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Pointwise inequalities for Sobolev functions and some applications

by

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Abstract. We get a class of pointwise inequalities for Sobolev functions. As a corollary we obtain a short proof of Michael–Ziemer's theorem which states that Sobolev functions can be approximated by C^m functions both in norm and capacity.

1. Introduction. In this paper, we prove some pointwise inequalities for Sobolev functions, i.e. functions in the Sobolev classes $W^{m,p}(\Omega)$, where m is an integer, $p \geq 1$, and Ω is an open subset of $\mathbb{R}^n$. For simplicity we restrict the discussion to the case $\Omega = \mathbb{R}^n$ and $mp < n$. The generalized derivatives $D^\alpha f$, $|\alpha| \leq m$, are defined as equivalence classes of measurable functions. For our pointwise estimates, presented in a form valid for each point of the domain Ω, it is essential to select a representative in each class which is a Borel function, i.e. a function well defined at each point of its domain, essentially by an everywhere convergent limiting process of sequences of continuous or continuously differentiable real-valued functions. This is best illustrated by the well known procedure of selecting a Borel function $\widetilde{f}(x)$ for the class of real-valued Lebesgue spaces $L^p_{\mathrm{loc}}(\mathbb{R}^n)$ using the formula

$$\widetilde{f}(x) = \limsup_{r\to 0} \fint_{B(x,r)} f(y)\,dy = \limsup_{r\to 0} f_r(x), \qquad r > 0\,,$$

where $f_r(x)$ are the Steklov means of the Lebesgue function f. Note that the above limiting process is rather delicate and should be applied with extreme care; in particular, it is not additive, and in general $\widetilde{f}(x) \neq -(\widetilde{-f})(x)$.

An important remark is that our main pointwise inequalities for the Borel function $\widetilde{f}(x)$ may be formulated in terms of the averaged Steklov type

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functions $f_r(x)$, with the right hand side of these estimates independent of the averaging parameter r. The pointwise estimates of this paper may be obtained from those inequalities for the averages $f_r(x)$ by the pointwise limiting process as $r \to 0$. A systematic exposition of this rather crucial, in our opinion, point of view is deferred to a subsequent paper.

We have used the Borel functions of type $\widetilde{f}(x)$ since this makes the exposition much shorter, allowing more direct references to the existing literature. Let us remark also that our pointwise estimates for functions in $W^{m,p}$ are rather sharp. In particular, after local integration, for $p > 1$, they imply the local Sobolev imbedding inequalities in the most general and precise form.

This is the first of a series of papers on the local geometric theory of Sobolev spaces. The analogues of the above results for Sobolev spaces of fractional order $W^{m,p}$, m real, $p > 1$, and the related theory of the Sobolev trace operator on submanifolds will be discussed in subsequent publications.

Theorem 1 of Section 2 generalizes the classical result. We prove that the inequalities in Theorem 1 hold everywhere. In the literature the first inequality was proved to hold almost everywhere (a.e.), but the second one seems to be missing even in the a.e. form.

In the rest of Section 2 we give the pointwise estimate of the remainder in the formal Taylor formula for a Sobolev function (Theorem 2). This result will be used in Section 5 to give a short proof of Michael and Ziemer's version of the Calderón–Zygmund theorem ([MZ]).

In Section 3 we deal with an integral representation of Sobolev functions. The results of this section are somehow parallel to (but independent of) the results of Section 2. In Section 4, included here for completeness' sake, and for the convenience of the reader, we recall some necessary results concerning the Bessel capacity estimates for the Lebesgue points of a Sobolev function. Theorems 4 and 5 are well known. Theorem 7 is due to Ziemer ([Z1] and [Z2]).

Basic notations. By $|A|$ we denote the Lebesgue measure of the set A. By Q we denote a cube in $\mathbb{R}^n$. $f_Q = \fint_Q f = |Q|^{-1} \int_Q f$ is used to denote the mean value of f on Q. Moreover,

$$T_x^k u(y) = \sum_{|\alpha| \le k} D^\alpha u(x) \frac{(y-x)^\alpha}{\alpha!}, \quad T_Q^k u(y) = \fint_Q T_x^k u(y)\, dx .$$

If $\int \varphi = 1$ then we set

$$T_\varphi^k u(y) = \int T_x^k u(y)\, \varphi(x)\, dx .$$

By $\nabla^m f$ we denote the vector with components $D^\alpha f$, $|\alpha| = m$. If f is a locally integrable function, then we define $\widetilde{f}$ at every point by the formula

$$\widetilde{f}(x) = \limsup_{r\to 0} \fint_{B(x,r)} f(y)\, dy\,. \tag{1}$$

Note that $\widetilde{\widetilde{f}}(x) = \widetilde{f}(x)$. In what follows, as a rule, we identify $\widetilde{f}$ with f and omit the tilde sign.

