f|(x) = \limsup_{y\to x} \frac{|f(y)-f(x)|}{|y-x|} < \infty \quad \text{a.e.}$$

We have

$$\begin{aligned} |D^+f|(x) &= \limsup_{r\to 0} \frac{1}{r} L_f(x,r) \\ &= \limsup_{r\to 0} \frac{l_f(x,r)}{r} \cdot \frac{L_f(x,r)}{l_f(x,r)} \\ &\leq H(x,f) \limsup_{r\to 0} \frac{1}{r} l_f(x,r). \end{aligned}$$

Observe that the ball centered at $f(x)$ with the radius $l_f(x,r)$ is contained in $f(B(x,r))$ and hence

$$\frac{l_f(x,r)^n}{r^n} \leq \frac{f(B(x,r))|}{|B(x,r)|}.$$

This yields

$$|D^+ f|^n(x) \leq H(x,f)^n \limsup_{r\to 0} \frac{|f(B(x,r))|}{|B(x,r)|} = H(x,f)^n \frac{d\mu_f}{dH^n}(x) < \infty,$$

almost everywhere. The proof is complete. □

Proof of Theorem 70. The differentiability a.e. follows form Lemma 73. The observation (23) yields condition 3. Thus we are left with the proof that $f \in W^{1,n}_{\rm loc}$.

We will prove that f is absolutely continuous on almost all lines paralell to coordinate axex. Then the theorem will follow from the ACL characterization of the Sobolev space.

Let $Q^n \subset \Omega$ be an open cube with faces paralell to coordinates. Let $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be the orthogonal projection in one of the coordinate directions. We define a Radon measure ν on the $(n-1)$-dimensional cube $\pi(Q^n)$ as follows

$$\nu(A) = |f(\pi^{-1}(A) \cap Q^n)|.$$

It follows form Theorem 71 that

$$\lim_{r\to 0} \frac{\nu(B(y,r))}{r^{n-1}} = \omega_{n-1} \frac{d\nu}{dH^{n-1}}(y) < \infty,$$

for a.e. $y \in \pi(Q^n)$. Here ω_{n-1} is the volume of the unit ball in $\mathbb{R}^{n-1}$. Next we prove that for every compact set $F \subset \pi^{-1}(y) \cap Q^n$

$$(H^1(\varphi(F)))^n \leq C(H^1(F))^{n-1} \frac{d\nu}{dH^{n-1}}(y). \tag{25}$$

Before we prove it observe that this impies absolute continuity of f along the segment $\pi^{-1}(y) \cap Q^n$, whenever $d\nu/dH^{n-1}(y) < \infty$ and hence te ACL property follows. Then the linear $Df(x)$ maps the unit ball onto an ellipsoid with axes $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$. The condition (23) means $\lambda_n \leq K\lambda_1$. Hence

$$|Df(x)|^n = \left(\sum_{i=1}^n \lambda_i^2\right)^{n/2} \leq C \sum_{i=1}^n \lambda_i^n \leq C' \lambda_1 \cdots \lambda_n = |J_f| \in L^1_{\rm loc}.$$

We conclude that $Df \in L^n_{\rm loc}$ which together with the ACL characterization of the Sobolev space implies $f \in W^{1,n}_{\rm loc}$.

Thus we are left with the proof of (25).

Since f is quasiconformal there exists a constant $H \geq 1$ such that (21) holds for all $x \in \mathbb{R}^n$.

For all sufficiently large integers k define

$$F_k \{x \in F : L_f(x,r) \leq 2H\, l_f(x,r), \quad \text{whenever } 0 < r < 1/k\} \,. \tag{26}$$

Obviously F_k are compact sets with $F_k \subset F_{k+1}$, $\bigcup_k F_= F$. We will estimate $H^1(f(F_k))$.

Fix $\varepsilon > 0$ and k. Then for all sufficiently small $0 < r < 1/k$ there exists an integer p such that F_k can be cobered by segments $I_1, \ldots, I_p$ with the following properties: all the segments have the same length $|I_i| = 2r$, all the segments are centered at F_k, each point of Fk belongs to no more than two segments I_i and $pr < H^1(F_k) + \varepsilon$.

To see that there is such a covering we first cover F_k by an open set U with $H^1(U \setminus F_k) < \varepsilon$, then choose $r < \text{dist}\,(F_k, \partial U)$, cover F_k by all segments of diameter $2r$ centered at F_k, choose a finite subcovering and then remove all redundant segments.

Observe that p depends on r and in general p bigger as r smaller.

Let a_i be the center of I_i. set $B_i = B^n(a_i, r)$. Inequality of (26) gives

$$\text{diam}\, f(B_i) \leq 4^n \omega_n^{-1} H^n(f(B_i))$$

Hence the definition of $H1$ and the Hölder inequality yields

$$\begin{aligned}
H^1(f(F_k)) &\leq \lim_{r\to 0} \sum_{i=1}^{p} \text{diam}\, f(B_i) \\
&\leq \lim_{r\to 0} \left(p^{n-1} \sum_{i=1}^{p} (\text{diam}\, f(B_i))^n \right)^{1/n} \\
&\leq 4\omega_n^{-1/n} H \lim_{r\to 0} \left(p^{n-1} \sum_{i=1}^{p} |f(B_i)| \right)^{1/n} \\
&\leq C(n) H (H^1(F_k) + \varepsilon)^{(n-1)/n} \lim_{r\to 0} \left(r^{1-n} \sum_{i=1}^{p} |f(B_i)| \right)^{1/n} \qquad (27)
\end{aligned}$$

The last inequality follows form the estimate $pr < H^1(F_k) = \varepsilon$. now observe that all the balls B_i are contained in $\Delta = \pi^{-1}(B^{n-1}(y, r)) \cap Q^n$ and hence (27) gives

$$H^1(f(F_k)) \leq C(n) H (H^1(F_k) + \varepsilon)^{(n-1)/n} \left(\frac{d\nu}{dH^{n-1}}(y) \right)^{1/n} .$$

Now inequality (25) follows by assing to the limits as $\varepsilon \to 0$ and then $k \to \infty$. The proof is complete. □

Now we can prove that the classes of quasiconformal and quasisymmetric mappings coincide. namely we will prove.

Theorem 74 *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a homeomorphism. Then the following conditions are equaivalent*

1. *f is quasiconformal,*
2. *f is quasisymmetric,*
3. *$r \in W^{1,n}_{\rm loc}(\mathbb{R}^n, \mathbb{R}^n)$, J_f des not change sign i.e. $J_f \geq 0$ a.e. or $J_f \leq 0$ a.e. and there is a constant $K \geq 1$ such that*

$$\max_{|\xi|=1} |Df(x)\xi| \leq K \min_{|\xi|=1} |Df(x)\xi| \ \textit{a.e.} \tag{28}$$

Remark. The assumption that J_f does not change sign is superflous as one can prove that any $W^{1,n}$ homeomorphism is differentiable a.e. and hence J_f does not change sign by Lemma 72.

Proof of Theorem 74. The implication 2. ⇒ 1. is obvious and the implication 1. ⇒ 3. follows from Theorem 70. Throough the proof C, C', C'' will denote general constants depending on n only. Observe that (28) implies that

$$\|Df(x)\|^n \leq K^{n-1}|J_f| \text{ a.e.}$$

Define

$$\psi(z) = \begin{cases} 1 & \text{if } |z| \leq l, \\ (\log L/l)^{-1}\log(L/|z|) & \text{if } l \leq |z| \leq L, \\ 0 & \text{if } |z| \geq L. \end{cases}$$

and define $\varphi(z) = \psi(z - x)$. The function ξ is Lipschitz and

$$\int_{\mathbb{R}^n} |\nabla\varphi|^n = \int_{\mathbb{R}^n} |\nabla\psi|^n = (\log L/l)^{-n} \int_{B(0,L)\setminus B(0,l)} |x|^{-n} = n\omega_n(\log L/l)^{1-n}\,.$$

Let $E = f^{-1}(\overline{B}(0,l))$ and $F' = f^{-1}(\mathbb{R}^n \setminus B(0,L)) \cap \overline{B}(x,2r)$. Then E is a continuum that connects x to the boundary $\partial B(x,r)$ and F' contains a continuum F that connects $\partial B(x,r)$ to $\partial B(x,2r)$. Hence $H^1_\infty(E) \geq r$, $H^1_\infty(F) \geq r$ and $\operatorname{dist}(E,F) \leq r$.

The function $v = \varphi \circ f$ is continuous and belongs to $W^{1,n}_{\rm loc}$ (because φ is Lipschitz). Moreover $v = 1$ on E and $v = 0$ on F. Now we estimate the L^n norm of the gradient of v empying the change of variables formula.

$$\begin{aligned} \int_{\mathbb{R}^n} |\nabla v(x)|^n &\leq \int_{\mathbb{R}^n} |\nabla\varphi(f(x))|^n \|Df(x)\|^n \\ &\leq K^{n-1} \int_{\mathbb{R}^n} |\nabla\varphi(f(x))|^n |J_f(x)|^n \\ &= K^{n-1} \int_{\mathbb{R}^n} |\nabla\varphi|^n \\ &= n\omega_n K^{n-1}(\log L/l)^{1-n}\,. \end{aligned}$$

We want to show that the ratio L/l is bounded. To this end it suffices t show that the integral $\int |\nabla v|^n$ is bounded from below by an universal constant.

Recall that by Theorem 58 the pointwise inequality holds

$$|v(x) - v(y)| \leq C|x-y|^{1/n} \left(M^{1-1/n}|\nabla v|(x) + M^{1-1/n}|\nabla v|(y) \right) .$$

Taking $x \in E$ and $y \in F$ yields

$$1 \leq Cr^{1/n} \left(M^{1-1/n}|\nabla v|(x) + M^{1-1/n}|\nabla v|(y) \right) .$$

This implies that there is another constant C such that either $M^{1-1/n}|\nabla v|(x) \geq Cr^{-1/n}$ for all $x \in E$ or $M^{1-1/n}|\nabla v|(y) \geq Cr^{-1/n}$ for all $y \in F$. Assume the first case. The proof for the second case is analogous. By Lemma 57 we obtain

$$r \leq H^1_\infty(M^{1-1/n}|\nabla v| > Cr^{-1/n}) \leq C'r \int_{\mathbb{R}^n} |\nabla v|^n \leq C''rK^{n-1}(\log L/l)^{1-n} ,$$

and hence

$$L \leq l \exp(CK).$$

The proof is complete. □

Mostov rigidity theorem. One of the most celebrated applications of the theory of quasiconformal mappings is so called Mostow ridigidity theorem. Unfortunately the theorem goes far beyond the scope of the material presented in the lectures, so we will only sketch the main ideas giveing a fravour of the beauty and deepness of the result.

Let us first state the theorem. Then we will explain the statement and the main idea of the proof.

Theorem 75 (Mostow) *Let Y and Y' be compact Reimannian maniflds of dimmension $n \geq 3$ and of the constant sectional curvature -1. If Y and Y' are diffeomorphic, then they are isometric.*

The theorem is false if $n = 2$.

Quasiregular mappings.

Lecture 6

Gromov and the isoperimetric inequality. The aim of this section is to prove the following result of Federer, Fleming and Maz'ya.

Theorem 76 *For any $u \in W^{1,1}(\mathbb{R}^n)$ there is*

$$\left(\int_{\mathbb{R}^n} |u|^{\frac{n}{n-1}}\, dx \right)^{\frac{n-1}{n}} \leq n^{-1}\omega_n^{-1/n} \int_{\mathbb{R}^n} |\nabla u|, \tag{29}$$

where ω_n is the volume of the unit ball. Moreover the constant $n^{-1}\omega_n^{-1/n}$ cannot be replaced by a smaller one.

We have already proved the inequality, but with a worse constant. The exact value of the conatsnt has a deep geometric meaning: we will see that Theorem 76 implies the classical isoperimatric inequality.

Let $K \subset \mathbb{R}^n$ be a compact set. We define the Minkowski content as follows

$$\mu^+(K) = \liminf_{\varepsilon\to 0} \frac{|\{x \in \mathbb{R}^n :\, 0 < \mathrm{dist}\,(x,K) < \varepsilon\}|}{\varepsilon}. \tag{30}$$

If K is a closure of an open, bounded set with the C^2 boundary, then it easily follows that $\mu^+(K) = H^{n-1}(\partial K)$. Thus the Minkowski content is a generalization of the surface area.

The isoperimetric theorem states that among all the sets with the given volume, the ball has the smallest area of the boundary. It can be expressed in the following inequality.

Theorem 77 (Isoperimetric Inequality) *Let $K \subset \mathbb{R}^n$ be a compact set. Then*

$$|K|^{\frac{n-1}{n}} \leq n^{-1}\omega_n^{-1/n}\mu^+(K),$$

and the constant $n^{-1}\omega_n^{-1/n}$ cannot be replaced by any smaller constant.

Proof. First observe that if $K = B^n$, then we have the equality, so the constant cannot be smaller. For any compact set $K \subset \mathbb{R}^n$ define

$$\varphi(x) = \begin{cases} 1 - \varepsilon^{-1}\mathrm{dist}\,(x,K) & \text{if } \mathrm{dist}\,(x,K) \leq \varepsilon \\ 0 & \text{otherwise} \end{cases}$$

Then the function φ is Lipschitz. Since the Lipschitz constant of the function $x \mapsto \mathrm{dist}\,(x,K)$ is 1, we conclude that $|\nabla\varphi| \leq \varepsilon^{-1}$ a.e. Moreover $|\nabla\varphi(x)| = 0$ a.e. outside the strip $\{0 < \mathrm{dist}\,(x,K) < \varepsilon\}$. Hence applying Theorem 76 to $u = \varphi$ yields

$$\begin{aligned} |K|^{\frac{n-1}{n}} &\leq \left(\int_{\mathbb{R}^n} |\varphi|^{\frac{n}{n-1}}\,dx\right)^{\frac{n-1}{n}} \\ &\leq n^{-1}\omega_n^{-1/n}\int_{\mathbb{R}^n} |\nabla\varphi|\,dx \\ &\leq n^{-1}\omega_n^{-1/n}\frac{|\{x \in \mathbb{R}^n :\, 0 < \mathrm{dist}\,(x,K) < \varepsilon\}|}{\varepsilon}, \end{aligned}$$

and the theorem follows. □

Observe that the argument above implies that the constant in (29) cannot be smaller (otherwise we would prove the isoperimetric inequality with a smaller constrant). Thus we are left with the proof of (29).

The proof that we present below is due to Gromov and it is one of the most beautiful and the most unexpected proofs I have ever seen.

Proof of Theorem 76. Since $|\nabla u| = |\nabla |u||$ a.e. we may assume $u \geq 0$ a.e. Moreover by a standard approximation argument we may assume that $u \in C_0^\infty(\mathbb{R}^n)$. Finally we may require that the support of u is a ball and that u is strictly positive in that ball. Indeed, otherwise we take $0 \leq \varphi \in C_0^\infty(\mathbb{R}^n)$ whose support is a ball containing the support of u and pass to the limit as $\varepsilon \to 0$. We will need the following lemma.

Lemma 78 *Let $0 < f \in C^1(\overline{\Omega})$, where $\Omega \subset \mathbb{R}^n$ is an open, bounded and convex set. Then there exits a diffeomorphism $\Phi : \Omega \to B(0, \omega_n^{-1/n})$ such that*

1. *Φ is a triangular mappping i.e.*

$$\Phi(x_1, x_2, \ldots, x_n) = (\Phi_1(x_1), \Phi_2(x_1, x_2), \ldots, \Phi_n(x_1, x_2, \ldots, x_n)) .$$

2. *Derivatives $\partial\Phi_i/\partial x_i$ are strictly positive and*

$$\det D\Phi(x) = \prod_{i=1}^n \frac{\partial \Phi_i}{\partial x_i} = \frac{f(x)}{\int_\Omega f} , \tag{31}$$

for all $x \in \Omega$.

The radius of the ball $B(0, \omega_n^{-1/n})$ was choosen in order to have a ball with the unit volume.

The lemma is intereteng on it own: it shows that there is a diffeomorphism between a ball and an arbitrary open, bounded and convex set, with a priori prescribed Jacobian. In particular there exits a diffeomorphism between a unit cube and a ball of volume one whose Jacobian equals one.

The denominator on the right hand side of (31) is a constant choosen for a normalization. This is just to make a volume of the image of Φ equal to one. Indeed,

$$|\Phi(\Omega)| = \int_\Omega |\det D\Phi| = \frac{\int_\Omega f}{\int_\Omega f} = 1 .$$

Proof of Lemma 78. The proof is not much longer than a statement. We define the diffeomorphism Φ explicite.

In the first step we prove a slightly modified verion of the lemma that we will denote by Lemma 78'. Namely we replace $B(0, \omega_n^{-1/n})$ by a unit cube $Q = [0,1]^n$ in the statement. All the oter words in the statement remain the same.

The desired diffeomorphim between Ω and Q, denoted by Ψ is defined by an explicite formula.

$$\Psi_i(x_1, x_2, \ldots, x_i) = \frac{\int_{A_i(x)} f \, dz_i \ldots dz_n}{\int_{B_i(x)} f \, dz_i \ldots dz_n} \qquad i = 1, 2, \ldots, n ,$$

where the sets $A_i(x)$, $B_i(x)$ are defined as follows

$$A_i(x) = \{z = (z_1, \dots, z_n) : \, z_j = x_j \text{ for } j < i \text{ and } z_i \le x_i\} ,$$

$$B_i(x) = \{z = (z_1, \dots, z_n) : \, z_j = x_j \text{ for } j < i\} .$$

The best way to see the geometric interpretation of the sets is to make a three dimensional picture.

$B_1(x)$ is the the whole domain Ω.

$A_1(x)$ is a "left" part of Ω cut by a $n-1$ dimensional hyperplane perpendicular to the axis x_1 passing through a point x.

$B_2(x)$ is the interection of Ω and the $n-1$ dimensional hyperplane.

$A_2(x)$ is a part of $B_2(x)$ cut by a $n-2$ dimensional hyperplane given by fixing the first tow coordinates.

etc.

Observe that Ψ_i depends on variables $x_1, \dots, x_i$ only, Ψ_i is strictly increasing with respect to x_i and hence Ψ is a one to one mapping with $\partial\Psi_i/\partial x_i > 0$. It is easy to see that $\Psi(\Omega) = Q$. Thus Ψ is a C^1 homeomorphism. Since the matrix $D\Psi$ is triangular we conclude that the Jacobi determinant is a product of the diagonal elements and hence it is strictly positive. Thus Ψ is a diffeomorphism.

It follow from the definition of Ψ_i that

$$\frac{\partial \Psi_i}{\partial x_i} = \frac{\int_{B_{i+1}} f}{\int_{B_i} f} ,$$

and hence

$$\det D\Psi = \prod_{i=1}^{n} \frac{\partial \Psi_i}{\partial x_i} = \frac{\int_{B_{n+1}} f}{\int_{B_1} f} = \frac{f(x)}{\int_\Omega f} .$$

This completes the proof of Lemma 78'.

In particular we may apply the above contruction to $\widetilde{\Omega} = B(0, \omega_n^{-1/n})$ and $f \equiv 1$. Thus we get a triagular diffeomorphism

$$\widetilde{\Psi} : B(0, \omega_n^{-1/n}) \to Q$$

with Jacobian equal to one. Hence $\widetilde{\Psi}^{-1} : Q \to B(0, \omega_n^{-1/n})$ is a triangular diffeomorphism with the Jacobian equal to one a well. Now the diffeomorphsim $\Phi = \widetilde{\Psi}^{-1} \circ \Psi : \Omega \to B(0, \omega_n^{-1/n})$ verifies all the condition of lemma 78. The proof is complete. $\square$

Now we can complete the proof of Theorem 76.

Let $\overline{\Omega}$ be a support of u (Ω is a ball) and let $f = |u|^{n/(n-1)}$. By Lemma 78 there exits a diffeomorphism

$$\Phi : \Omega \to B(0, \omega_n^{-1/n})$$

such that

$$\det D\Phi = \prod_{i=1}^{n} \frac{\partial \Phi_i}{\partial x_i} = |u|^{n/(n-1)} / \int_\Omega |u|^{n/(n-1)} .$$

The arithmetic–geometric mean inequality yields

$$\frac{\operatorname{div}\Phi}{n} = n^{-1}\sum_{i=1}^{n} \frac{\partial \Phi_i}{\partial x_i} \geq \left(\prod_{i=1}^{n} \frac{\partial \Phi_i}{\partial x_i} \right)^{1/n} = |u|^{\frac{1}{n-1}} \|u\|_{\frac{n}{n-1}}^{\frac{-1}{n-1}} .$$

Since the vector field $u\Phi \in C^1(\Omega)$ vanishes at the boundary of Ω, integration by parts gives

$$0 = \int_\Omega \operatorname{div}(u\Phi) = \int_\Omega u \operatorname{div}\Phi + \int_\Omega \nabla u \cdot \Phi .$$

Invoking that $u > 0$ in Ω we obtain

$$n\int_\Omega |u||u|^{\frac{1}{n-1}} \|u\|_{\frac{n}{n-1}}^{\frac{-1}{n-1}} \leq \int_\Omega |u| \operatorname{div}\Phi \leq \int_\Omega |\nabla u||\Phi| \leq \omega_n^{-1/n} \int_\Omega |\nabla u| ,$$

and hence inequality (29) follows. The proof is complete. □

Lecture 7

Caccioppoli estimates. In the previous lectures we proved the existence of a weak solution to the Dirichlet problem. Now we prove that this solution is in fact the classical harmonic function. This important result is known as the Weyl lemma.

Theorem 79 (Weyl lemma) *If $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ is a weak solution to the Laplace equation $\Delta u = 0$, then $u \in C^\infty(\Omega)$ and hence u is a classical harmonic function.*

Proof. By the definition u is a weak solution to $\Delta u = 0$ if and only if

$$\int_\Omega \nabla u \cdot \nabla \varphi = 0 \qquad \forall \varphi \in C_0^\infty(\Omega). \tag{32}$$

Note that (32) holds also for $\varphi \in W^{1,2}(\Omega)$ with compact support — simply by approximating such φ by compactly supported smooth functions.

In the first step of the proof we will derive the so called Caccioppoli type inequality. The idea is very simple and it easily generalizes to more complicated elliptic equations or systems, where it is frequently employed.

Fix concentric balls $B(r) \subset\subset B(R) \subset\subset \Omega$ and let $\eta \in C_0^\infty(B(R))$, $0 \leq \eta \leq 1$, $\eta|_{B(r)} \equiv 1$, $|\nabla \eta| \leq 2/(R-r)$ be a cutoff function.

Applying $\varphi = (u-c)\eta^2$ to (32) we obtain

$$\int_\Omega \nabla u \cdot \Big(\nabla u \eta^2 + 2(u-c)\eta \nabla \eta \Big) = 0,$$

so

$$\int_\Omega |\nabla u|^2 \eta^2 \leq 2 \int_\Omega |u-c|\eta |\nabla u|\, |\nabla \eta| \leq 2 \left(\int_\Omega |u-c|^2 |\nabla \eta|^2 \right)^{1/2} \left(\int_\Omega |\nabla u|^2 \eta^2 \right)^{1/2}.$$

Hence

$$\int_\Omega |\nabla u|^2 \eta^2 \leq 4 \int_\Omega |u-c|^2 |\nabla \eta|^2,$$

which yields the following Caccioppoli inequality

$$\int_{B(r)} |\nabla u|^2 \leq \frac{16}{(R-r)^2} \int_{B(R)\setminus B(r)} |u-c|^2\,, \tag{33}$$

for any $c \in \mathbb{R}$. In particular we get

$$\int_{B(r)} |\nabla u|^2 \leq C(R,r) \int_{B(R)} |u|^2. \tag{34}$$

Assume for a moment that we already know that $u \in C^\infty$. Then also the derivatives of u are harmonic and hence (34) applies to derivatives of u, so for $r < r' < R$ we get

$$\int_{B(r)} |\nabla^2 u|^2 \leq C_1 \int_{B(r')} |\nabla u|^2 \leq C_2 \int_{B(R)} |u|^2.$$

Repeating the argument with higher order derivatives we obtain

$$\int_{B(R/2)} |\nabla^k u|^2 \, dx \leq C(R,k) \int_{B(R)} |u|^2,$$

for $k = 1, 2, 3, \ldots$ In other words

$$\|u\|_{W^{k,2}(B(R/2))} \leq C(R,k) \|u\|_{L^2(B(R))}. \tag{35}$$

The inequality was proved under the asumption that $u \in C^\infty$. We shall prove now that (35) holds also for any weakly harmonic function $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$.

Let $u_\varepsilon(x) = \int u(x-y)\phi_\varepsilon(y)\, dy$ be a standard molifier approximation where $\phi_\varepsilon(y) = \varepsilon^{-n}\phi(y/\varepsilon)$, $\phi \in C_0^\infty$, $\phi \geq 0$, $\int \phi = 1$.

Then $u_\varepsilon \in C^\infty(\Omega_\varepsilon)$, where Ω_ε consists of points in Ω with the distance to the boundary bigger than ε. Note that u_ε is harmonic in Ω_ε. Indeed, for any $\varphi \in C_0^\infty(\Omega_\varepsilon)$ we have

$$\begin{aligned} \int_\Omega \nabla u_\varepsilon(x) \nabla \varphi(x)\, dx &= \int \left(\int u(y) \nabla_x \phi_\varepsilon(x-y)\, dy \right) \nabla \varphi(x)\, dx \\ &= \int \left(\int \nabla_x u(x-y) \phi_\varepsilon(y)\, dy \right) \nabla \varphi(x)\, dx \\ &= \int \left(\int \nabla_x u(x-y) \nabla \varphi(x)\, dx \right) \phi_\varepsilon(y)\, dy = 0. \end{aligned}$$

Hence u_ε is a weakly harmonic function and since u_ε is C^∞ smooth we conclude that u_ε is a classical harmonic function. Thus (35) holds for each u_ε. Since $u_\varepsilon \to u$ in $L^2(B(R))$, we obtain that u_ε is a Cauchy sequence in $W^{k,2}(B(R/2))$ and passing to the limit yields (35) for u and all $k = 1, 2, 3, \ldots$ Hence by the Sobolev embedding theorem (Corollary 42) $u \in C^\infty$. The proof is complete. □

We will show now some typical applications of the Caccioppoli inequaity. However for the clarity of presentation we will concentrate on very simple examples only.

Taking $R = 2r$ in (33) and applying Poincaré inequality (Corollary 39) we get

$$\begin{aligned}\int_{B(r)} |\nabla u|^2\, dx &\leq \frac{16}{r^2} \int_{B(2r)\setminus B(r)} |u - u_{B(2r)\setminus B(r)}|^2\, dx \\ &\leq C(n) \int_{B(2r)\setminus B(r)} |\nabla u|^2\, dx.\end{aligned}$$

We have obtained the estimate of the integral over $B(r)$ by an integral over an annulus. Now we add $C(n) \int_{B(r)} |\nabla u|^2$ to both sides of the inequality to fill the hole in the annulus. We get

$$\int_{B(r)} |\nabla u|^2\, dx \leq \underbrace{\frac{C(n)}{C(n)+1}}_{<1} \int_{B(2r)} |\nabla u|^2\, dx. \tag{36}$$

For obvious reasons the argument is called *hole–filling*.

It is crucial that the coefficient in (36) is strictly less than 1. We will show some applications of this fact.

Theorem 80 *If u is a harmonic function on $\mathbb{R}^n$ with $|\nabla u| \in L^2(\mathbb{R}^n)$, then u is constant.*

Proof. Passing to the limit as $r \to \infty$ in inequality (36) yields

$$\int_{\mathbb{R}^n} |\nabla u|^2\, dx \leq \theta \int_{\mathbb{R}^n} |\nabla u|^2, \qquad \theta < 1\,.$$

Hence $|\nabla u| = 0$ a.e. and thus u is constant. □

Corollary 81 *Any bounded harmonic function on $\mathbb{R}^2$ is constant.*

Proof. By Caccioppoli inequality (33) we get

$$\int_{B(r)} |\nabla u|^2 \leq \frac{C}{r^2} \int_{B(2r)} |u|^2 \leq C'.$$

Since u is bounded, the constant C' does not depend on r. Passing to the limit as $r \to \infty$ we conclude that $\int_{\mathbb{R}^2} |\nabla u|^2\, dx < \infty$ and hence u is constant by the previous result. □

Remark. The corollary holds in $\mathbb{R}^n$ for any $n \geq 1$, but the proof is different. We will prove this and more general results in Lecture 8.

More generally we say that $u = (u_1, \ldots, u_m) \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$, $\Omega \subset \mathbb{R}^n$, $1 < p < \infty$ is a weak solution to the *p-harmonic system*, (u is a p-harmonic mapping) if

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u_i) = 0\,, \qquad i = 1, 2, \ldots, m\,,$$

i.e.

$$\int |\nabla u|^{p-2}\langle \nabla u_i, \nabla\varphi\rangle = 0 \qquad \forall \varphi \in C_0^\infty(\Omega) \text{ and } i = 1, 2, \ldots, m. \tag{37}$$

Here $|\nabla u| = (\sum_{ij}(\partial u_i/\partial x_j)^2)^{1/2}$. Taking $\varphi_i = (u_i - c_i)\eta^p$ in (37) and adding up resulting identities for all i leads to the following generalization of (33)

$$\int_{B(r)} |\nabla u|^p \leq \frac{C}{(R-r)^p}\int_{B(R)\setminus B(r)} |u - c|^p\,,$$

for any $c = (c_1, c_2, \ldots, c_m)$. By the hole–filling argument

$$\int_{B(r)} |\nabla u|^p \leq \theta \int_{B(2r)} |\nabla u|^p\,, \qquad \theta < 1\,. \tag{38}$$

Fix $\Omega' \subset\subset \Omega$ and let $R = \operatorname{dist}(\Omega', \partial\Omega)$. Let $x \in \Omega'$ and $r < R$. Take the integer k such that $R/2 \leq 2^k r < R$. Then $B(x, 2^k r) \subset \Omega$ and $\theta^k \approx r^{-\log_2\theta}$. Hence iterating (38) yields

$$\int_{B(x,r)} |\nabla u|^p \leq C r^{-\log_2\theta}\int_\Omega |\nabla u|^p\,.$$

Since θ is slightly less than 1 we get $\alpha = -\log_2\theta \in (0,1)$ and hence

$$\int_{B(x,r)} |\nabla u|^p \leq C r^\alpha\,,$$

for all $x \in \Omega'$ and $r < R$. If $p = n$, then by Morrey's lemma (Corollary 59) we obtain that u is Hölder continuous in Ω'. Thus we have proved

Corollary 82 *n-harmonic mappings in $\Omega \subset \mathbb{R}^n$ are locally Hölder continuous.*

In general, in contrast to the case $p = 2$, the p-harmonic mappings need not be C^∞ smooth. The optimal smoothness result (even for $m = 1$) is the following very difficult theorem

Theorem 83 *If $u \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$, $1 < p < \infty$, $\Omega \subset \mathbb{R}^n$, is p-harmonic, then $u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ for some $\alpha \in (0,1)$.*

In Lecture 8 we will sketch the proof of Hölder continuity of p-harmonic functions (i.e. when $m = 1$) for any $1 < p < \infty$.

As we will see the p-harmonic equations also appear as the equations for the minimizers of some variational functionals.

Euler Lagrange Equations. The Dirichlet principle says that it is the same to seek for the minimizer of the Dirichlet integral or for the solution to the Laplace equation. The Laplace equation was derived by taking directional derivatives of the functional I in all the directions $\varphi \in C_0^\infty$. We took directional derivatives in the infinite dimensional space of functions. Thus roughly speaking we can say that u is a minimizer of the functional I if the "derivative" of I in point u is zero. We will generalize this observation to the abstract setting of functionals on Banach spaces.

Let X be a Banach space and let $I : X \to \mathbb{R}$. The directional derivative of I at $u \in X$ in the direction $h \in X$, $h \neq 0$ is defined as

$$D_h I(u) = \lim_{t \to 0} \frac{I(u+th) - I(u)}{t}\,.$$

We say that I is *differentiable in the sense of Gateaux* at a point $u \in X$ if for every $h \in X$, $h \neq 0$ the directional derivative $D_h I(u)$ exists and the function $h \mapsto D_h I(u)$ is linear and continuous. This defines functional $DI(u) \in X^*$ by the formula $\langle DI(u), h\rangle = D_h I(u)$. We call $DI(u)$ the *Gateaux differential.*

Proposition 84 *If $I : X \to \mathbb{R}$ is Gateaux differentiable and $I(\overline{u}) = \inf_{u\in X} I(u)$, then $DI(\overline{u}) = 0$.* □

Later we will see that this is an abstract statement of the so called Euler–Lagrange equations. For example we will see that when we differentiate the functional $I(u) = \int |\nabla u|^2$, we obtain the equation $\Delta u = 0$. This looks like the Dirichlet principle.