We say that x is a *Lebesgue point* of f if

$$\lim_{r\to 0} \fint_{B(x,r)} |f(y) - f(x)|\, dy = 0$$

($f(x)$ is defined by (1)). Some variants of the Hardy–Littlewood maximal functions are used:

$$M_R f(x) = \sup_{r<R} \fint_{B(x,r)} |f(y)|\, dy\,, \qquad Mf = M_\infty f\,,$$

$$M_R^\sharp f(x) = \sup_{r<R} \fint_{B(x,r)} |f(y) - f(x)|\, dy\,, \quad M^\sharp f = M_\infty^\sharp f\,.$$

Note that for all x,

$$M_R^\sharp f(x) \le 2M_R f(x) \le 2Mf(x)\,.$$

We use the following definition of the Sobolev space:

$$W^{m,p}(\Omega) = \{f \in \mathcal{D}'(\Omega) : D^\alpha f \in L^p(\Omega),\ |\alpha| \le m\},$$
$$\|f\|_{m,p} = \sum_{|\alpha|\le m} \|D^\alpha f\|_p\,,$$

where $\|\cdot\|_p$ denotes the L^p-norm. Analogously we define the corresponding local space $W^{m,p}_{\mathrm{loc}}$. Obviously $W^{m,p} \subset W^{m,1}_{\mathrm{loc}}$. By C we denote the general constant; it may vary even in the same proof.

2. Pointwise inequalities. Let $Q \subset \mathbb{R}^n$ be a cube and let $f \in C^1(Q)$. The following inequality is well known to be true for all $x \in Q$:

$$|f(x) - f_Q| \le C \int_Q \frac{|\nabla f(y)|}{|x-y|^{n-1}}\, dy \tag{2}$$

(see e.g. [GT], Lemma 7.16).

Now we show, by extending the method used in [B1] (see also [R]), how to obtain the stronger inequality involving the derivatives of any order m. Even for $m = 1$ this inequality will be more sophisticated than (2).

Let $f \in C^m(Q)$ and let $a = (a_\alpha)_{|\alpha|=m}$ be a family of real numbers. Let

$$\varphi(y;x) = \sum_{|\alpha|\le m-1} D^\alpha f(y)\frac{(x-y)^\alpha}{\alpha!} + \sum_{|\alpha|=m} a_\alpha \frac{(x-y)^\alpha}{\alpha!}.$$

Obviously $\varphi(x;x) = f(x)$ and

$$\frac{\partial\varphi}{\partial y_i}(y;x) = \sum_{|\alpha|=m-1} D^{\alpha+\delta_i} f(y)\frac{(x-y)^\alpha}{\alpha!} - \sum_{|\alpha|=m-1} a_{\alpha+\delta_i}\frac{(x-y)^\alpha}{\alpha!},$$

where $\delta_i = (0,\ldots,1,\ldots,0)$ (the ith component is equal to 1).

Directly from the definition we have

$$T_Q^m f(x) = \fint_Q \varphi(y;x)\,dy + \fint_Q \sum_{|\alpha|=m} (D^\alpha f(y) - a_\alpha)\frac{(x-y)^\alpha}{\alpha!}\,dy.$$

Hence, applying (2) to the function φ, we get

$$\begin{aligned}|f(x) - T_Q^m f(x)| &\le |\varphi(x) - \varphi_Q| + C\fint_Q |\nabla^m f(y) - a|\,|x-y|^m\,dy\\ &\le C'\int_Q \frac{|\nabla^m f(y) - a|}{|x-y|^{n-m}}\,dy,\end{aligned}$$

where $(\nabla^m f(y) - a) = (D^\alpha f(y) - a_\alpha)_{|\alpha|=m}$ is treated as a vector.

Thus we have

LEMMA 1. *If $f \in C^m(Q)$ and $a = (a_\alpha)_{|\alpha|=m}$ then*

$$|f(x) - T_Q^{m-1} f(x)| \le C\int_Q \frac{|\nabla^m f(y)|}{|x-y|^{n-m}}\,dy,$$

and

$$|f(x) - T_Q^m f(x)| \le C\int_Q \frac{|\nabla^m f(y) - a|}{|x-y|^{n-m}}\,dy; \tag{3}$$

in particular, substituting $a = \nabla^m f(x)$ we have

$$|f(x) - T_Q^m f(x)| \le C\int_Q \frac{|\nabla^m f(y) - \nabla^m f(x)|}{|x-y|^{n-m}}\,dy,$$

where the constants C depend on n and m only.

Remark. As far as we know, the second inequality is missing in the literature.

Proof of Lemma 1. We have proved the second inequality. The first follows easily from the second by taking $a = 0$. Indeed,

$$|f(x) - T_Q^{m-1} f(x)| \le |f(x) - T_Q^m f(x)| + C \fint_Q |\nabla^m f(y)|\, |x-y|^m \, dy$$
$$\le C' \int_Q \frac{|\nabla^m f(y)|}{|x-y|^{n-m}}\, dy\,.$$

Now we will deal with the Sobolev functions. If $f \in W^{m,1}_{\mathrm{loc}}$, then we can choose natural Borel representatives of this function and its derivatives defined at every point by the formula

$$D^\alpha f(x) = \limsup_{r \to 0} \fint_{B(x,r)} D^\alpha f(y)\, dy \tag{4}$$

(compare (1)).

Now we prove the following extension of Lemma 1.

THEOREM 1. *There exists a constant $C_{m,n}$ such that if $f \in W^{m,1}(Q)$ is defined at every point by* (4) (*with $\alpha = 0$*) *and $a = (a_\alpha)_{|\alpha|=m}$ is an arbitrary family of real numbers, then, at each $x \in Q$,*

$$|f(x) - T_Q^{m-1} f(x)| \le C_{m,n} \int_Q \frac{|\nabla^m f(y)|}{|x-y|^{n-m}}\, dy\,, \tag{5}$$

$$|f(x) - T_Q^m f(x)| \le C_{m,n} \int_Q \frac{|\nabla^m f(y) - a|}{|x-y|^{n-m}}\, dy\,. \tag{6}$$

Remarks. 1) It seems to be a new and very important fact (as our applications show) that inequalities (5) and (6) hold everywhere, and not only a.e. 2) We prove this theorem with the constants $C_{m,n}$ larger than their counterparts in Lemma 1. 3) This proof extends the method used in [H2]. See also [B2].

Proof of Theorem 1. The same argument as in the proof of Lemma 1 shows that the first inequality follows from the second, so it suffices to prove the latter. A standard approximation argument implies that (3) holds a.e. for $f \in W^{m,1}(Q)$. Integrating both sides of that inequality over a ball we have

$$\Big| \fint_{B(x,r)} f(y)\, dy - \fint_{B(x,r)} T_Q^m f(y)\, dy \Big| \le C \int_Q \fint_{B(x,r)} \frac{|\nabla^m f(z) - a|}{|y-z|^{n-m}}\, dy\, dz\,. \tag{7}$$

We can estimate the right hand side of this inequality by Lemma 2 below. To our knowledge, the estimates of Lemma 2 have been first used by O. Frostman [Fr] (see also [La]). Here we present the simple proofs of Lemma 2 as well as of Lemma 3 for the sake of completeness.

LEMMA 2. *If $\alpha > 0$, then there exists a constant $C_{\alpha,n}$ such that for all $x, z \in \mathbb{R}^n$ and all $r > 0$,*

$$\fint_{B(x,r)} |y-z|^{\alpha-n}\,dy \le \begin{cases} C_{\alpha,n}|x-z|^{\alpha-n} & \text{if } \alpha \le n\,, \\ C_{\alpha,n}(r+|x-z|)^{\alpha-n} & \text{if } \alpha \ge n\,. \end{cases}$$

Proof. We can assume that $z = 0$. If $r < \frac{1}{2}|x|$, then $\frac{1}{2}|x| < |y| < \frac{3}{2}|x|$, hence $|y|^{\alpha-n} < C_{\alpha,n}|x|^{\alpha-n}$, and the lemma follows. If $r \ge \frac{1}{2}|x|$ then $B(x,r) \subset B(0,3r)$ and hence

$$\fint_{B(x,r)} |y|^{\alpha-n}\,dy \le Cr^{-n} \int_{B(0,3r)} |y|^{\alpha-n}\,dy = C'r^{\alpha-n}\,.$$

If $\alpha \le n$, then $r^{\alpha-n} \le 2^{n-\alpha}|x|^{\alpha-n}$. This ends the proof of the lemma.

As we have already mentioned, Lemma 2 leads to an estimate of the right hand side of (7), and hence, by passing to the limit, the theorem follows.

In the sequel we need the following lemma of Hedberg [He]:

LEMMA 3. *If $\alpha > 0$, then there exists a constant $C_{n,\alpha}$ such that for all $u \in L^1(Q)$ and all $x \in Q$,*

$$\int_Q \frac{|u(y)|}{|x-y|^{n-\alpha}}\,dy \le C_{n,\alpha}(\operatorname{diam} Q)^\alpha M_{\operatorname{diam} Q}|u|(x)\,.$$

Proof. We can break the integral into the sum of the integrals over the "rings" $Q \cap (B(x, \operatorname{diam} Q/2^k) \setminus B(x, \operatorname{diam} Q/2^{k+1}))$. In each ring, we have $|x-y|^{\alpha-n} \sim (\operatorname{diam} Q/2^k)^{\alpha-n}$. Now we estimate the integral over the ring by the integral over the ball $B(x, \operatorname{diam} Q/2^k)$ and the lemma follows easily.

Let $f \in W^{m,1}_{\mathrm{loc}}(\mathbb{R}^n)$ and its derivatives be defined at every point by (4). Let $x, y \in \mathbb{R}^n$ be such that $|f(y)| < \infty$ and $|D^\alpha f(x)| < \infty$ for $|\alpha| \le m-1$. Moreover, let $Q \subset \mathbb{R}^n$ be any cube such that $x, y \in Q$.