Theorem 85 *Let $I : X \to \mathbb{R}$ be Gateaux differentiable. Then the following conditions are equivalent.*

1. *I is convex,*
2. *$I(v) - I(u) \geq \langle DI(u), v - u\rangle$ for all $u, v \in X$,*
3. *$\langle DI(v) - DI(u), v - u\rangle \geq 0$ for all $u, v \in X$.*

Proof. 1. $\Rightarrow$ 2. Convexity implies that

$$\frac{I(u + t(v-u)) - I(u)}{t} \leq I(v) - I(u)$$

for $t \in (0,1)$ and hence the claim follows by passing to the limit as $t \to 0$. 2. $\Rightarrow$ 3. It follows directly from the assumption that $\langle -DI(u), v-u\rangle \geq I(u)-I(v)$ and $\langle DI(v), v-u\rangle \geq I(v)-I(u)$. Adding both inequalities we obtain the desired inequality. 3. $\Rightarrow$ 1. We have to prove that for any two points $u, v \in X$, the function $f(t) = I(u+t(v-u))$ is convex. To this end it suffices to prove that $f'(t)$ is increasing. This follows easily from the assumed inequality and the formula for f'. $\square$

In the case of convex functionals the necesary condition given in Proposition 84 is also sufficient.

Proposition 86 *If $I : X \to \mathbb{R}$ is convex and Gateaux differentiable, then $I(\overline{u}) = \inf_{u\in X} I(u)$ if and only if $DI(\overline{u}) = 0$.*

Proof. It remains to prove the implication $\Leftarrow$. By the convexity and the above theorem $I(u) - I(\overline{u}) \geq \langle DI(\overline{u}), u - \overline{u}\rangle = 0$, hence $I(u) \geq I(\overline{u})$. $\square$

Note that we have similar situation in the case of the Dirichlet principle. The condition $\Delta u = 0$ is necessary and sufficient for u to be the minimizer.

Below we give some important examples of functionals differentiable in the Gateaux sense.

Lemma 87 *Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be measurable in $x \in \Omega$ (for every $u \in \mathbb{R}$) and C^1 in $u \in \mathbb{R}$ (for almost every $x \in \Omega$). Moreover assume the following growth conditions*

$$|f(x,u)| \leq a(x) + C|u|^p, \qquad |f'_u(x,u)| \leq b(x) + C|u|^{p-1}$$

where $a \in L^1(\Omega)$, $b \in L^{p/(p-1)}(\Omega)$ and $1 < p < \infty$. Then the functional $I(u) = \int_\Omega f(x, u(x))\, dx$ is Gateaux differentiable as defined on $L^p(\Omega)$ and

$$\langle DI(u), v\rangle = \int_\Omega f'_u(x,u)v$$

Proof. The growth condition implies that I is defined and finite on $L^p(\Omega)$. We have

$$\frac{I(u+tv) - I(u)}{t} = \int_\Omega \frac{f(x, u+tv) - f(x,u)}{t} = \int_\Omega \frac{1}{t}\int_0^t f'_u(x, u+sv)v\, ds\, dx. \qquad (39)$$

Observe that

$$\frac{1}{t}\int_0^t f'_u(x, u+sv)v\, ds \to f'_u(x,u)v \text{ a.e.},$$

and that by the growth condition

$$\left|\frac{1}{t}\int_0^t f'_u(x, u+sv)v\, ds\right| \leq C\left(|b|^{p/(p-1)} + |u|^p + |v|^p\right),$$

where C does not depend on t. Hence we may pass to the limit in (39) and we get

$$\lim_{t\to 0}\frac{I(u+tv)-I(u)}{t}=\int_\Omega f'_u(x,u)v\,dx.$$

The proof is complete. $\square$

It is also easy to prove the following

Lemma 88 *For $1<p<\infty$ the functional $I_p=\int_\Omega|\nabla u|^p$ is Gateaux differentiable on $W^{1,p}(\Omega,\mathbb{R}^m)$ and*

$$\langle DI_p(u),v\rangle = p\int_\Omega |\nabla u|^{p-2}\langle\nabla u,\nabla v\rangle.$$

Since the functionl I_p is convex we get that

$$I_p(\overline{u})=\inf_{u\in W^{1,p}_w(\Omega,\mathbb{R}^m)} I_p(u)\,, \tag{40}$$

if and only if $DI_p(\overline{u})=0$ i.e. if and only if $\overline{u}$ is a weak solution to the following system

$$\mathrm{div}\,(|\nabla u|^{p-2}\nabla u_i)=0 \qquad u_i-w_i\in W^{1,p}_0(\Omega)\,, \qquad i=1,2,\ldots,m\,. \tag{41}$$

Here $w=(w_1,w_2,\ldots,w_m)\in W^{1,p}(\Omega,\mathbb{R}^m)$. Moreover the strict convexity guarantess the uniqueness of the minimizer or equivalently the uniqueness of the solution to the system.

This is a version of the Dirichlet principle. Thus the problem of solving (41) reduces to finding the minimizer of (40), which is easy, see Corollary 7. Equations (41) are called Euler–Lagrange system for the minimizer of I_p. The variational approach easily generalizes to more complicated elliptic equations or systems.

Now for the simplicity of the notation we will be cencerned with the case $m=1$.

Theorem 89 *Let $\Omega\subset\mathbb{R}^n$ be a bounded domain with Lischitz boundary and $1<p<\infty$. Assume that the function $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is measurable in $x\in\Omega$, C^1 in $u\in\mathbb{R}$ and satisfies*

$$|f(x,u)|\le a(x)+C|u|^q \qquad |f'_u(x,u)|\le b(x)+C|u|^{q-1}$$

where $q=np/(n-p)$ if $p<n$ and $q<\infty$ is any exponent if $p\ge n$. Then the functional

$$I(u)=\int_\Omega \frac{1}{p}|\nabla u|^p+f(x,u) \tag{42}$$

is Gateaux differentiable as defined on the space $W^{1,p}(\Omega)$ and

$$\langle DI(u),v\rangle=\int_\Omega|\nabla u|^{p-2}\langle\nabla u,\nabla v\rangle+\int_\Omega f'_u(x,u)v \tag{43}$$

for every $v\in W^{1,p}(\Omega)$.

Proof. It easily follows from Lemma 87, Lemma 88 and the Sobolev embedding $W^{1,p}(\Omega) \subset L^q(\Omega)$. $\square$

Remark. If we define I on $W_0^{1,p}(\Omega)$ only, Ω can be an arbitrary open and bounded set, because then we still have $W_0^{1,p}(\Omega) \subset L^q(\Omega)$. In this case (43) holds for $v \in W_0^{1,p}(\Omega)$.

Fix $w \in W^{1,p}(\Omega)$. If u is a minimizer of (42) in $W_w^{1,p}(\Omega)$, then u solves the equation

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = f'_u(x,u)\,, \qquad u - w \in W_0^{1,p}(\Omega)\,. \tag{44}$$

Indeed, if $J(v) = I(v+w)$, then $v = u - w$ is a minimizer of J on $W_0^{1,p}(\Omega)$ and hence $DJ(v) = 0$ i.e.,

$$0 = \langle DJ(v), \varphi\rangle = \langle DI(\underbrace{v+w}_{u}), \varphi\rangle \qquad \forall \varphi \in W_0^{1,p}(\Omega)\,,$$

i.e. (44) holds. Equation (44) is called Euler–Lagrange equation for (42). Conversly one can prove the existence of a solution to (44) by proving the existexce of a minimizer to (42). This is our next aim.

Recall that if $1 < p < n$, then $p^* = np/(n-p)$ is the Sobolev exponent. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Assume that a function $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is measurable in $x \in \Omega$ and continuous in $u \in \mathbb{R}$. If the following growth condition holds

$$|g(x,u)| \le C(1 + |u|^q) \tag{45}$$

where $q \le p^* - 1$ when $1 < p < n$ and $q < \infty$ when $n \le p < \infty$, then the equation

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = g(x,u)\,, \qquad u - w \in W_0^{1,p}(\Omega) \tag{46}$$

is the Euler–Lagrange equation for the minimizer of the functional

$$I(u) = \int_\Omega \frac{1}{p}|\nabla u|^p + G(x,u)\,, \tag{47}$$

where $G(x,u) = \int_0^u g(x,t)\,dt$, defined on $W_w^{1,p}(\Omega)$. This follows from the estimate $|G(x,u)| \le C(1 + |u|^{q+1})$ and from Theorem 89.

In order to prove the existence of a minimizer of I we need find assumptions that would guarantee that I is coercive and swlc.

Oberve that if $g(x,u)u \ge 0$ for all u, then $G(x,u) \ge 0$ and hence I is coercive. This condition for g is too strong. We will relax it now.

Asume that

$$g(x,u)\frac{u}{|u|} \ge -\gamma |u|^{p-1} \tag{48}$$

for all $|u| \ge M$ and a.e. $x \in \Omega$. Here M and γ are constants such that $M \ge 0$ and $\gamma < \mu_1^{-1}$, where μ_1 is the best (the least) constant in the Poincaré inequality

$$\int_\Omega |u|^p \le \mu_1 \int_\Omega |\nabla u|^p \qquad \forall u \in C_0^\infty(\Omega)\,.$$

We claim that under conditions (45) and (48) the functional I is coercive on $W^{1,p}_w(\Omega)$. Indeed, both the conditions imply that

$$G(x,u) \geq -C - \frac{\gamma}{p}|u|^p$$

for all $u \in \mathbb{R}$. Hence

$$I(u) \geq \frac{1}{p}\int_\Omega |\nabla u|^p - \frac{\gamma}{p}\int_\Omega |u|^p - C \geq \frac{1}{p}(1-\gamma\mu_1)\int_\Omega |\nabla u|^p - C \to \infty$$

as $\|u\|_{1,p} \to \infty$ and $u \in W^{1,p}_w(\Omega)$.

Condition (48) is optimal. Namely at the end of the lecture we will show that the functional constructed for $g(x,u) = -\mu_1^{-1}u|u|^{p-2}$ is not coercive on $W^{1,p}_0(\Omega)$.

Now in order to have the swlc condition we need slightly relax condition (45).

Theorem 90 *Let g satifies (45) and (48) and let the functional I be defined on $W^{1,p}_w(\Omega)$ a above. Then the functional is coercive. If in addition $q < p^* - 1$ when $1 < p < n$ (in the case $p \geq n$ we do not change the assumption) then the functional I is swlsc and hence it assumes the minimum which solves the Dirichlet problem (46) with the boundary condition $u \in W^{1,p}_w(\Omega)$.*

Proof. We have already proved that the functional is coercive. Now assume that $q < p^* - 1$ when $1 < p < n$ and $q < \infty$ when $p \geq n$.

Let $v_k \rightharpoonup v$ weakly in $W^{1,p}_0(\Omega)$. We have to prove that

$$I(v+w) \leq \liminf_{k\to\infty} I(v_k + w). \tag{49}$$

Under the additional assumption about q, the embedding $W^{1,p}_0(\Omega) \subset L^{q+1}(\Omega)$ is compact and hence $v_k + w \to v + w$ in L^{q+1}. Since $|G(x,u)| \leq C(1+|u|^{q+1})$ we conclude that

$$\int_\Omega G(x, v_k + w) \to \int_\Omega G(x, v+w)\,. \tag{50}$$

This follows easily form the following version of the dominated convergence theorem.

Lemma 91 *Let $|f_k| \leq g_k$, $g_k \to g$ in L^1 and $f_k \to f$ a.e. Then $\int f_k \to \int f$.* □

Observe that

$$\int_\Omega |\nabla(v+w)|^p \leq \liminf_{k\to\infty} \int_\Omega |\nabla(v_k + w)|^p. \tag{51}$$

This is a direct consequence of Theorem 5. Now (50) and (51) impliy the swlsc property (49) and then the theorem follows directly form Theorem 4. □

We leave as an exercise the proof of the following variant of the above result.

Theorem 92 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $1 < p < \infty$. Then for every $f \in (W_0^{1,p}(\Omega, \mathbb{R}^m))^*$ there exists the unique solution to the following system*

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u_i) = f \qquad u_i \in W_0^{1,p}(\Omega), \qquad i = 1, 2, \ldots, m.$$

Are the solutions Hölder continuous when $p = n$?

At the begining of the lecture we have proved that weak solutions to the Laplace equation are C^∞ smooth. Now we show a tricky and powerful method of proving the regularity results for weak solutions to the nonlinear equations of the form

$$\Delta u = g(x, u), \qquad u \in W^{1,2}_{\rm loc}(\Omega). \tag{52}$$

The method is called *bootstrap*. First we need the following result due to Calderón and Zygmund.

Theorem 93 (Calderón–Zygmund) *If $u \in W^{1,2}_{\rm loc}(\Omega)$ solves $\Delta u = f$, where $f \in W^{m,p}_{\rm loc}(\Omega)$, $m = 0, 1, 2, \ldots$, $1 < p < \infty$, then $u \in W^{m+2,p}_{\rm loc}(\Omega)$.*

This is a very deep result and we will not prove it. □

Assume now that $g \in C^\infty(\Omega \times \mathbb{R})$. Let u be a solution to (52). Then $g(x, u) \in W^{1,2}_{\rm loc}$ and hecne by Theorem 93 $u \in W^{3,2}_{\rm loc}$. This implies in turn that $g(x, u) \in W^{3,2}_{\rm loc}$ and then $u \in W^{5,2}_{\rm loc} \ldots$ Itetating this argument yields $u \in W^{k,2}_{\rm loc}$ for any k. Hence by the Sobolev embedding theorem $u \in C^\infty$. Thus we have proved

Theorem 94 *If u is a solutioin to (52) with $g \in C^\infty(\Omega \times \mathbb{R})$, then $u \in C^\infty(\Omega)$.*

Remark. We employed here the special case of Therem 93 in which $p = 2$. The proof in this case is much easier than in the general one.

Assume now that g is measurable in x, continuous in u and that it satisfies the following growth condition

$$|g(x, u)| \le C(1 + |u|^q), \qquad 1 \le q < 2^* = \frac{n+2}{n-2}. \tag{53}$$

Theorem 95 *If u is a solution to (52) with the growth condition (53), then $u \in C^{1,\alpha}_{\rm loc}$ for any $0 < \alpha < 1$.*

Proof. Since $u \in W^{1,2}_{\rm loc}$, Sobolev embedding yields $u \in L^{2n/(n-2)}_{\rm loc}$ and hence $g(x, u) \in L^{p_1}_{\rm loc}$, where $p_1 = \frac{2n}{(n-2)q} > 1$. Now by Theorem 93 $u \in W^{2,p_1}_{\rm loc}$ and again by Sobolev embedding $g(x, u) \in L^{p_2}_{\rm loc}$, $p_2 = \frac{2n}{((n-2)q-4)q} > p_1$. The inequality $p_2 > p_1$ follows from the fact that $q < (n+2)/(n-1)$. Iterating this argument finitely many times yields $u \in W^{2,r}_{\rm loc}$ for any $r < \infty$. Hence by the Sobolev embedding theorem $u \in C^{1,\alpha}_{\text{loc}}$ for all $\alpha < 1$. The proof is complete. □

Eigenvalue problem. We start with considering the following two problems.

1. The best constant in the Poincaré inequality.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $1 < p < \infty$. Find the smallest constant μ such that the inequality

$$\int_\Omega |u|^p \leq \mu \int_\Omega |\nabla u|^p \tag{54}$$

holds for every $u \in W^{1,p}_0(\Omega)$.

Of course such a constant exists and is positive. This follows from the Poincaré inequality. The smallest constant is called the best constant in the Poincaré inequality and it will be denoted by μ_1 throhought the lecture.

The second problem looks much different.

2. The first eigenvalue of the p-Laplace operator.

We say that λ is an *eigenvalue* of the p-Laplace operator $-\text{div}\,(|\nabla u|^{p-2}\nabla u)$ on $W^{1,p}_0(\Omega)$, where $1 < p < \infty$, if there exists $0 \neq u \in W^{1,p}_0(\Omega)$ such that

$$-\text{div}\,(|\nabla u|^{p-2}\nabla u) = \lambda u|u|^{p-2}. \tag{55}$$

Such a function u is called *eigenfunction.* If $p = 2$, then we have the classical eigenvalue problem for the Laplace operator.

Observe that every eigenvalue is positive. Indeed, by the definition of the weak solution, (55) means that

$$\int_\Omega |\nabla u|^{p-2}\langle \nabla u, \nabla v\rangle = \lambda \int_\Omega |u|^{p-2}uv$$

for every $v \in C^\infty_0(\Omega)$ and hence for every $v \in W^{1,p}_0(\Omega)$. Taking $v = u$ we obtain

$$\int_\Omega |\nabla u|^p = \lambda \int_\Omega |u|^p\,. \tag{56}$$

Thus $\lambda > 0$. Note that inequality (56) implies that $\lambda \geq \mu_1^{-1}$. This gives the lower bound for the eigenvalues. The following theorem says much more.

Theorem 96 *There exists the smallest eigenvalue λ_1 of the problem (55) and it satisfies $\lambda_1 = \mu_1^{-1}$, where μ_1 is the best constant in the Poincaré inequality (54).*

The eigenvalue λ_1 is called *the first eigenvalue.*

Note that (55) is the Euler–Lagrange equation of the functional

$$I(u) = \frac{1}{p}\int_\Omega (|\nabla u|^p - \lambda |u|^p) \tag{57}$$

The similar situation was in Theorem 90, there are however two esential differences. First of all Theorem 90 guarantees the existence of a solution but it does not say anything about the properties of the solution. We already know that a solution of (55) exits — a function constant equal to zero. However we are not interested in that solution, we are looking for another one. In such a situation Theorem 90 cannot help. The second difference is that that functional defined by (57) need not be coercive and in fact in the most interesting case it is not.

Proof of Theorem 96. We know that $\lambda \geq \mu_1^{-1}$. It remains to prove that there exists $0 \neq u_0 \in W_0^{1,p}(\Omega)$ such that

$$-\mathrm{div}\,(|\nabla u|^{p-2}\nabla u) = \mu_1^{-1} u|u|^{p-2}. \tag{58}$$

First note that $u_0 \in W_0^{1,p}(\Omega)$ satisfies (58) if and only it satisfies

$$\int_\Omega |u|^p = \mu_1 \int_\Omega |\nabla u|^p. \tag{59}$$

Assume that u_0 satisfies (58). Integrating both sides of (58) against the test function u_0 we obtain (59). In the oposite direction, if u_0 satisfies (59), then u_0 is a minimizer of the functional

$$E(u) = \frac{1}{p}\int_\Omega \left(|\nabla u|^p - \mu_1^{-1}|u|^p\right).$$

Indeed, $E(u_0) = 0$ and always $E(u) \geq 0$ because of the Poincaré inequality. Hence u_0 satisfies Euler–Lagrange equations (58).

Thus it remains to prove that there exists a nontrivial minimizer $0 \neq u_0 \in W_0^{1,p}(\Omega)$ of the functinal E.

Let M consint of all $u \in W_0^{1,p}(\Omega)$ such that $\int_\Omega |u|^p = 1$. Minimize the functional $I_p(u) = \int_\Omega |\nabla u|^p$ over M. Let $u_k \in M$, $I_p(u_k) \to \inf_M I_p = \mu_1^{-1}$. By reflexivity of the space $W_0^{1,p}(\Omega)$ we can select a weakly convergent subsequence $u_k \rightharpoonup u_0 \in W_0^{1,p}(\Omega)$. Then $\int_\Omega |\nabla u_0|^p \leq \mu_1^{-1}$. Since the imbedding $W_0^{1,p}(\Omega) \subset L^p(\Omega)$ is compact we conclude that $u_0 \in M$. Hence $E(u_0) = 0$ and thus u_0 is the desired minimizer. This completes the proof. □

Observe that the functional $E(u)$ is not coercive. Indeed, if u_0 is as in the above proof then $E(tu_0) = 0$, while $\|tu_0\|_{1,p} \to \infty$. This also shows that condition (48) in Theorem 90 are optimal: here $g(x,u) = -\mu_1^{-1}u|u|^{p-2}$ satisfies (45) and (48) with $\gamma = -\mu_1^{-1}$ which is not allowed.

Observe also that the functional E has infinitely many minimizers — each of the functions tu_0.

There are many open problems concerning the eigenvalues of the problem (55). In the linear case $p = 2$ the eigenvalues form an infinite discrete set $0 < \lambda_1 < \lambda_2 < \ldots$, $\lambda_i \to \infty$. No such result is known for $p \neq 2$. It is easy to show that the set of eigenvalues of (55) is closed. Moreover one can prove that the set of eigenvalues is infinite, unbounded and that the first eigenvalue is isolated.

We conclude the discussion about the eigenfunctions with the following theorem that will be used in the next lecture.

Theorem 97 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $1 < p < \infty$. Then each eigenfunction is continuous and bounded. Moreover the eigenfunction corresponding to the first eigenvalue is either positive or negative everywhere is Ω.*

We will not prove it. $\square$

Lecture 8

General variational integrals. Consider the functional of the form

$$I(u) = \int_\Omega F(x, \nabla u),$$

where $\Omega \subset \mathbb{R}^n$ is an open set, and $F : \Omega \times \mathbb{R}^n \to \mathbb{R}$ is measurable in x for all $\xi \in \mathbb{R}^n$ and C^1 and strictly conves in $\xi \in \mathbb{R}^n$ for almost all $x \in \Omega$. Assume in addition that F satisfies the following growth condition

$$\alpha|\xi|^p \leq F(x, \xi) \leq \beta|\xi|^p,$$

for almost all $x \in \Omega$ and all $\xi \in \mathbb{R}^n$, where $\alpha, \beta > 0$, $1 < p < \infty$ are fixed.

Under the above assumptions I is defined and finite on $W^{1,p}(\Omega)$. Moreover I is strictly conves. Hence for every $w \in W^{1,p}(\Omega)$ there is the unique $\overline{u} \in W^{1,p}_w(\Omega)$ such that

$$I(\overline{u}) = \inf_{u \in W^{1,p}_w(\Omega)} I(u). \tag{60}$$

Functional I is Gateaux differentiable on $W^{1,p}(\Omega)$. We compute its differential

$$\langle DI(u), \varphi \rangle = \frac{d}{dt}|_{t=0} \int_\Omega F(x, \nabla u + t\nabla\varphi)\, dx = \int_\Omega \nabla_\xi F(x, \nabla u) \cdot \nabla\varphi\, dx\,.$$

Of course taking the differentiation under the sigen of the integral requires a proof, but it is not very difficule and we leave it to the reader. Since I is strictly convex, we get that u is a minimizer of I if and only if $DI(\overline{u}) = 0$ i.e. if and only if $\overline{u}$ is a solution to the following problem

$$\nabla_\xi F(x, \nabla u) = 0\,, \qquad u - w \in W^{1,p}_0(\Omega).$$

Denote $A(x, \xi) = \nabla_\xi F(x, \xi)$. It is not difficult to prove that

1. $A(x,\xi)$ is a Carathéodory function i.e. it is measurable in x and continuous in ξ,
2. $|A(x,\xi)| \le b|\xi|^{p-1}$, $A(x,\xi)\cdot\xi \ge a|\xi|^p$, for some $a,b>0$,
3. $(A(x,\xi)-A(x,\eta))(\xi-\eta)>0$ for all $\xi\neq\eta$ a.e.

Thus we have proved the following result.

Theorem 98 *Let $F:\Omega\times\mathbb{R}^n\to\mathbb{R}$ be a function satisfying the conditions as above. Given arbitrary $w\in W^{1,p}(\Omega)$, the functional*

$$I(u)=\int_\Omega F(x,\nabla u)\,,\qquad u\in W^{1,p}(\Omega) \tag{61}$$

has the unique minimizer in the class $W^{1,p}_w(\Omega)$ Moreover u is a minimizer of (61) in $W^{1,p}_w(\Omega)$ is and only if u is a solution to the following problem

$$\operatorname{div} A(x,\nabla u)=0\,,\qquad u-w\in W^{1,p}_0(\Omega)\,,$$

where $A(x,\xi)=\nabla_\xi F(x,\xi)$ satisfiers the above conditions 1., 2., 3.

The result easily generalizes to the case of functionals defined on $u\in W^{1,p}(\Omega,\mathbb{R}^m)$ i.e., when $F:\Omega\times\mathbb{R}^{m\times n}\to\mathbb{R}$. We leave details to the reader.

We give not two important examples where similar functionals appear in the applications of the calculus of variations.

Conformal mappings. Let $\Omega\subset\mathbb{R}^n$ be an open set. A diffeomorphism $\varphi:\Omega\to\mathbb{R}^n$ of the class C^1 is called conformal if it preserves the angles in Ω i.e. if the linear transformation $D\varphi(x):\mathbb{R}^n\to\mathbb{R}^n$ preserves the angles for all $x\in\Omega$ i.e., if the columns of the matrix $D\varphi(x)$ (as images of the orthonormal basis of $\mathbb{R}^n$) are arthogonal and of equal length, say $\lambda(x)>0$, for all $x\in\Omega$ i.e. if

$$(D\varphi(x))^T D\varphi(x)=\lambda(x)^2 I \qquad \text{for all } x\in\Omega.$$

It is easy to see that $\lambda(x)=|J_\varphi(x)|^{1/n}$ and hence we get

Lemma 99 *A C^1 diffeomorphism $\varphi:\Omega\to\mathbb{R}^n$ is conformal if and only if*

$$(D\varphi(x))^T D\varphi(x)=|J_\varphi(x)|^{2/n} I\,,$$

for all $x\in\Omega$.

In what follows we will assume that conformal mappings preserve the orientation i.e. $J_\varphi>0$ in Ω.

The following result is a celebrated theorem of Liouville (1850).

Theorem 100 (Liouville) *If $n \geq 3$, then every conformal mapping $\varphi : \Omega \to \mathbb{R}^n$ of the class C^3 is a composition of a finite number of reflections in planes and in spheres. By the reflection in the sphere $S^{n-1}(a,r)$ we mean the mapping*

$$\psi(x) = \frac{r^2(x-a)}{|x-a|^2} + a.$$

The theorem does not hold when $n = 2$. Indeed, any holomorphic diffeomorphism is conformal.

It was very essential in the Liouville's proof that φ is at least of the class C^3. It took then about 100 years until Hartman has proved

Theorem 101 (Hartman) *Liouville theorem is still true if we assume that $\varphi \in C^1$.*

Then Gehring and Reshetnyak replaced the requirement $\varphi \in C^1$ by a weaker one: φ is 1-quasiconformal. The notion of the quasiconformality will be explained in one of the further lectures.

We will sketch the proof of Hartman's theorem using heavily the methods of calculus of variations, nonlinear elliptic P.D.E. and Sobolev spaces.

Now we show a connection between conformal mappings and calculus of variations. For $f \in W^{1,n}(\Omega, \mathbb{R}^n)$ we set

$$I_n(f) = \int_\Omega |\nabla f|^n$$

As we have already seen, if f is a minimizer of I_n in the class $W^{1,n}_w(\Omega, \mathbb{R}^n)$, where $w \in W^{1,n}(\Omega, \mathbb{R}^n)$, then f solves the following n-harmonic system

$$\operatorname{div}\,(|\nabla f|^{n-2}\nabla f_i) = 0\,, \qquad i = 1, 2, \ldots, n.$$

Proposition 102 *Let $\Omega \subset \mathbb{R}^n$ be bounded and let $\varphi \in C^1(\overline{\Omega}, \mathbb{R}^n)$ be conformal. Then*

$$I_n(\varphi) = \inf_{\psi \in W^{1,n}_\varphi(\Omega, \mathbb{R}^n)} I_n(\psi),$$

and hence components of, φ are solutions to the n-harmonic system.

We do not prove the theorem. □

Minimal surfaces. Another interesting example of the variational functional arise in he theory of minimal surfaces. Assume that

Lecture 9

The aim of this lecture is to develope the theory of Sobolev spaces on domains with the irregular boundary.

Sobolev spaces on John domains. We say that a bounded domain $\Omega \subset \mathbb{R}^n$ is a *John domain* if there is a constant $C_J \geq 1$ and a distinguished point $x_0 \in \Omega$ so that each point $x \in \Omega$ can be joined to x_0 (inside Ω) by a curve (called John curve) $\gamma : [0,1] \to \Omega$ such that $\gamma(0) = x$, $\gamma(1) = x_0$ and

$$\mathrm{dist}\,(\gamma(t), \partial\Omega) \geq C_J^{-1}|x - \gamma(t)|\,,$$

for every $t \in [0,1]$.

Lemma 103 *Every bounded Lipschitz domain is John.* □

One can easily construct John domains with the fractal boundary of the Hausdorff dimension strictly greater than $n-1$ (for example two-dimensional von Koch snowflake domain). Thus the class of John domains is much larger than the class of Lipschitz domains.

The bove definition is slightly different than the usual definition of the John domain, but it is equivalent.

The following lemma follows from the results of Lecture 2. Since we present different approach to Sobolev inequalities we prove it one more time using different argument.

Lemma 104 *Let $u \in C^1(B)$, where $B \subset \mathbb{R}^n$ is a ball of radius r. Then for $1 \leq p < \infty$*

$$\left(⨍_B |u - u_B|^p\, dx\right)^{1/p} \leq C(n,p) r \left(⨍_B |\nabla u|^p\, dx\right)^{1/p}.$$

Proof. We can assume that B is centred at the origin. For $x, y \in B$ we have

$$\begin{aligned} |u(y) - u(x)| &= \left|\int_0^1 \frac{d}{dt} u(x + t(y-x))\, dt\right| = \left|\int_0^1 \langle \nabla u(x + t(y-x)), y - x\rangle\, dt\right| \\ &\leq 2r \int_0^1 |\nabla u(x + t(y-x))|\, dt. \end{aligned}$$

Hence integrating with respect to y and then applying Hölder's inequality yield

$$\begin{aligned} |u(x) - u_B| &\leq Cr \int_0^1 ⨍_B |\nabla u(x + t(y-x))|\, dy\, dt \\ &\leq Cr \left(\int_0^1 ⨍_B |\nabla u(x + t(y-x))|^p\, dy\, dt\right)^{1/p}. \end{aligned}$$

Now

$$\int_B |u - u_B|^p \leq C \frac{r^p}{|B|} \int_0^1 \int_B \int_B |\nabla u(x + t(y-x))|^p\, dy\, dx\, dt.$$

Changing variables $(x,y)\in B\times B$ to $(\xi,\eta)\in B\times 2B$ by the formula

$$\xi = x+t(y-x), \qquad \eta = y-x,$$

we easily see that the Jacobian of the transformation is one and hence

$$\int_B |u-u_B|^p \leq C\frac{r^p}{|B|}\int_0^1\int_{2B}\int_B |\nabla u(\xi)|^p\, d\xi\, d\eta\, dt = Cr\int_B |\nabla u(\xi)|^p\, d\xi.$$

□

The following is a generalizetion of Lemma 25.

Theorem 105 *Let* $\Omega\subset \mathbb{R}^n$ *be a John domain. Then for every* $u\in C^1(\Omega)$ *and all* $x\in\Omega$

$$|u(x)-u_\Omega| \leq C(C_J,n)\int_\Omega \frac{|\nabla u(z)|}{|x-z|^{n-1}}\, dz.$$

Proof. Let $x_0\in\Omega$ be a central point. Let $B_0 = B(x_0, \operatorname{dist}(x_0,\partial\Omega)/4)$. We will prove that there is a constant $M=M(C_J,n)>0$ such that to every $x\in\Omega$ there is a sequence of balls (chain) $B_i = B(x_i,r_i)\subset\Omega$, $i=1,2,\ldots$ such that

1. $|B_i\cup B_{i+1}|\leq M|B_i\cap B_{i+1}|$, $i=0,1,2,\ldots$
2. $\operatorname{dist}(x,B_i)\leq Mr_i$, $r_i\to 0$, $x_i\to x$ as $i\to\infty$,
3. No point of Ω belongs to more than M balls B_i.

To prove it assume first that x is far enough from x_0, say $x\in\Omega\setminus 2B_0$. let γ be a John curve that joins x with x_0. We construct a chain of balls as follows.

All balls in the chain are centred on γ. Ball B_0 is already defined. Assume that balls $B_0,\ldots,B_i$ are defined. Starting from the center x_i of B_i we trace along γ toward x untill we leave B_i for the last time. Denote by x_{i+1} the point on γ when it happends and define $B_{i+1} = B(x_{i+1}, |x-x_{i+1}|/4C_J)$.

The property 1. and the inequality $\operatorname{dist}(x,B_i)\leq Cr_i$ in 2. follows form the fact that consecutive balls have comparable radii and that the radii are comparable to the distance of centers to x.

To prove 3. suppose that $y\in B_{i_1}\cap\ldots\cap B_{i_k}$. Observe that the radii of the balls B_{i_j}, $j=1,2,\ldots,k$ are comparable to $|x-y|$. Hence it follows from the construction that distances between centers of the balls B_{i_j} are comparable to $|x-y|$. The number of the points in $\mathbb{R}^n$ with pairwise comparable distances is bounded i.e. if $z_1,\ldots,z_N\in\mathbb{R}^n$ satisfy $c^{-1}r<\operatorname{dist}(z_i,z_j)<cr$ for $i\neq j$, then $N\leq C(c,n)$. Hence k is bounded by a constant depending on n and C_J, so 3. follows.

Now 3. easily implies that $r_i \to 0$ and hence $x_i \to x$ as $i \to \infty$, which completes the proof of 2.

The case $x \in 2B_0$ is easy and we leave it to the reader.