As a direct consequence of the triangle inequality we have

$$\begin{aligned} (8) \qquad |f(y) - T^{m-1}_x f(y)| &\le |f(y) - T^{m-1}_Q f(y)| \\ &\quad + \sum_{|\alpha| \le m-1} \left| D^\alpha f(x)\frac{(y-x)^\alpha}{\alpha!} - \frac{(y-x)^\alpha}{\alpha!} T^{m-1-|\alpha|}_Q D^\alpha f(x) \right| \\ &\quad + \left| T^{m-1}_Q f(y) - \sum_{|\alpha|\le m-1} \frac{(y-x)^\alpha}{\alpha!} T^{m-1-|\alpha|}_Q D^\alpha f(x) \right|. \end{aligned}$$

The last term is identically zero since it is the difference of two equal polynomials. Indeed, the first polynomial is $T^{m-1}_Q f(y)$ and the second is Taylor's expansion of the first one.

The remaining terms on the right hand side of (8) can be estimated by Theorem 1. This leads to the following

THEOREM 2. *Let $f \in W^{m,1}_{\mathrm{loc}}(\mathbb{R}^n)$ and its derivatives be defined at every point by (4), and let $x, y \in \mathbb{R}^n$ be such that $|f(y)|, |D^\alpha f(x)| < \infty$, $|\alpha| \leq m-1$. If $x, y \in Q$, then for some constant $C = C(n,m)$,*

$$|f(y) - T_x^{m-1} f(y)| \leq C\Big(I_Q^m(|\nabla^m f|)(y) + \sum_{i=0}^{m-1} (\operatorname{diam} Q)^i I_Q^{m-i}(|\nabla^m f|)(x)\Big),$$

where $I_Q^k(g)(x) \equiv \int_Q |x-y|^{k-n}\, g(y)\, dy$ is the local Riesz potential of order k.

If we now apply Lemma 3, we obtain

$$|f(y) - T_x^{m-1} f(y)| \leq C(M_{\operatorname{diam} Q}|\nabla^m f|(y) + M_{\operatorname{diam} Q}|\nabla^m f|(x))(\operatorname{diam} Q)^m\,. \tag{9}$$

Analogously, we can obtain the inequality

$$|f(y) - T_x^m f(y)| \leq C(M_{\operatorname{diam} Q}|\nabla^m f - a|(y) + M_{\operatorname{diam} Q}|\nabla^m f - b|(x))(\operatorname{diam} Q)^m\,, \tag{10}$$

where $a = (a_\alpha)_{|\alpha|=m}$, $b = (b_\alpha)_{|\alpha|=m}$. We note a difference in the proof of (10): the highest order terms are estimated directly without the use of Theorem 1 and Lemma 3. Namely,

$$\begin{aligned}\sum_{|\alpha|=m} \Big| D^\alpha f(x)\frac{(y-x)^\alpha}{\alpha!} - \frac{(x-y)^\alpha}{\alpha!} T_Q^{m-|\alpha|} D^\alpha f(x)\Big| &\leq C|y-x|^m |\nabla^m f(x) - (\nabla^m f)_Q| \\ &\leq C(\operatorname{diam} Q)^m (|\nabla^m f(x) - b| + |(\nabla^m f)_Q - b|) \\ &\leq C'(\operatorname{diam} Q)^m M_{\operatorname{diam} Q}|\nabla^m f - b|(x)\,.\end{aligned}$$

Thus we have the following

THEOREM 3. *If $f \in W^{m,1}_{\mathrm{loc}}(\mathbb{R}^n)$ and its derivatives are defined at every point by (4), then the following inequalities are satisfied.*

1) *If $|f(y)| < \infty$ and $|D^\alpha f(x)| < \infty$ for $|\alpha| \leq m-1$, then*

$$|f(y) - T_x^{m-1} f(y)| \leq C(M_{|x-y|}|\nabla^m f|(y) + M_{|x-y|}|\nabla^m f|(x))|x-y|^m\,.$$

2) *If $|f(y)| < \infty, |D^\alpha f(x)| < \infty$ for $|\alpha| \leq m$ and $a = (a_\alpha)_{|\alpha|=m}$, $b = (b_\alpha)_{|\alpha|=m}$ are taken arbitrarily, then*

$$|f(y) - T_x^m f(y)| \leq C(M_{|x-y|}|\nabla^m f - a|(y) + M_{|x-y|}|\nabla^m f - b|(x))|x-y|^m\,,$$

where the constant C depends on m and n only.

P r o o f. There exists a cube $Q \subset \mathbb{R}^n$ such that $x, y \in Q$ and $\operatorname{diam} Q \sim |x-y|$. Now the theorem follows from (9) and (10).

Remarks. 1) We will prove in the next section (Lemma 4) that if $M_1|\nabla^k f|(x) < \infty$, then $|D^\alpha f(x)| < \infty$ for $|\alpha| \le k$. 2) Inequalities of the type considered in Theorem 3 have been obtained in [H1], but the proof presented here is a direct generalization of the method given in [B2] for the special case of $m = 1$ (see also [H2]).