Since $u_{B_i} = \fint_{B_i} u \to u(x)$ as $i \to \infty$ we get

$$
\begin{aligned}
|u(x) - u_{B_0}| &\leq \sum_{i=0}^{\infty} |u_{B_i} - u_{B_{i+1}}| \\
&\leq \sum_{i=0}^{\infty} |u_{B_i} - u_{B_i \cap B_{i+1}}| + |u_{B_{i+1}} - u_{B_i \cap B_{i+1}}| \\
&\leq \sum_{i=0}^{\infty} \frac{|B_i|}{|B_i \cap B_{i+1}|} \fint_{B_i} |u - u_{B_i}| + \frac{|B_{i+1}|}{|B_i \cap B_{i+1}|} \fint_{B_{i+1}} |u - u_{B_{i+1}}| \\
&\quad \text{(propery 1.)} \\
&\leq C \sum_{i=0}^{\infty} \fint_{B_i} |u - u_{B_i}| \\
&\quad \text{(Lemma 104)} \\
&\leq C \sum_{i=0}^{\infty} \int_{B_i} \frac{|\nabla u(z)|}{r_i^{n-1}}\, dz.
\end{aligned}
$$

Observe that 2. implies that for $z \in B_i$ there is $|x - z| \leq C r_i$ and hence $1/r_i^{n-1} \leq C/|x-z|^{n-1}$. Thus

$$
|u(x) - u_{B_0}| \leq C \sum_{i=0}^{\infty} \int_{B_i} \frac{|\nabla u(z)|}{|x - z|^{n-1}}\, dz \leq C \int_{\Omega} \frac{|\nabla u(z)|}{|x-z|^{n-1}}\, dz. \tag{62}
$$

The last inequality follows form 3. Now it is easy to complete the proof. Since

$$
|u(x) - u_{\Omega}| \leq |u(x) - u_{B_0}| + |u_{B_0} - u_{\Omega}|, \tag{63}
$$

it remainds to prove the estimate for $|u_{B_0} - u_{\Omega}|$. We have

$$
|u_{B_0} - u_{\Omega}| \leq \fint_{\Omega} |u - u_{B_0}| \leq C \int_{\Omega} \fint_{\Omega} \frac{|\nabla u(z)|}{|x-z|^{n-1}}\, dx\, dz \leq C|\Omega|^{-(n-1)/n} \int_{\Omega} |\nabla u(z)|\, dz\,. \tag{64}
$$

The last inequality follows form the following standard argument

$$
\fint_{\Omega} \frac{dx}{|x-z|^{n-1}} \leq |\Omega|^{-1} \int_{B} \frac{dx}{|x-z|^{n-1}} = C|\Omega|^{-(n-1)/n}, \tag{65}
$$

where B is a ball centred at z with $|B| = |\Omega|$. By the John condition we have

$$
C|\Omega|^{1/n} \geq \operatorname{dist}(x_0, \partial\Omega) \geq C_J^{-1}|x - x_0|.
$$

Taking the supremum over $x \in \Omega$ yields

$$
\operatorname{diam}\Omega \leq C(n, C_J)|\Omega|^{1/n},
$$

and hence

$$|\Omega|^{-(n-1)/n} \le \frac{C}{|x-z|^{n-1}},$$

for all $z \in \Omega$. This and (64) gives

$$|u_{B_0} - u_\Omega| \le C \int_\Omega \frac{|\nabla u(z)|}{|x-z|^{n-1}}\, dz. \tag{66}$$

Now the theorem follows form the estimates (63), (62) and (66). The proof is complete. □

Observe that te above result easily implies the Poincaré inequality

Theorem 106 (Poincaré inequality) *Let $\Omega \subset \mathbb{R}^n$ be a John domain and $u \in C^1(\Omega)$. Then for $1 \le p < \infty$*

$$\left(\fint_\Omega |u-u_\Omega|^p \right)^{1/p} \le C(n,p,C_J)|\Omega|^{1/n} \left(\fint_\Omega |\nabla u|^p \right)^{1/p}.$$

Proof. It is a direct consequence of the above theorem and Lemma 27. □

Sobolev embedding theorem. The aim of this section is the proof of the following theorem.

Theorem 107 *If $\Omega \subset \mathbb{R}^n$ is a John domain, and $1 \le p < n$, then*

$$\left(\fint_\Omega |u-u_\Omega|^{p^*} \right)^{1/p^*} \le C(C_J,n,p)|\Omega|^{1/n} \left(\fint_\Omega |\nabla u|^p \right)^{1/p},$$

for $u \in W^{1,p}(\Omega)$, where $p^* = np/(n-p)$.

In the proof we will need the following lemma whose proof goes back to Santalo. I learned to form Jan Malý.

Lemma 108 *Let $\Omega \subset \mathbb{R}^n$ be open and $g \in L^1(\Omega)$. Then*

$$\sup_{t>0} |\{x \in \mathbb{R}^n :\, I_1^\Omega g(x) > t\}| t^{n/(n-1)} \le C \left(\int_\Omega |g| \right)^{n/(n-1)}.$$

Proof. Replacing g by g/t we may assume that $t=1$. let $E = \{I_1^\Omega g > 1\}$. Then

$$|E| \le \int_E I_1^\Omega g = \int_\Omega \int_E \frac{dx}{|x-z|^{n-1}}\, g(z)\, dz \le C|E|^{1/n} \int_\Omega |g|.$$

The last inequality follows from a standard trick as in (65). The proof is complete. □

Lemma 108 and Lemma 105 imply

Lemma 109 *If $\Omega \subset \mathbb{R}^n$ is a John domain and $u \in W^{1,1}(\Omega)$, then*

$$\sup_{t>0} |\{x \in \Omega : \, |u(x) - u_\Omega| > t\}| t^{n/(n-1)} \leq C(C_J, n) \left(\int_\Omega |\nabla u| \right)^{n/(n-1)} .$$

Inequalities stated in the above two lemmas might seem strange, so we explain their meaning now.

Since our comments will be of a general nature, we assume for a while that we deal with functions defined on a space equipped with a σ-finite measure μ.

If $u \in L^p(X)$, $1 \leq p < \infty$, then it follows from Chebyschev's inequality that

$$\sup_{t>0} \mu(\{x \in X : \, |u(x)| > t\}) t^p \leq \int_X |u|^p \, d\mu.$$

This suggests the following definition.

We say that a measurable function u belongs to the *Marcinkiewicz space* $L^p_w(X)$ (weak L^p) if there is $m \geq 0$ such that

$$\sup_{t>0} \mu(\{x \in X : \, |u(x)| > t\}) t^p \leq m. \tag{67}$$

hence $L^p(X) \subset L^p_w(X)$. In general, the space L^p_w is larger that L^p. Indeed, $x^{-1} \in L^1_w(\mathbb{R}) \setminus L^1(\mathbb{R})$. However we have the following

Lemma 110 *If $\mu(X) < \infty$ then $L^p_w(X) \subset L^q(X)$ for all $0 < q < p$. Moreover if u satisfies (67) then*

$$\|u\|_{L^q(X)} \leq 2^{1/q} \left(\frac{qm}{p-q} \right)^{1/p} \mu(X)^{1/q - 1/p}. \tag{68}$$

Proof. In the proof we need the following

Lemma 111 (Cavalieri pronciple) *If $q > 0$ and u is measurable, then*

$$\int_X |u|^q \, d\mu = q \int_0^\infty t^{q-1} \mu(|u| > t) \, dt.$$

Proof. Apply Fubini's theorem to $X \times [0, \infty)$. □

Now fix $t_0 > 0$. We use estimates $\mu(|u| > t) \leq \mu(X)$ for $t \leq t_0$ and $\mu(|u| > t) \leq mt^{-p}$ for $t > t_0$. We get

$$\int_X |u|^q \, d\mu \leq q \left(\int_0^{t_0} t^{q-1} \mu(X) \, dt + m \int_{t_0}^\infty t^{q-p-1} \, dt \right) = t_0^q \mu(X) + \frac{qm}{p-q} t_0^{q-p}.$$

Now the inequality (68) follows by substitution $t_0 = (qm/(p-q))^{1/p} \mu(X)^{-1/p}$. □

Hence Lemma 109 implies only that for every $q < n/(n-1)$

$$\left(\fint_\Omega |u - u_\Omega|^q \, dx \right)^{1/q} \leq C|\Omega|^{1/n} \fint_\Omega |\nabla u| \, . \tag{69}$$

Thus it is surprising that one can actually take $q = n/(n-1)$ in (69) as Theorem 107 says.

First we will prove the following modified version of the theorem

Lemma 112 *Let $\Omega \subset \mathbb{R}^n$ be a John domain and let $u \in W^{1,p}(\Omega)$, $1 \leq p < n$. If $|\{u = 0\}| \geq |\Omega|/2$, then*

$$\left(\int_\Omega |u|^{p^*} \right)^{1/p^*} \leq C(C_J, n, p) \left(\int_\Omega |\nabla u|^p \right)^{1/p} .$$

em Proof. For a function v and $0 < t_1 < t_2 < \infty$ we define the truncation between levels t_1 and t_2 as

$$v_{t_1}^{t_2} = \min\{\max\{0, v - t_1\}, t_2 - t_1\}\} \, .$$

Then obviously $v_{t_1}^{t_2} \in W^{1,p}$ if $v \in W^{1,p}$ and

$$\nabla v_{t_1}^{t_2} = \nabla u \chi_{\{t_1 < v \leq t_2\}} \text{ a.e.}$$

Assume first that $p = 1$. We have

$$\begin{aligned} \int_\Omega |u|^{n/(n-1)} &\leq \sum_{k=-\infty}^{\infty} 2^{kn/(n-1)} |\{2^{k-1} < |u| \leq 2^k\}| \\ &\leq \sum_{k=-\infty}^{\infty} 2^{kn/(n-1)} |\{|u| geq 2^{k-1}\}| \\ &\leq \sum_{k=-\infty}^{\infty} 2^{kn/(n-1)} |\{|u|_{2^{k-2}}^{2^{k-1}} \geq 2^{k-2}\}| \\ &= \diamondsuit \end{aligned} \tag{70}$$

Lemma 113 *Let $\mu(X) < \infty$. If w is a measurable function such that $\mu(\{w = 0\}) \geq \mu(X)/2$, then for every $t > 0$*

$$\mu(\{|w| \geq t\}) \leq 2\mu(\{|w - w_X| \geq t/2\}) \, .$$

We leave the proof as an exercise. $\square$

In our situation $w^k = |u|_{2^{k-2}}^{2^{k-1}}$ is nonnegative and vanishes on at least on half of Ω. Hence applying the lemma to (70) and then applying Lemma 109 we get

$$
\begin{aligned}
\diamondsuit &\leq \sum_{k=-\infty}^{\infty} 2^{kn/(n-1)} \left(2^{k-3}\right)^{-n/(n-1)} 2\left|\left\{|w^k - w^k_\Omega| \geq 2^{k-3}\right\}\right| \left(2^{k-3}\right)^{n/(n-1)} \\
&\leq C \sum_{k=-\infty}^{\infty} \left(\int_\Omega |\nabla w^k|\right)^{n/(n-1)} \\
&= C \sum_{k=-\infty}^{\infty} \left(\int_{\{2^{k-2}<|u|\leq 2^{k-1}\}} |\nabla u|\right)^{n/(n-1)} \\
&\leq C \left(\int_\Omega |\nabla u|\right)^{n/(n-1)}.
\end{aligned}
$$

Hence Lemma 112 follows when $p = 1$. Now assume that $1 < p < n$. Let $v = |u|^{p(n-1)/(n-p)}$. Then $|v|^{n/(n-1)} = |u|^{p^*}$ and hence applying the case $p = 1$ of the lemma yields

$$\int_\Omega |u|^{p^*} = \int_\Omega |v|^{n/(n-1)} \leq C \left(\int_\Omega |\nabla v|\right)^{n/(n-1)}. \tag{71}$$

Observe that

$$|\nabla v| = \frac{p(n-1)}{n-p} |u|^{n(p-1)/(n-p)} |\nabla u|.$$

Placing it in (71) and applying Hölder's inequality yields the desired inequality. □

Proof of Theorem 107. Choose $b \in \mathbb{R}$ such that

$$|\{u \geq b\}| \geq |\Omega|/2 \qquad \text{and} \qquad |\{u \leq b\}| \geq |\Omega|/2.$$

Then functions $v_+ = \max\{u - b, 0\}$ and $v_- = \min\{u - b, 0\}$ satisfy the assumptions of Lemma 112 and hence

$$\left(\int_\Omega |u-b|^{p^*}\right)^{1/p^*} \leq C \left(\int_\Omega |\nabla u|^p\right)^{1/p}.$$

Now the theorem follows from the following obsevation

Lemma 114 *If $\mu(X) < \infty$ and $q \geq 1$, then*

$$\inf_{c\in\mathbb{R}} \left(\int_X |u-c|^q \, d\mu\right)^{1/q} \leq \left(\int_X |u - u_X|^q \, d\mu\right)^{1/q} \leq 2 \int_{c\in\mathbb{R}} \left(\int_X |u-c|^q \, d\mu\right)^{1/q}.$$

Proof. Apply Hölder's inequality. □

The proof of the theorem is complete □

Generalized Rellich–Kondrachov theorem. Our next aim is a generalization of the Rellich–Kondrachov theorem to almost arbitrary domain.

General extension theorem.

Lecture 10

Beyond classical Sobolev spaces. As we have seen, the classical theory of Sobolev spaces has numerous applications to calculus of variations, linear and nonlinear partial differential equations and geometry. There are still another applications like those to algebraic topology that we did not discuss here because of the lack of the time.

Although the language of classical Sobolev spaces is quite universal it is sometimes not general enough. We will show now few examples that will motivate generalization of the Sobolev space to the setting of metric spaces that will be given later on.

Graphs. Let $\Gamma = (V, E)$ be a graph, where V is the vertex set and E the set of edges. We say that $x, y \in V$ are neighbours if they are joined by an edge; we denote this by $x \sim y$. Assume that the graph is connected in the sense that any two vertices can be connected by a sequence of neighbours. We let the distance between two neighbours to be 1. This induces a geodesic metric on V denoted by ϱ. The graph is endowed with the counting measure. The measure of a set $E \subset V$ is simply the number $V(E)$ of elements of E. For a ball $B = B(x, r)$ we use also the notation $V(B) = V(x, r)$. The length of the gradient of a function u on V at a point x is defined by

$$|\nabla_\Gamma u|(x) = \sum_{y \sim x} |u(y) - u(x)|.$$

Many graphs have the following two properties

1. The counting measure is doubling i.e., there is a cnstant $C_d \geq 1$ such that

$$V(x, 2r) \leq C_d V(x, r), \tag{72}$$

   for every $x \in V$ and $r > 0$.

2. The following inequality holds

$$\frac{1}{V(B)} \sum_{x \in B} |u(x) - u_B| \leq Cr \left(\frac{1}{V(\sigma B)} \sum_{x \in \sigma B} |\nabla_\Gamma u|^p(x) \right)^{1/p}, \tag{73}$$

   where $C > 0$, $\sigma \geq 1$ are fixed constants and B is any ball.

Observe that inequality (73) is similar to the Poincaré inequality. Although one might think the more natural counterpart of the Poincaré inequality is

$$\left(\frac{1}{V(B)} \sum_{x \in B} |u(x) - u_B|^p \right)^{1/p} \leq Cr \left(\frac{1}{V(B)} \sum_{x \in B} |\nabla_\Gamma u|^p \right)^{1/p}, \tag{74}$$

inequality (73) seems weaker and hence it is more convenient to require (73). Actually one can prove that inequalities (73) and (74) are equivalent. We will cone back to this question later, in a more general setting.

We may ask whether a Sobolev–Poincaré inequality holds as well. This would be an inquality of the form

$$\left(\frac{1}{V(B)}\sum_{x\in B}|u(x)-u_B|^q\right)^{1/q} \leq Cr\left(\frac{1}{V(\sigma B)}\sum_{x\in\sigma B}|\nabla_G u|^p(x)\right)^{1/p}, \tag{75}$$

with some $q > p \geq 1$. If such inequality is true, then how to determine the best possible exponent q? Can we replace σB on the right hand side by B? What would be a counterpart of the Rellich–Kondrachov theorem?

One can define the notion of the harmonic function on the graph. There are two possible approaches. The first one is based on the so called mean value property:

We say that $u : V \to \mathbb{R}$ is harmonic if for every $x \in V$,

$$u(x) = \frac{1}{d(x)}\sum_{y\sim x} u(y), \tag{76}$$

where $d(x)$ is a number of neighbours of x.