COROLLARY 1. *If $f \in W^{m,1}_{\mathrm{loc}}$ and its derivatives are defined at every point by* (4), $|D^\alpha f(x)| < \infty$ *and* $|D^\alpha f(y)| < \infty$ *for* $|\alpha| \le m$, *then*

$$|D^\alpha f(y) - T_x^{m-|\alpha|} D^\alpha f(y)| \le C(M^\sharp_{|x-y|}|\nabla^m f|(y) + M^\sharp_{|x-y|}|\nabla^m f|(x))|x-y|^{m-|\alpha|}.$$

Proof. It suffices to put $a = \nabla^m f(y)$, $b = \nabla^m f(x)$ and apply the second inequality from Theorem 3 to $D^\alpha f \in W^{m-|\alpha|,1}_{\mathrm{loc}}$.

If we apply inequality 1) from Theorem 3 to $f \in W^{m,1}_{\mathrm{loc}}(\mathbb{R}^n)$ and to all its derivatives (in a way similar to Corollary 1), then Whitney–Glaeser's extension theorem ([M], Th. 3.6) implies that for every $\varepsilon > 0$ there exists a function $h \in C^{m-1,1}_{\mathrm{loc}}(\mathbb{R}^n)$ (C^{m-1} function with locally Lipschitz $(m-1)$-derivatives) such that $|\{x : h(x) \ne f(x)\}| < \varepsilon$. But as follows from another theorem of Whitney ([F], Th. 3.1.15), for every $\delta > 0$ there exists a function $g \in C^m(\mathbb{R}^n)$ such that $|\{x : g(x) \ne h(x)\}| < \delta$. Therefore, we have proved the theorem of Calderón and Zygmund ([CZ], Th. 13) which can be stated as follows.

COROLLARY 2. *If $f \in W^{m,1}_{\mathrm{loc}}(\mathbb{R}^n)$, then for every $\varepsilon > 0$ there exists a function $g \in C^m(\mathbb{R}^n)$ such that $|\{x : f(x) \ne g(x)\}| < \varepsilon$.*

In Section 5 we show how Corollary 1 can be used to get a short proof of the generalization of Corollary 2—the theorem of Michael and Ziemer.

COROLLARY 3. *If $f \in W^{m,p}_{\mathrm{loc}}(\mathbb{R}^n)$ where $1 < p < mp$, then*

$$\Big(\fint_Q |f(x) - T_Q^{m-1} f(x)|^{p^*}\,dx\Big)^{1/p^*} \le C_{m,n,p}|\operatorname{diam} Q|^m \Big(\fint_Q |\nabla^m f(y)|^p\,dy\Big)^{1/p}$$

for each cube $Q \subset \mathbb{R}^n$ with the constant $C_{m,n,p}$ depending on m, n, p only.

This is the precise form of the Sobolev imbedding inequality for the spaces $W^{m,p}(\mathbb{R}^n)$, $mp < n$. Here $p^* = pn/(n - mp)$. The proof is a direct consequence of the Hardy–Littlewood–Sobolev inequality for Riesz potentials [S], [Z2].

The global Sobolev imbedding inequality for the spaces $W^{m,p}(\Omega)$ for a large class of domains $\Omega \subset \mathbb{R}^n$, including in particular the class of so-called John domains, which may not have a rectifiable boundary, has been obtained in [B3], as a consequence of Corollary 3. See also [GR].

3. Integral representations and Taylor's formula for Sobolev functions. If $f \in C^\infty(Q)$, then, integrating the Taylor formula multiplied by a weight $\varphi \in C_0^\infty(Q)$, $\int \varphi = 1$, we have

$$\begin{aligned}(11)\quad & f(y) - T_\varphi^{m-1} f(y) \\ &= m \int_0^1 \sum_{|\alpha|=m} \int_Q D^\alpha f(x+(y-x)t) \frac{(y-x)^\alpha}{\alpha!} \varphi(x)\, dx\, (1-t)^{m-1}\, dt\,,\end{aligned}$$

or, in an equivalent form,

$$\begin{aligned}(12)\quad f(y) - T_\varphi^m f(y) = m \int_0^1 \sum_{|\alpha|=m} \int_Q & (D^\alpha f(x+(y-x)t) - D^\alpha f(x)) \\ & \times \frac{(y-x)^\alpha}{\alpha!} \varphi(x)\, dx\, (1-t)^{m-1}\, dt.\end{aligned}$$

Now we prove

LEMMA 4. *Let $f \in W^{m,1}(Q)$. If $M_1|\nabla^m f|(y) < \infty$, then the limit*

$$\lim_{r\to 0} \fint_{B(y,r)} f(x)\, dx$$

exists and is finite. If we set $f(y)$ equal to this limit (compare (1)), *then formulas* (11) *and* (12) *hold at y.*