The other approach is based on a version of the Dirichlet principle: $u : V \to \mathbb{R}$ is harmonic if it locally minimizes the Dirichlet sum

$$E(u) = \sum_{x\in V}|\nabla_\Gamma u|^2(x). \tag{77}$$

This is to say that if $v : V \to \mathbb{R}$ is a function which coincides with u outside a finite set $Y \subset X$, then

$$\sum_{x\in Y}|\nabla_\Gamma v|^2 \geq \sum_{x\in Y}|\nabla_\Gamma u|^2.$$

Observe that according to this definition, if u locally minimizes the Dirichlet sum, then the series (77) need not converge. It is not difficult to prove that the above two definitions of a harmonic function are equivalent. The mean value property is then a version of the Euler–Lagrange equation for the energy functional (77). We leave it as an exercise.

There are many questions that arise right away. Can the above mentioned Poincaré or Sobolev-Poincaré type inequalties — if true — be used to prove some properties of harmonic functions on graphs like for example Harnack inequality?

A sample of such result is the following theorem due to Holopainen and Soardi.

Theorem 115 (Harnack inequality) *Suppose that the graph* Γ *satisfies both, the doubling property (72) and the Poincaré inequality (73) with* $p = 2$. *Then there is a constant* $C \geq 1$ *such that*

$$\max_{x \in B} u(x) \leq C \min_{x \in B} u(x),$$

whenever u *is a positive harmonic function in* $12B$.

The doubling property along with the Poincaré inequality are really crucial for the proof of the Harnack inequality.

Corollary 116 (Liouville Theorem) *Under the above assumptions about the graph any bounded harmonic function is constant.*

Proof. Replacing u by $\sup_V u - u$ we may assume that $\inf_V u = 0$. Hence Harnack inequality implies that $\sup_V u = 0$, and thus u is constant. $\Box$

The proof of the Harnack inequality involves a Sobolev–Poincaré inequality (75) with $q > p = 2$. Thus in particular one has to show that Poincaré inequality (73) for $p = 2$ together with (72) implies such Sobolev–Poincaré inequality. We will come back to the question later.

Although there is an obvious analogy between inequalities (73), (74), (75) and the classical Poincaré and Sobolev–Poincaré inequalities, they cannot be deduced form the classical results.

A very important class of graphs is given by finite generated groups. Let G be a group with generators $\{g_1, \ldots, g_n\}$. This is to say every element of G can be represented as $g_{i_1}^{\pm 1} g_{i_2}^{\pm 1} \cdots g_{i_k}^{\pm 1}$. We associate a graph with G as follows. We identify the vertex set V with the set of all elements of the group G. Then we connect two vertices $x_1, x_2 \in V$ by an edge if and only if $y = g_i^{\pm 1} x$ for some generator g_i. Inequalities of Sobolev and Poincaré type play the essential role in the analysis and geometry of finite generated groups.

Upper gradient. Now we turn to another class of examples The gradinet of a Lipschitz function $u \in \mathrm{Lip}\,(\Omega)$ satisfies the inequality

$$|u(\gamma(b)) - u(\gamma(a))| \leq \int_a^b |\nabla u(\gamma(t))|\, dt,$$

whenever $\gamma : [a, b] \to \Omega$ is 1-Lipschitz curve i.e. $|\gamma(t) - \gamma(s)| \leq |t - s|$ for all $a \leq s \leq t \leq b$, or equivalently γ is absolutely continuous with $|\dot{\gamma}| \leq 1$ a.e.

Moreover $|\nabla u|$ is the smallest among all locally integrable functions such that

$$|u(\gamma(b)) - u(\gamma(a))| \leq \int_a^b g(\gamma(t))\, dt, \qquad \text{for all 1-Lipschtz curves } \gamma. \tag{78}$$

Namely we have

Lemma 117 *If u is a Lipschitz function defined on an open set $\Omega \subset \mathbb{R}^n$, then any measurable function g such that $g \geq |\nabla u|$ everywhere satisfies (78). On the other hand if $g \in L^p(\Omega)$ and $u \in L^p(\Omega)$ satisfy (78), then $u \in W^{1,p}(\Omega)$ and $g \geq |\nabla u|$ a.e.*

Proof. The first part is obvious. The second part follows from the ACL characterization of the Sobolev space that was discussed during the second lecture. □

The above lemma motivates the following generalization of the notion of gradient to the setting of metric spaces.

Let (X, d, μ) be a metric space (X, d) equipped with a Borel measure μ. *Till the end of the lectures* we will assume that the measure μ of any ball is strictly positive and finite.

Following Heinonen and Koskela we say that a Borel function $g : X \to [0, \infty]$ is an *upper gradinet* of another Borel function $u : X \to \mathbb{R}$ if for every 1-Lipschitz curve $\gamma : [a, b] \to X$ there is

$$|u(\gamma(b)) - u(\gamma(a))| \leq \int_a^b g(\gamma(t))\, dt. \tag{79}$$

Note that $g \equiv \infty$ is an upper gradient of any Borel function u. A more clever example is provided by the following result.

Lemma 118 *If u is a locally Lipschitz function on X, then $|\nabla^+ u|(x) = \limsup_{y\to x} |u(y) - u(x)|/d(x, y)$ is an upper gradient of u.*

Proof. Let $\gamma : [a, b] \to \Omega$ be 1-Lipschitz. The function $u \circ \gamma$ is Lipschitz and hence differentiable a.e. It easily follows that $|(u \circ \gamma)'(t)| \leq |\nabla^+ u(\gamma(t))|$ whenever $u \circ \gamma$ is differentiable at t. Hence

$$|u(\gamma(b)) - u(\gamma(a))| \leq \int_a^b |(u \circ \gamma)'(t)|\, dt \leq \int_a^b |\nabla^+ u(\gamma(t))|\, dt.$$

The proof is complete. □

We may ask then whether on a given space (X, d, μ) a Poincaré type inequality holds between u and its upper gradient.

We say that the space (X, d, μ) supports p-Poincaré inequality, $1 \leq p < \infty$ if there exists $C > 0$ and $\sigma \geq 1$ such that every pair (u, g), of a continuous function u and its upper gradient g satisfies the following Poincaré type inequality

$$\fint_B |u - u_B|\, d\mu \leq Cr \left(\fint_{\sigma B} g^p\, d\mu \right)^{1/p} \qquad \text{for all balls } B \subset X. \tag{80}$$

If the space supports 1-Poincaré inequality, then it supports p-Poincaré inequality for all $1 \leq p < \infty$ simply by the Hölder inequality. Obviously $\mathbb{R}^n$ supports 1-Poincaré inequality.

As we will see, sometimes it is easier to prove inequality like (80) with $\sigma > 1$, than that with $\sigma = 1$. For this reason we allow $\sigma > 1$ in the above definition.

If the space supports the p-Poincaré inequality, then it provides a lot of information about the geometry of the space. It says in a quantitative way that there are a lot of rectifiable curves. Indeed, assume that there are no rectifiable curves at all in the space (X, d, μ) (this is for example the case when we deal with a metric space associated with a graph). Then the function $g \equiv 0$ is an upper gradient of any continuous function, because all the 1-Lipschitz curves $\gamma : [a, b] \to X$ are constant. This, however, means (80) cannot be true! Thus, in particular, no graph supports p-Poincaré inequality.

We say that the space (X, d, μ) is Q-regular $Q > 0$ if it is a complete metric space and $C_1 r^Q \leq \mu(B(x, r)) \leq C_2 r^Q$, whenever $x \in X$ and $r < \operatorname{diam} X$.

Heinonen and Koskela developped the theory of quasiconformal mappings between metric spaces that are Q-regular for some $Q > 1$, support the Q-Poincaré inequality and that every two points $x, y \in X$ can be connected by a curve whose length is less than or equal to $Cd(x, y)$.

It is usually a very difficult problem to prove that a metric space supports the p-Poincaré inequality (if it does). We state one such result due to Semmes.

Let X be a connected Q-regular metric space, $Q \geq 2$ integer, that is also Q-dimensional orientable topological manifold. Assume that X is *locally linearly contractible* in the following sense: there is $C \geq 1$ such that for every $x \in X$ and every $r \leq C^{-1} \operatorname{diam} X$, the ball $B(x, r)$ can be contracted to a point inside $B(x, Cr)$.

Theorem 119 *Under the above assumptions the space supports* 1*-Poincaré inequality.*

The proof of the theorem is more than 100 pages long! □

Vector fields. Classical Sobolev spaces were very usefull in studing existence and regularity properties of solutions to elliptic equations of the form

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial u}{\partial x_j} \right) = 0, \tag{81}$$

provided the uniform ellipticity condition is satisfied

$$\sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \geq C|\xi|^2. \tag{82}$$

Weak solutions to (81) belong to $W^{1,2}_{\mathrm{loc}}(\Omega)$. This class of equations include for example the Laplace equation. Now loot at

$$\frac{\partial^2 u}{\partial x^2} + x^2 \frac{\partial^2 u}{\partial y^2} = 0 \tag{83}$$

This equation can be written in the form (81), but the condition (82) is not satisfied. For this reason the methods developped to deal with equation (81) do not work. It is possible, however, to adopt them, but this requires a generalization of the Sobolev space. Roughly speaking one has to deal with the Sobolev space on a metric space where the metric reflects the structure of the operator (83).

Equation (83) is a special case of a large class of sub-Laplace operators that we next describe.

Let $X_1, X_2, \ldots, X_k$ be vector fields defined on $\Omega \subset \mathbb{R}^n$ with real, Lipschitz continuous coeficients. Each such vector field can be identified with a first order differential operator

$$X_i u = \langle X_i, \nabla u \rangle.$$

We will write $Xu = (X_1 u, \ldots, X_k u)$, $|Xu| = (\sum_i |X_i u|^2)^{1/2}$. By X_i^* we will denote a formal adjoint of X_i that is X_i^* is defined by the identity

$$\int_\Omega u X_i v = \int_\Omega (X_i^* u) v \,, \qquad \forall u, v \in C_0^\infty(\Omega)\,.$$

If $X_i = \sum_j c_{ij}(x)\partial/\partial x_j$, then the integration by parts yields $X_i^* = -X_i + f_i$, where $f_i = -\sum_j \partial c_{ij}/\partial x_j$.

Now the *generalized sub-Laplace operator* is defined as

$$L_X u = -\sum_i X_i^* X_i u.$$

Observe that equation (83) is of the form $L_X u = 0$, where $X_1 = \partial/\partial x$, $X_2 = x\partial/\partial y$.

A class of generalized sub-Laplace operators is to large and one cannot prove anything about the regularity of the sloutions to the equation $L_X u = 0$. Indeed, if all the coefficients of X_i's are equal to zero, than any function is a solution to the equation $L_X u = 0$!

What do we have to assume about X_i's to have a resonable regularity theory for the solutions to $L_X u = 0$? Trying to answer the question we will have to deal with Poincaré and Sobolev inequalities on metric spaces, where the metric will be induced by the geometry of the vector fields.

Carnot–Carathéodory metric. Now we will show how to associate a metric with a given family of vector fields. Let, as before, $X = (X_1, \ldots, X_k)$ be a family of Lipschitz continuous vector fields in $\Omega \subset \mathbb{R}^n$.

Willing to prove estimates for solutions to $L_X u = 0$ we would like to use as a technical tool Sobolev–Poincaré inequalities of the form

$$\left(\int_G |u - u_G|^q \, dx \right)^{1/q} \leq C \left(\int_G |Xu|^p \, dx \right)^{1/p} \tag{84}$$

with $q > p \geq 1$ for a sufficently large class of domains G, or at, least a weaker, Poincaré inequality (i.e. $p \geq q = 1$). How does one prove Poincaré inequality for the pair u, $|Xu|$? The natural approach is to bound u by integrals of $|Xu|$ along curves and then average resulting one-dimensional integrals to obtain desired Poincaré inequality.

In order to have such bounds for u in terms of integrals of $|Xu|$ one would like to know that $|Xu|$ is an upper gradient of u. Unfortunately it is seldom the case.

For example if we have only one vector field in $\mathbb{R}^2$, $X_1 = \partial/\partial x_1$ and $\gamma(t) = (0, t)$, $u(x_1, x_2) = x_2$, then $|u(\gamma(1)) - u(\gamma(0))| = 1$, while $|Xu| \equiv 0$, so $|Xu|$ is not an upper gradient of u. It is not an upper gradient even up to a constant factor. Roughly speaking the problem is caused by the fact that $\dot{\gamma}$ is not spanned by X_j's.

There is a brilliant idea that allows one to avoid the problem by introducing a new metric (that is described below) in Ω that makes $|Xu|$ an upper gradient of u on a new metric space. The metric is such that it restricts the class of 1-Lipschitz curves to those for which $\dot{\gamma}$ is a linear combinatin of X_j's. To be more precise it is not always a metric as it allows the distance to be equal to infinity.

We say that an absolutely continuous curve $\gamma : [a, b] \to \Omega$ is *admissible* if there exist measurable functions $c_j(t)$, $a \leq t \leq b$ satisfying $\sum_{j=1}^k c_j(t)^2 \leq 1$ and $\dot{\gamma}(t) = \sum_{j=1}^k c_j(t) X_j(\gamma(t))$.

Note that if the vector fields are not linearly idependent at a point, then the coefficients c_j are not unique.

Lemma 120 *Every curve 1-Lipschitz with respect to ρ is Lipschitz with respect to the Euclidean metric.*

Proof. It follows from the Schwartz inequality. □

Then we define the distance $\rho(x, y)$ between $x, y \in \Omega$ as the infimum of those $T > 0$ such that there exists an admissible curve $\gamma : [0, T] \to \Omega$ with $\gamma(0) = x$ and $\gamma(T) = y$. If there is no admissible curve that joins x and y, then we set $\rho(x, y) = \infty$.

Note that the space (Ω, ρ) splits into a (possibly infinite) family of metric spaces $\Omega = \bigcup_{i \in I} A_i$, where $x, y \in A_i$ if and only if x and y can be connected by an admissible curve. Obviously (A_i, ρ) is a metric space and the distance between distinct A_i's equals infinity.

The distance function ρ is given many names in the literature. We will use the name *Carnot–Carathéodory distance.* The space equipped with the Carnot–Carathéodory distance is called *Carnot–Carathéodory space.*

Proposition 121 *The mapping $\gamma : [0, T] \to \Omega$ is admissible if and only if it is 1-Lipschitz with respect to the distance ρ i.e. $\rho(\gamma(s), \gamma(t)) \leq |s - t|$.*

The implication $\Rightarrow$ follows directly form the definition of ρ. The opposite implication is more difficult and we do not prove it. $\square$

The following two results generalize Lemma 117.

Proposition 122 *$|Xu|$ is an upper gradient of $u \in C^\infty(\Omega)$ on the space (Ω, ρ).*

The space (Ω, ρ) is not necessarely a metric space since the distance ρ can be euqal to infinity. However the definition of upper gradient can be generalized (without any changes) to such degenerate metric spaces.

Proof. Let $\gamma : [a, b] \to (\Omega, \rho)$ be 1-Lipschitz. Then γ is admissible and hence by Lemma 120 γ is Lipschitz. Now $u \circ \gamma$ is Lipschitz and thus

$$|u(\gamma(b)) - u(\gamma(a))| = \left| \int_a^b \langle \nabla u(\gamma(t)), \dot{\gamma}(t) \rangle \, dt \right| \leq \int_a^b |Xu(\gamma(t))| \, dt.$$

The inequality follows from the fact that γ is admissible and from the Schwartz inequality. The proof is complete. $\square$

The following result is much more difficult.

Theorem 123 *Let $0 \leq g \in L^1_{\rm loc}(\Omega)$ be an upper gradient of a continuous function u on (Ω, ρ). Then the distributional derivatives $X_j u$, $j = 1, 2, \ldots, k$ are locally integrable and $|Xu| \leq g$ a.e.*

Now we come back to the question posed before: How does one prove Poincaré inequality of the type (84) with $q = 1$? The idea is the following: We estimate oscillations of u over admissible curves by integrals of $|Xu|$ and then the Poincaré inequality follows by averaging resulting line integrals. Thus the above idea — if it works — leads to inequalities of the type

$$\fint_{\widetilde{B}} |u - u_{\widetilde{B}}| \, dx \leq Cr \left(\fint_{\sigma \widetilde{B}} |Xu|^p \, dx \right)^{1/p}, \tag{85}$$

where $\widetilde{B}$ is a ball with respect ot the distance $\widetilde{B}$.

The idea seems simple but in general it is very difficult to handle it. In the next lecture we will show some examples of vector fields for which the Poincaré inequality (85) holds.

Lecture 11

The Hörmander condition. During the last lecture we defined so called Carnot–Carathéodory distance associated with a given system of vector fields. In general this need not be a metric as the distance between two points can be equal to infinity.

We describe now a large class of examples where both, the C.-C. distance is a genuine metric and inequlaities like (85) are true. This is the class of vector fields satisfying so called Hörmander's condition.