Proof. For notational reasons we assume that $\operatorname{diam} Q = 1$. We will be concerned only with (12). The proof in this case includes (11) as well. We can assume that f is defined in a neighborhood of Q. Let f_ε be a standard convolution approximation of f (with a C_0^∞ kernel). First we prove that the right hand side of (12) applied to f_ε converges as $\varepsilon \to 0$ to the same expression but with f_ε replaced by f. This will be a direct consequence of the Lebesgue Dominated Convergence Theorem if we prove that for $|\alpha| = m$ the integrals

$$\int_Q |D^\alpha f_\varepsilon(x+(y-x)t)|\, dx \quad \text{and} \quad \int_Q |D^\alpha f_\varepsilon(x)|\, dx$$

are bounded by a constant independent of ε and t.

The estimate for the second integral follows directly from

LEMMA 5. *If g_ε is a convolution approximation of g and E is a measurable set, then*

$$\int_E |g_\varepsilon(x)|\, dx \le C \int_{E_\varepsilon} |g(x)|\, dx\,,$$

where $E_\varepsilon = \{x : \operatorname{dist}(x, E) \le \varepsilon\}$.

Proof.

$$\int_E |g_\varepsilon(x)|\,dx = \int_E \Big|\int g(y)\psi_\varepsilon(x-y)\,dy\Big|\,dx \le C\varepsilon^{-n}\int_E \int_{B(x,\varepsilon)} |g(y)|\,dy\,dx\,.$$

This inequality follows from the observation that $\operatorname{supp}\psi_\varepsilon \subset B(0,\varepsilon)$ and $|\psi_\varepsilon| \le C\varepsilon^{-n}$. If $x \in E$ and $y \in B(x,\varepsilon)$, then $y \in E_\varepsilon$ and $x \in B(y,\varepsilon)$. Hence, using Fubini's theorem, the lemma follows.

Now we estimate the first integral. We have

$$\int_Q |D^\alpha f_\varepsilon(x+(y-x)t)|\,dx = (1-t)^{-n}\int_{J_y^{1-t}Q} |D^\alpha f_\varepsilon(z)|\,dz\,,$$

where J_a^s denotes the homothety with center a and scale factor s.

If $1-t > \varepsilon$, then it follows readily from Lemma 5 and $M_1|\nabla^m f|(y) < \infty$ that this integral is bounded by a constant independent of ε and t.

If $1-t \le \varepsilon$, then

$$\begin{aligned}(1-t)^{-n}&\int_{J_y^{1-t}Q} |D^\alpha f_\varepsilon(z)|\,dz \\ &\le C(1-t)^{-n}\varepsilon^{-n}\int_{J_y^{1-t}Q}\int_{B(z,\varepsilon)} |D^\alpha f(v)|\,dv\,dz \\ &\le C'(1-t)^{-n}\varepsilon^{-n}(2\varepsilon)^n\int_{J_y^{1-t}Q}\fint_{B(y,2\varepsilon)} |D^\alpha f(v)|\,dv\,dz \\ &\le C'' \qquad (\text{notice that } B(z,\varepsilon)\subseteq B(y,2\varepsilon))\,.\end{aligned}$$

We have proved the desired convergence. Hence the left hand side of (12) is convergent and, in consequence, $f_\varepsilon(y)$ is convergent as $\varepsilon \to 0$ and the limit is independent of the choice of the C_0^∞ convolution kernel. The averaged integral $\fint_{B(y,r)} f(x)\,dx$ is a convolution with kernel $\omega = |B|^{-1}\chi_B$. Since such a kernel can be approximated by C_0^∞ kernels, one can prove that the formula analogous to (12) holds if we replace $D^\alpha f$ with $(D^\alpha f)*\omega_\varepsilon$ for $|\alpha| \le m$ in both sides of this inequality. Namely, we write (12) for $f*\psi_k$ instead of f, where ψ_k is a sequence of C_0^∞ kernels approximating ω. Then we pass to the limit as $k \to \infty$.

Now we can prove, just as in the smooth kernel case, that the limit of $(f*\omega_\varepsilon)(y)$ exists and is equal to that for smooth kernels. This ends the proof of Lemma 4.

Remark. One can prove Theorem 3 using Lemma 4 instead of Theorem 1 and Lemma 3.

Formula (11) plays a key role in the proof of the so-called integral representation formula for Sobolev functions ([KA], p. 437, Lemma 1a; [Bu]; [Ma], Th. 1.1.10).

It is easy to show that (11) holds a.e. for a Sobolev function. We have proved much more. Namely we have exhibited the points (probably not all) at which this formula holds. Hence we are in a position to prove the following stronger form of the integral representation theorem.

THEOREM 4. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain, starshaped with respect to a ball $B \subseteq \Omega$. Let $\varphi \in C_0^\infty(B)$, $\int \varphi = 1$. There exists a smooth, bounded function $\omega_\varphi \in C^\infty(\Omega \times \Omega)$ such that if $f \in W^{m,p}(\Omega)$ and*

$$M_1|\nabla^m f|(y) < \infty\,,$$

then

$$f(y) - T_\varphi^{m-1} f(y) = \int\limits_\Omega \sum_{|\alpha|=m} D^\alpha f(x) \frac{(y-x)^\alpha}{|y-x|^n} \omega_\varphi(y,x)\, dx\,.$$

Remark. To be more accurate, we should replace $M_1|\nabla^m f|(y)$ by $M_d|\nabla^m f|(y)$, where $d = \operatorname{dist}(y, \partial\Omega)$.

Proof of Theorem 4. This follows easily from formula (11) by a linear change of variables in the integral on the right hand side. The details can be found in [KA], pp. 438–439, [Bu] or, in a slightly modified form, in [Ma], Th. 1.1.10.

Remark. Following the same ideas as in the proof of Theorem 2 we can obtain Taylor's formula for Sobolev functions. Namely, we can represent Taylor's remainder $f(y) - T_x^{m-1} f(y)$ as a potential type integral operator involving derivatives of the highest order m only.

4. Bessel capacity and Lebesgue points. In this section we recall some results concerning Bessel capacity. In the previous sections we were concerned with the set of Lebesgue points of a Sobolev function. It is well known that almost all points of the domain of a locally integrable function are Lebesgue points. In the case of Sobolev functions we can say more. Namely, we prove (see Theorem 8 below) Ziemer's theorem ([Z1], [Z2], Th. 3.10.2 and Remark 3.10.3), which generalizes the fact that if $f \in W^{k,p}$, then $B_{k,p}$-almost all points are Lebesgue points, where $B_{k,p}$ denotes Bessel capacity (Corollary 4).

The results of this section (with almost the same proofs) can be found in [Z2]. We added this section for the sake of completeness.

Let G_α, $\alpha > 0$, be the kernel of the operator $(1-\Delta)^{-\alpha/2}$.

Let $L^{\alpha,p}(\mathbb{R}^n) = \{G_\alpha * g : g \in L^p(\mathbb{R}^n)\}$ denote the space of Bessel potentials. If $f \in L^{\alpha,p}$, $f = G_\alpha * g$, then we define the norm of f as $\|f\|_{\alpha,p} = \|g\|_p$.

THEOREM 5. *If $k \in \mathbb{N}$ and $1 < p < \infty$, then*

$$L^{k,p}(\mathbb{R}^n) = W^{k,p}(\mathbb{R}^n).$$

The proof of this classical theorem can be found in [S], Chapter 5, Th. 3.

The *Bessel capacity* of a set E is defined by

$$B_{\alpha,p}(E) = \inf\{\|g\|_p^p : G_\alpha * g \ge 1 \text{ in } E,\ g \ge 0\}.$$

Sets of zero $B_{\alpha,p}$ capacity have Hausdorff dimension less than or equal to $n - \alpha p$. Namely, one can prove the following

THEOREM 6. *If $\alpha p < n$, then*

$$H_{n-\alpha p}(E) < \infty \ \Rightarrow\ B_{\alpha,p}(E) = 0,$$
$$B_{\alpha,p}(E) = 0 \ \Rightarrow\ \forall_{\varepsilon>0}\ H_{n-\alpha p+\varepsilon}(E) = 0.$$

If $\alpha p > n$, then there exists a constant $C > 0$ such that

$$E \ne \emptyset \ \Rightarrow\ B_{\alpha,p}(E) > C$$

(H_s denotes Hausdorff measure).

The proof can be found in [Me], Theorems 20 and 21.

In the sequel we need the following

THEOREM 7. *If $1 < p < \infty$, $\alpha > 0$, and $f \in L^{\alpha,p}(\mathbb{R}^n)$, then*

$$B_{\alpha,p}(\{x : Mf(x) > t\}) \le \frac{C}{t^p}\|f\|_{\alpha,p}^p$$

where the constant C depends on p and n only.

Proof. Let $f = G_\alpha * g$, $\|f\|_{\alpha,p} = \|g\|_p$. Let $\omega_r = |B(0,r)|^{-1}\chi_{B(0,r)}$. We have

$$\fint_{B(x,r)} |f(y)|\,dy = \omega_r * |f|(x) \le \omega_r * G_\alpha * |g|(x)$$
$$= G_\alpha * \omega_r * |g|(x) \le G_\alpha * Mg(x).$$

Hence

$$Mf(x) \le G_\alpha * Mg(x).$$

And so, by the definition of $B_{\alpha,p}$,

$$B_{\alpha,p}(\{Mf > t\}) \le B_{\alpha,p}(\{G_\alpha * Mg > t\}) \le \|M(g/t)\|_p^p \le \frac{C}{t^p}\|f\|_{\alpha,p}^p.$$

The last inequality follows from the Hardy–Littlewood maximal function theorem.