Let $\Omega \subset \mathbb{R}^n$ be a domain and let $X = (X_1, X_2 \ldots, X_k)$ be vector fields defined in Ω with real, C^∞ smooth coefficients. We say that the vector fields X satisfy the *Hörmander condition* if there is a positive integer d such that commutators of $X_1, \ldots, X_k$ up to the length d span the tangent space $\mathbb{R}^n$ at every point $x \in \Omega$.

The *commutator* of two vector fields X, Y is the differential operator defined by $[X, Y] = XY = YX$. One can easily prove that the second order differentiations cancel and $[X, Y]$ is a homogeneous first order differential operator i.e. it is a vector field.

The condition that commutators span $\mathbb{R}^n$ means that for every $x \in \Omega$ the vectors

$$X_1(x), \ldots, X_k(x), [X_{i_1}, X_{i_2}](x), \ldots [X_{i_1}, [X_{i_2}, [\ldots, X_{i_d}], \ldots]](x)$$

span $\mathbb{R}^n$

Example. The vector fields $X_1 = \partial/\partial x$, $X_2 = x^d \partial/\partial y$, d-positive integer, span $\mathbb{R}^2$ everywhere except the line $x = 0$. Now compute

$$[X_1, X_2] = \frac{\partial}{\partial x}\left(x^d \frac{\partial}{\partial y}\right) - x^d \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\right) = dx^{d-1}\frac{\partial}{\partial y}.$$

Hence by induction we see that taking a commutator of length $d+1$ (d fields X_1 and one field X_2) we get $d!\partial/\partial y$. Thus commutators of length $d+1$ span $\mathbb{R}^2$ at every point of $\mathbb{R}^2$. □

Let $X_1, \ldots X_k$ satisfy the Hörmander condition. Then the operator $L_X = -\sum_{i=1}^k X_i^* X_i$ is called *sub-Laplace operator.*

Solutions to sub-Laplace operator posseses many properties similar to those of the solutions to Laplace operator: they are C^∞ smooth and they satisfy a version of the Harnack inequliaty.

In the case of vecotr fields satisfying Hörmander's condition, the Carnot–Carathéodory distance is a genuine metric. The following result is a combination of efforts of many people including Carathéodory, Chow, Rashevsky, Fefferman, Phong, Lanconelli, Nagel, Stein, Wainger and many others.

In what follows by $\widetilde{B}$ we will denote a ball with respect to the $C.-C.$ distance. We will call $\widetilde{B}$ metric ball.

Theorem 124 *Given a domain $\Omega \subset \mathbb{R}^n$ and a system of vector fields satisfying Hörmander's condition in Ω. Then any two points in Ω can be connected by a piecewise smooth admissible curve and hence the C.-C. distance is a genuine metric. Moreover for every compact set $K \subset \Omega$ there exist constants C_1 and C_2 such that*

$$C_1|x-y| \le \rho(x, y) \le C_2|x-y|^{1/d},$$

for every $x, y \in K$*. There also exist* $r_0 > 0$ *and* $C \geq 1$ *such that*

$$|\widetilde{B}(x, 2r)| \leq C|\widetilde{B}(x, r)|,$$

whenever $x \in K$*,* $r \leq r_0$ *and* $\widetilde{B}$ *is the ball in the metric* ρ*.*

The second inequality states that the Lebesgue measure is locally doubling with respect to the C.-C. metric.

The Hörmander condition is so good that the Poincaré inequality holds. This is a result due to Jerison.

Theorem 125 (Jerison) *Let* $X = (X_1, \ldots, X_k)$ *be vector fields satisfying Hörmander's condition in* Ω*. Then for every compact set* $K \subset \Omega$ *there are constants* $C > 0$ *and* $r_0 > 0$ *such that for all* $u \in C^\infty(2\widetilde{B})$

$$\fint_{\widetilde{B}} |u - u_{\widetilde{B}}| \, dx \leq Cr \fint_{2\widetilde{B}} |Xu| \, d\mu, \tag{86}$$

whenever $\widetilde{B}$ *is a metric ball centred at* K *and of radius* $r \leq r_0$*.*

Then Jerison proved that $2\widetilde{B}$ on the right hand side can be replaced by $\widetilde{B}$. Actually we will prove later a stronger result: inequality (86) has the self improving property in a sense that if the Poincaré inequalities (86) hold on every metric ball, then the family of Sobolev–Poincaré inequalities

$$\left(\fint_{\widetilde{B}} |u - u_{\widetilde{B}}|^q \, dx\right) \leq Cr \left(\fint_{\widetilde{B}} |Xu|^p \, dx\right)^{1/p}, \tag{87}$$

for any $p \geq 1$ and some $q > p$ hold on any metric ball as well. Similar phenomenon has been mentioned in the context of inequalities on graphs. In the next lecture we will prove the self improving property in a very general setting of metric spaces. This will cover the cases of graphs, vector fields, and many others.

As we have already mentioned in the setting of graphs, the crucial technical tools employed in the proof of the Harnack inequality were the doubling property and the Poincaré inequality. Then it was also essential that both the doubling and the Poincaré imply the Sobolev–Poincaré inequality. We meet the same phenomenon in the context of Hörmander's vctor fields. Thus mimiking the usual proof of the Harnack inequality we obtain

Theorem 126 (Harnack inequality) *There is* $C > 0$ *such that if* u *is a positive solution to* $L_X = 0$ *in* $2\widetilde{B} \subset\subset \Omega$*, then*

$$\sup_{\widetilde{B}} u \leq C \inf_{\widetilde{B}} u.$$

Actually one can prove the Harnack inequality for generalized sub-Laplace equations (i.e. when the vector fields do not necessarely satisfy Hörmander's condition), provided the doubling condition and the Poincaré inequality holds.

Carnot groups. Now we will discuss a special case of vecotr fields satisfying Hörmander's condition that have a simple structure but still provide a lot of non-trivial examples. Those vector fields are left invariant vector fields on the so called *Carnot group* (*stratified Lie group*.

Recall that Lie G group is a smooth manifold with such a group structure that the multiplication and taking an inverse element are smooth mappings. The neutral element is usually denoted by e and the group law by a multiplicative notation gh.

If $g \in G$, then $l_g(h) = gh$, the left multiplication by g is a diffeomorphism. We say that a vector field X on G is left invariant is for every $g \in G$, $X(g) = Dl_g(e)X(e)$. Thus all the left invariant vector fields can be identified with the tangent space to G at e. We use notation $\mathfrak{g} = T_eG$ and writting $\mathfrak{g}$ we mean the identification of the tangent space with all the left invariant vector fields.

The comutator of two vector fields X and Y is defined by $[X, Y] = XY - YX$. One can prove that again this is a vecotr field and that if the vecor fields X and Y were left invariant then $[X, Y]$ is left invariant as well. Thus the comutator induces *Lie algebra* structure in $\mathfrak{g}$: $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ with the following properties:

$$[X, Y] = -[Y, X], \qquad [X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

We say that a measure μ on G is a left invariant Haar measure if the left multiplication by g is a measure preserving transformation for all g. The left invariant Haar measure is unique up to a constant factor.

The exponential mapping $\exp : \mathfrak{g} \to G$ is defined as follows: Let $X \in \mathfrak{g}$. Take an integral curve $\gamma(t)$ of the vector field X passing through e at $t = 0$. Then define $\exp(X) = \gamma(1)$. It is easy to see that $D\exp(e)$ is an isomorphism and hence exp is a diffeomorphism in a neighborhood of e.

The right rultiplication by $g \in G$ is also a diffeomorphism on G. By analogy we define the right invariant Haar measure. It is unique up to a constant factor. However the left and the right invariant measures may be different. If they are equal (up to a constant factor), then we say *the group is unimodal*.

A *Carnot group* is a connected and simply connected Lie group G whose Lie algebra $\mathfrak{g}$ admits a *stratification* $\mathfrak{g} = V_1 \oplus \cdots \oplus V_m$, $[V_1, V_i] = V_{i+1}$. $V_i = \{0\}$, for $i > m$. Carnot groups are also known as *stratified groups*.

Note that the basis of V_1 generates the whole Lie algebra $\mathfrak{g}$ and hence the left invariant vector fields generated by the basis of V_1 satisfy Hörmander's condition.

The structure of the Carnot group is particularly simple. One can prove that the exponential map is a global diffeomrphism from $\mathfrak{g}$ to G. Hence G is diffeomorphic to the Euclidean space $\mathbb{R}^n$.

This diffeomorphism allows us to identify elements of G with elements of $\mathfrak{g}$. Denote the group law by "$\circ$". Let $X, Y \in \mathfrak{g}$ be identified (via exp) with elements of G. Then one can prove that

$$X \circ Y = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}([X, [X, Y]] - [Y, [X, Y]]) + \dots$$

where on the right hand side we have a finite linear combination of commutators of higher orders. The dots denote commutators of order at least 4.

Let

Lecture 12

Sobolev inequalities on metric spaces. Today we will show how to develop a theory of Sobolev spaces on metric spaces equipped with a measure. The generality of our approach is motivated by examples given during previous twe lectures: we want to develope a theory that will cover all those examples.

Now we fix a setting in which we will work. Let (X, d, μ) be a complete metric space equipped with a Borel measure. We will assume that $0 < \mu(B) < \infty$, for any ball B. Moreover we assume that μ satisfies a doubling condition i.e. there is a constant $C_d \geq 1$ such that

$$\mu(2B) \leq \mu(B)$$

for every ball $B \subset X$.

We state now some results form the measure theory on metric spaces with doubling measure.

The doubling measure implies a lower bound for the growth of a measure of a ball.

Lemma 127 *If $Y \subset X$ is a bounded set, then*

$$\mu(B(x, r)) \geq (2 \operatorname{diam} Y)^{-s} \mu(Y) r^s, \tag{88}$$

for $s = \log_2 C_d$, $x \in Y$ and $r \leq \operatorname{diam} Y$.

If we take μ to be the Lebesgue measure in $\mathbb{R}^n$, then $C_d = 2^n$ and hence $s = n$. This shows that the exponent in (88) is sharp.

Proof. Let $x \in Y$. Then

$$\mu(B(x, 2^k, r) \leq C_d \mu(B(x, 2^{k-1} r) \leq \dots \leq C_d^k \mu(B(x, r)).$$

Now we take the least k such that $Y \subset B(x, 2^k r)$. Hence $\mu(Y) \le \mu(B(x, 2^k r)$. We estimate k comparing diameter of G with r and then the lemma follows. □

We will also need a generalization of the Hardy–Littlewood theorem. Let $g \in L^1_{\rm loc}(X, \mu)$. Then

$$Mg(x) = \sup_{r>0} \fint_{B(x,r)} |g|\, d\mu$$

is called *Hardy–Littlewood maximal function.*

Lemma 128 *The maximal function is bounded in L^p when $1 < p \le \infty$ i.e.*

$$\|Mg\|_p \le C\|g\|_p. \tag{89}$$

We do not prove the lemma here. Inequality (89) is not true for $p = 1$. The fact that the measure is doubling plays essential role in the proof. It also plays essential role in the proof of the following generalization of the Lebesgue theorem.

Lemma 129 *Let $u \in L^1_{\rm loc}(X, \mu)$. Fix $c \ge 1$. Then for a.e. $x \in X$*

$$\lim_i \fint_{E_i} u\, d\mu = u(x),$$

whenever $E_i \subset B(x, r_i)$ and $\mu(B(x, r_i)) \le c\mu(E_i)$ for some $r_i \to 0$.

We do not prove the lemma.

Now we are ready to deal with Sobolev spaces on metric spaces.

We say that $u \in M^{1,p}(X, d, \mu)$ if $u \in L^p$ and there is $0 \le g \in L^p$ such that

$$|u(x) - u(y) \le d(x, y)(g(x) + g(y)) \text{ a.e.} \tag{90}$$

The space is equipped with a norm

$$\|u\|_{M^{1,p}} = \|u\|_p + \inf_g \|g\|_p.$$

This definition has been mentioned in Lecture 3 and employed in the study of extension domains in Lecture 8. The definition is based on Theorem 63 which provedes a characterization of Sobolev spaces.

Another definition of the Sobolev space on a metric space is based on another characterization of the Sobolev space without using derivatives.

Theorem 130 *Let $u \in L^p(\mathbb{R}^n)$, $1 \le p < \infty$. Then $u \in W^{1,p}(\mathbb{R}^n)$ is and only if there are constants $C > 0$, $\sigma \ge 1$ and a function $0 \le g \in L^p(\mathbb{R}^n)$ such that*

$$\fint_B |u - u_B|\, dx \le Cr \fint_B g\, dx \tag{91}$$

for every ball B. Moreover (91) imlies that $|\nabla u| \le g$ a.e. for some conatnt C.

We skip the proof of the theorem. □

Observe that Theorem 130 includes the case $p = 1$, while the case was excluded from Theorem 63.

It follows form the theorem that for $u \in W^{1,p}(\mathbb{R}^n)$ there is $\|\nabla u\|_p \equiv \inf_g \|g\|_p$, where the infimum is obiously taken over the set of all g as in (91). This result suggests one could define a Sobolev space on a metric space.

Fix $C > 0$, $\sigma \geq 1$ and $1 \leq p < \infty$. By $P^p_{C,\sigma}(X, d, \mu)$ we will denote the class of functions $u \in L^p(X, \mu)$ such that there is $0 \leq g \in L^p(X, \mu)$ for which the inequality

$$\fint_B |u - u_B|\, dx \leq Cr \fint_B g\, dx,$$

holds on evey ball B of radius r. We endowe the space with a norm

$$\|u\|_{P^p_{C,\sigma}} = \|u\|_p + \inf_g \|g\|_p.$$

Now we will show that when $p > 1$ one can provide some other equivalent definitions.

We say that $u \in C^p(X, d, \mu)$ if $u \in L^p$ and $u_1^{\#} \in L^p$, where

$$u_1^{\#}(x) = \sup_{r>0} r^{-1} \fint_{B(x,r)} |u - u_B|\, d\mu.$$

The space is endowed with the norm

$$\|u\|_{C^p} = \|u\|_p + \|u_1^{\#}\|_p.$$

Theorem 131 *Let $p > 1$, then $P^p_{C,\sigma} = M^{1,p} = C^p$ and the norms are equivalent.*

Proof. Let $u \in M^{1,p}$. Then integrating inequality (90) with respect to $y \in B$ and then with respect to $x \in B$ we get te inequality

$$\fint_B |u - u_B\, d\mu \leq Cr \fint_B g\, d\mu$$

which easily implies that $u \in P^p_{C,\sigma}$ for all $C > 0$ and σ.

Now let $u \in P^p_{C,\sigma}$. Then

$$r^{-1} \fint_B |u - u_B|\, d\mu \leq C \fint_{\sigma B} g\, d\mu \leq CMg, \tag{92}$$

where Mg is the Hardy–Littlewood maximal function. Taking the supremum on the left hand side of (92) yields

$$u_1^{\#} \leq CMg \tag{93}$$

Now Lemma 128 together with (93) implies that $u_1^\# \in L^p$ and hence $u \in C^p$.

Assume that $u \in C^p$.

Reference comments

We do not provide here a complete list of references of the topics discussed during the lectures. It is simply impossible. The list would be to large. For the topincs which are standard we provide references to books. For the other topics which are not so standard we refer to the original papers, where the reader may find details and further references. We want to emphasize and appologise that the list of references is not complete. It was prepared in hurry.

The direct method described in Chapter I is standard, see e.g. Evans [9], Dacorogna [8], De Figeiredo [12], Giaquinta [22], Struwe [65].

There are many excelent sources for the theory of Sobolev spaces described in Chapters II and III. Our approach is close to that in Evans and Gariepy [10], Gilbarg and Trudinger [23], Maly and Ziemer [56], Ziemer [71].

The approach to pointwise inequalities is rather new. It has recently been discovered by many authors. However some ideas go back to old papers of Calderón and Zygmund, [4], [5]. Theorem62 is due to Liu, [51]. The characterization of the Sobolev space, Theorem 63, and the definition of the Sobolev space on a metric space is due to Hajłasz [28].

The theory of elliptic equation developped in Chapters IV and V is standard, see e.g. De Figueiredo [12], Giaquinta [22], Gilbarg and Trudinger [23], Giusti [24, Appendix], Heinonen Kilpeläinen and Martio [39], Malý and Ziemer [56], Struwe [65].

John domains are named after F. John, [46]. The theory of Sobolev spaces in John domains was originated in particular in Bojarski [2], Goldshtein and Reshetn'yak [25], Hurri [42], Martio [57], and Smith and Stegenga [64]. There are several papers devoted to the study of Sobolev spaces in domains with the irregular boundary that we do not mention here. Theorem 105 is taken form Martio [57] and Theorem 107 from Bojarski [2]. Proofs of both of the theorems are different and simpler than the original ones. They are taken from Hajłasz and Koskela [32] and [33]. The truncation argument employed in the proof of Theorem 107 is based on some ideas of Maz'ya [58] and Long and Nie [52]. Similar truncation method has recently beed rediscovered by many authors dealing with various generalizations of Sobolev inequalities.

For further applications of the Sobolev spaces to the boundar behaviour of conformal and quasiconformal mappings, see Hajłasz [27], Koskela, Manfredi and Villamor [49], Koskela and Rhode [50], Malý and Martio [55] and references therein.

There are several monographs devoted to the theory of quasiconformal mappings, see Heinonen Kilpeläinen and Martio [39], Iwaniec [43], Reshetn'yak [60], Rickman [61], Väisälä [67], Vuorinen [70]. The approach to quasiconformal mappings through the Sobolev spaces is the standard one.

The general Rellich–Kondrachov theorem as in Lecture 7 is taken from Hajłasz and Koskela [32]. The statement seems new, but the method of the proof employes standard arguments only.

The characterization of domains with the extension property is due to Hajłasz and Martio [34]. Extension theorem for uniform domains is due to Jones [45].

Theorem 115 is due to Holopainen and Soardi [40]. The reader will find there further references to a wide area of analysis on graphs.

Upper gradients and quasiconformal mappings between metric spaces were introduced and investigated in Heinonen and Koskela [38]. Theorem 119 is due to Semmes [63].

There is a huge number of papers devoted to the analysis of vector fields satisfying Hörmander's condition. The theory of sub-Laplace equations originates from a celebrated paper of Hörmander [41]. Theorem 124 is due to Nagel, Stein and Wainger [59] and Theorem 124 is due to Jerison [44]. For further results see a book of Varopoulos, Saloff–Coste and Coulhon [69]. See also Buckley, Koskela and Lu [3], Capogna Danielli Garofalo [6], Chernikov and Vodopyanov [7], Franchi, Lu and Wheeden [?], Hajłasz and Strzelecki [35], and references therein for the generalizations to nonlinear equations.

A vary nice introduction to Carnot groups is provided in Folland and Stein [13] and in Heinonen [37].

The case of general vector fields which do not necessarely satisfy Hörmander's condition is developped in particular in Franchi, Guttiérez and Wheeden, [14], Garofalo and Nhieu, [20], [21], Hajłasz and Koskela, [33] and references therein.

The role of the Poincaré inequality, Sobolev inequality and the doubling property in the proof of the Harnack inequality has been emphasized by Fabes, Kenig and Serapioni, [11] and in a more general setting by Heinonen, Kilpeläinen and Martio, [39]. It was Franchi and Lanconelli who suggested the approach to Harnack inequalities for generalized sub-Lapalce equations by the way of Sobolev–Poincaré inequalities and the doubling property for on metric balls, After their work a large developement has been undergone.

As explained above was clear that the crucial role in the proof of the Harnack inequality in various situations is played by the Sobolev–Poincaré inequality and the doubling property. It seems that Grigor'yan [26] and independently Saloff-Coste, [62] have discovered that in some situations the Sobolev inequality follows from the Poincaré inequality and the doubling property. Hence in order to prove Harnack inequality it is enough to prove doubling inequality and Poincaré inequality. Later, the fact that

Poincaré and doubling implies Sobolev–Poincaré also has been established in a very general setting by many other authors, Biroli and Mosco [1], Franchi, Lu and Wheeden [18], Franchi, Pérez and Wheeden, [19], Garofalo and Nhieu [21], Hajłasz and Koskela [31], [33], Maheux and Saloff-Coste [54], Sturm [66] and the others. In Lecture 11 we presented a general theorem of this type following Hajłasz and Koskela, [31], [33].

Theory of Sobolev spaces on metric space has been originated in Hajłasz [28]. The space defined there was our $M^{1,p}$. Then several other equivalent approaches has been developped. The space P^p has been introduced in Hajłasz and Koskela [31], [33] and the space C^p in Hajłasz and Kinnunen [30].

The characterization Theorem 130 and its further generalizations has been obtained in Franchi, Hajłasz and Koskela [15], Hajłasz and Koskela [33], Koskela and MacManus [48].

The references to Sobolev spaces on metric spaces include also Hajłasz [29], Heinonen and Koskela [38], Kałamajska [36], Kilpeläinen, Kinnunen and Martio, [?], Franchi Lu and Wheeden [18], Koskela and MacManus [48], MacManus and Pérez, [53], Semmes [63].

References

[1] Biroli, M. Mosco, U.: Sobolev inequalities on homogeneous spaces. *Pot. Anal.* 4 (1995), 311–324.

[2] Bojarski, B.: Remarks on Sobolev imbedding inequalities, in: *Proc. of the Conference in Complex Analysis, Joensuu 1987*, pp. 52–68, Lecture Nothes in Math. 1351, Springer 1988.

[3] Buckley, S. M., Koskela, P., Lu, G.: Subelliptic Poincare inequalities: the case $p < 1$. *Publ. Mat.* 39 (1995), 313–334.

[4] Calderón, A. P.: Estimates for singular integral operators in terms of maximal functions. *Studia Math.* 44 (1972), 563–582.

[5] Calderón, A.P., Zygmund, A.: Local properties of solutions of elliptic partial differential equations, *Studia Math.* 20 (1961), 171–225.

[6] Capogna, L., Danielli, D., Garofalo, N.: Capacitary estimates and the local behavior of solutions to nonlinear subelliptic equations. *Amer. J. Math.* 118 (1996), 1153–1196.

[7] Chernikov, V. M., Vodop'yanov, S. K.: Sobolev Spaces and hypoelliptic equations I,II. *Siberian Advances in Mathematics.* 6 (1996) no. 3, 27–67, no. 4, 64–96. Translation from: Trudy In-ta matematiki RAN. Sib. otd-nie. 29 (1995), 7–62.

[8] Dacorogna, B.: *Direct Methods in the Calculus of Variations*, Springer–Verlag 1989.

[9] Evans, L. C.: *Weak convergence methods for nonlinear partial differential equations.* CBMS Regional Conference Series in Mathematics, vol. 74 Amer. Math. Soc., Providence, RI, 1990.

[10] Evans, L. C., and Gariepy, R. F.: *Measure Theory and Fine Properties of Functions.* Studies in Advanced Mathematics, 1992.

[11] Fabes, E. B., Kenig, C. E., Serapioni, R. P.: The local regularity of solutions of degenerate elliptic equations. *Comm. Partial Diff. Eq.* 7 (1982) 77–116.

[12] De Figueiredo, D. G.: *Lectures on the Ekeland variational principle with applications and detours.* Tata Institute of Fundamental Research Lectures on Mathematics and Physics, 81. Springer, 1989.

[13] Folland, G. B., Stein, E. M.: *Hardy spaces on homogeneous groups.* Mathematical Notes vol. 28, Princeton Univ. Press, Princeton, New Jersey, 1982.

[14] Franchi, B., Gutiérrez, C., Wheeden, R.L.: Weighted Sobolev-Poincaré inequalities for Grushin type operators. *Comm. Partial Differential Equations* 19 (1994), 523–604.

[15] Franchi, B., Hajlasz, P., Koskela, P.: Definitions of Sobolev classes on metric spaces. Preprint.

[16] Franchi, B., Lanconelli, E.: Hölder regularity theorem for a class of non uniformly elliptic operators with measurable coefficients. *Ann. Scuola Norm. Sup. Pisa* 10 (1983), 523–541.

[17] Franchi, B., Lu, G., Wheeden, R.: Representation formulas and weighted Poincare inequalities for Hörmander vector fields. *Ann. Inst. Fourier (Grenoble)* 45 (1995), 577-604.

[18] Franchi, B., Lu, G., Wheeden, R. L.: A relationship between Poincaré type inequalities and representation formulas in spaces of homogeneous type. *Int. Mat. Res. Notices* (1996), No. 1, 1–14.

[19] Franchi, B., Pérez, C., Wheeden R.L.: Self–improving iroperties of John–Nirenberg and Poincaré inequalities on spaces of homogeneous type. *J. Funct. Anal.* (to appear).

[20] Garofalo, N., Nhieu, D. M.: Lipschitz continuity, global smooth approximations and extension theorems for Sobolev functions in Carnot–Carathéodory spaces, (preprint).

[21] Garofalo, N., Nhieu, D. M.: Isoperimetric and Sobolev inequalities for Carnot-Carathéodory spaces and the existence of minimal surfaces, *Comm. Pure Appl. Math.* 49 (1996), 1081–1144.

[22] Giaquinta, M.: *Introduction to regularity theory for nonlinear elliptic systems.* Lectures in Mathematics ETH Zurich. Birkhauser Verlag, Basel, 1993.

[23] Gilbarg, D., Trudinger, N.: *Elliptic Partial Differential Equations of Second Order*, Springer–Verlag 1983.

[24] E. Giusti, *Minimal surfaces and functions of bounded variation* (Birkhäuser Boston, Inc., 1984).

[25] Gol'dshtein, V. M., Reshetnyak, Y. G.: *Quasiconformal mappings and Sobolev spaces* (in Russian). Nauka, Moscow, 1983; English ed., Kluwer Acad. Publishers 1990.

[26] Grigor'yan, A.: The heat equation on noncompact Riemannian manifolds. *Mat. Sb.* 182, (1991), 55–87 (Russian).

[27] Hajłasz, P.: Boundary behaviour of Sobolev mappings. *Proc. Amer. Math. Soc.*, 123 (1995), 1145–1148.

[28] Hajłasz, P.: Sobolev spaces on an arbitrary metric space. *Potential Analysis*, 5 (1996), 403–415.

[29] Hajłasz, P.: Geometric approach to Sobolev spaces and badly degenerated elliptic equations. *The Proceedings of Banach Center Minisemester: Nonlinear Analysis and Applications.* (N.Kenmochi, M.Niezgódka, P.Strzelecki, eds.) GAKUTO International Series; Mathematical Sciences and Applications, vol. 7 (1995), 141–168.

[30] Hajłasz, P., Kinnunen, J.: Hölder quasicontinuity of Sobolev functions on metric spaces. *Rev. Mat. Iberoamericana*, (to appear).

[31] Hajłasz, P., Koskela, P.: Sobolev meets Poincaré. *C. R. Acad. Sci. Paris* 320 (1995), 1211–1215.

[32] Hajłasz, P., Koskela, P.: Isoperimetric inequalities and imbedding theorems in irregular domains, to appear in *J. London Math. Soc.*

[33] Hajłasz, P., Koskela, P.: Sobolev met Poincaré. Preprint.

[34] Hajłasz, P., Martio, O.: Traces of Sobolev functions on fractal type sets and characterization of extension domains. *J. Funct. Anal.* 143 (1997), 221–246.

[35] Hajłasz, P., Strzelecki, P.: Subelliptic p-harmonic maps into spheres and the ghost of Hardy spaces. *Math. Ann.* (to appear).

[36] Kałamajska, A.: On compactness of embedding for Sobolev spaces defined on metric spaces. *Ann. Acad. Sci. Fenn. Math.* (to appear).

[37] Heinonen, J.: Calculus on Carnot groups. *Ber. Univ. Jyväskylä Math. Inst.* 68 (1995), 1–31.

[38] Heinonen, J., Koskela, P.: Quasiconformal maps on metric spaces with controlled geometry. *Acta Math.* (to appear).

[39] Heinonen, J., Kilpeläinen, T., Martio, O.: *Nonlinear Potential Theory of Degenerate Elliptic Equations*, Oxford Univ. Press, 1993.

[40] Holopainen, I., Soardi, P. M.: A strong Liouville theorem for p-harmonic functions on graphs. *Ann. Acad. Sci. Fenn. Math.* 22 (1997), 205–226.

[41] Hörmander, L.: Hypoelliptic second order differential equations, Acta Math. 119 (1967), 147–171.

[42] Hurri, R.: Poincaré domains in $\mathbb{R}^n$. *Ann. Acad. Sci. Fenn. Ser. A I. Math. Dissertationes*, 71 (1988), 1–42.

[43] Iwaniec, T.: *Lectures on Quasiconformal Mappings.* Unpublished Lecture Notes.

[44] Jerison, D.: The Poincaré inequality for vector fields satisfying Hörmander's condition. *Duke Math. J.* 53 (1986) 503–523.

[45] Jones, P. W.: Quasiconformal mappings and extendability of functions in Sobolev spaces, *Acta Math.* 147 (1981), 71–88.

[46] John, F.: Rotation and strain. *Comm. Pure and Appl. Math.* 4 (1961) 391–414.

[47] Kinnunen, J., Martio, O.: The Sobolev capacity on metric spaces. *Ann. Acad. Sci. Fenn. Math.* 21 (1996), 367–382.

[48] Koskela, P., MacManus, P.: Quasiconformal mappings and Sobolev spaces. *Studia Math.* (to appear).

[49] Koskela, P., Manfredi, J., Villamor, E.: Regularity theory and traces of A-harmonic functions. *Trans. Amer. Math. Soc.* 348 (1996), 755–766.

[50] Koskela, P., Rohde, S.: Hausdorff dimension and mean porosity. *Math. Ann.* 309 (1997), 593–609.

[51] Liu, F.C.: A Lusin type property of Sobolev functions, *Indiana Univ. Math. Journ.* 26 (1977), 645–651.

[52] Long, R. L., Nie, F. S.: Weighted Sobolev inequality and eigenvalue estimates of Schrödinger operators. *In: Harmonic Analysis.* (Tianjin, 1988). Lecture Notes in Math. 1494, Springer, 1991.

[53] MacManus, P., Pérez, C.: Generalizad Poincaré inequalities: sharp self-improving properties. International Math. res. Notices 1988, No2, 101–116.

[54] Maheux, P., Saloff-Coste, L.: Analyse sur les boules d'un opérateur sous-elliptique. *Math. Ann.* , 303 (1995), 713–740.

[55] Malý, J., Martio, O.: Lusin's condition (N) and mappings of the class $W^{1,n}$. *J. Reine Angew. Math.* 458 (1995), 19–36.

[56] Malý, J., Ziemer, W.: *Fine regularity of solutions of elliptic partial differential equations.* Mathematical Surveys and Monographs, vol. 51. Amer. Math. Soc. Providence, RI, 1997.

[57] Martio, O.: John domains, bilipschitz balls and Poincaré inequality. *Rev. Roumaine Math. Pures Appl.*, 33 (1988), 107–112.

[58] Maz'ya, V. G.: A theorem on the multidimensional Schrödinger operator. *Izv. Akad. Nauk.* 28 (1964), 1145–1172 (in Russian).

[59] Nagel, A., Stein, E. M., Wainger, S.: Balls and metrics defined by vector fields I: Basic properties. *Acta Math.* 155, (1985), 103–147.

[60] Reshetnyak, Yu, G.: *Space mappings with bounded distorsion.* Translation of Mathematical Monographs, vol. 73, American Mathematical Society, Providence 1989.

[61] Rickman, S.: *Quasiregular mappings.* Results in Mathematics and Related Areas (3), vol. 26. Springer, 1993.

[62] Saloff-Coste, L.: A note on Poincaré, Sobolev, and Harnack inequalities, Internat. Mat. Res. Notices, 1992 no. 2, pp. 27–38.

[63] Semmes, S.: Finding curves on general spaces through quantitative topology, with applications to Sobolev and Poincare inequalities. *Selecta Math. (N.S.)* 2 (1996), 155–295.

[64] W.Smith and D.A.Stegenga, 'Hölder domains and Poincaré domains', *Trans. Amer. Math. Soc.* 319 (1990) 67–100.

[65] Struwe, M.: *Variational methods. Applications to nonlinear partial differential equations and Hamiltonian systems.* Second edition. Results in Mathematics and Related Areas vol. 34. Springer, 1996.

[66] Sturm, K. T.: Analysis on local Dirichlet spaces III. The parabolic Harnack inequality. *J. Math. Pures Appl.* 75 (1996), 273–297.

[67] Väisälä, J.: *Lectures on n-dimensional quasiconformal mappings*, Lecture Notes in Math. 299, Springer 1971.

[68] Varopoulos, T.: Fonctions harmoniques sur les groupes Lie, *C. R. Acad. Sci. Paris* 309 (1987), 519–521.

[69] Varopoulos, T., Saloff-Coste, L., Coulhon, T.: *Analysis and geometry on groups* (Cambridge Univ. Press 1992).

[70] Vuorinen, M.: *Conformal geometry and quasiregular mappings.* Lecture Notes in Mathematics, vol. 1319 Springer, 1988.

[71] Ziemer, W.: *Weakly differentiable functions*, Graduate Texts in Mathematics 120 (Springer 1989).