THEOREM 8. *If $1 < p < \infty$, $\alpha > 0$, and $f \in L^{\alpha,p}(\mathbb{R}^n)$, then for every $\varepsilon > 0$ there exists an open set $U \subset \mathbb{R}^n$ such that $B_{\alpha,p}(U) < \varepsilon$ and*

$$M_R^\sharp f \xrightarrow[R\to 0]{} 0$$

uniformly in $\mathbb{R}^n \setminus U$.

Remark. The maximal function $M_R^\sharp f$ is defined everywhere since f is defined everywhere by (1).

Proof of Theorem 8. There exists $g \in C_0^\infty(\mathbb{R}^n)$ such that

$$\|f-g\|_{\alpha,p}^p < \varepsilon^{p+1}.$$

Let $h = f - g$. We have

$$M_R^\sharp f(x) \le M_R^\sharp g(x) + M_R^\sharp h(x) \le M_R^\sharp g(x) + 2Mh(x).$$

There exists $R > 0$ such that

$$M_R^\sharp g(x) < \varepsilon \quad \text{for every } x \in \mathbb{R}^n.$$

Hence, as follows from Theorem 7, we have

$$B_{\alpha,p}(\{M_R^\sharp f > 3\varepsilon\}) \le B_{\alpha,p}(\{2Mh > \varepsilon\}) < C\varepsilon.$$

If $\varepsilon_i = 2^{-i}\varepsilon/C$ and R_i is taken with respect to ε_i, then

$$B_{\alpha,p}(\{M_{R_i}^\sharp f > \varepsilon_i\}) < \frac{\varepsilon}{3\cdot 2^i}.$$

Let

$$V = \bigcup_i \{M_{R_i}^\sharp f > \varepsilon_i\}.$$

Evidently $B_{\alpha,p}(V) < \frac{1}{3}\varepsilon$.

Now, as is well known, there exists an open set $U \supseteq V$ such that $B_{\alpha,p}(U) < \varepsilon$ (see e.g. [Z2], Lemma 2.6.6).

COROLLARY 4. *If $f \in L^{\alpha,p}$, where $\alpha > 0$ and $1 < p < \infty$, is defined everywhere by* (1), *then $B_{\alpha,p}$-almost all points are Lebesgue points of f.*

5. A new proof of Michael and Ziemer's theorem. In this section we give a new, short proof of Michael and Ziemer's theorem ([MZ], [Z1], [Z2], Th. 3.11.6). This theorem extends an earlier result of Calderón and Zygmund (Corollary 2 in this paper) and of Liu [L]. In the proof we only need Corollary 1, Theorems 7 and 8 from the previous sections and Whitney's extension theorem ([W], [M], Th. 3.2, 3.5). This proof is independent of Section 3 and is based on the proof of a weaker result, given in [H1] (see also [B2]).

THEOREM 9. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set, $1 < p < \infty$, $1 \le m \le k$, m, k integers, and $f \in W^{k,p}_{\mathrm{loc}}(\Omega)$. Then, for every $\varepsilon > 0$, there exists a closed set $F \subset \Omega$ and a function $g \in C^m(\Omega)$ such that*

$$B_{k-m,p}(\Omega \setminus F) < \varepsilon, \tag{13}$$

$$D^\alpha f(x) = D^\alpha g(x) \quad \textit{for } x \in F \textit{ and } |\alpha| \le m, \tag{14}$$

$$f - g \in W_0^{m,p}(\Omega), \tag{15}$$

$$\|f-g\|_{m,p} < \varepsilon. \tag{16}$$

Remarks. 1) The derivatives $D^\alpha f$ are defined everywhere by (4).

2) $W_0^{m,p}(\Omega)$ stands for the completion of $C_0^\infty(\Omega)$ in the $W^{m,p}$ norm.

Proof of Theorem 9. First assume that $f \in W^{k,p}(\mathbb{R}^n)$ and f has a compact support in $\frac{1}{10}Q$ (the cube with the same center as Q and with $1/10$ of its side length). Let

$$E_s = \Big\{x : \sum_{|\alpha|\le m} M(D^\alpha f)(x) \le s\Big\}.$$

Evidently $|\mathbb{R}^n \setminus E_s|s^p \to 0$ as $s \to \infty$ (the maximal function belongs to L^p and this convergence follows directly from the Chebyshev inequality). Let U^s be the set, defined as in Theorem 8, with the following properties:

$$M_R^\sharp(D^\alpha f) \underset{R\to 0}{\longrightarrow} 0$$

uniformly in $\mathbb{R}^n \setminus U^s$ for all $|\alpha| = m$,

$$B_{k-m,p}(U^s) < \frac{1}{s},$$

and

$$|Q \cap U^s| < \frac{1}{s^{p+1}}. \tag{17}$$

Inequality (17) can be guaranteed since it follows easily from the fact that sets of zero capacity are also of zero Lebesgue measure. It also follows directly from the more sophisticated fact that a suitable power of the Lebesgue measure is dominated by $B_{\alpha,p}$ ([Me], Th. 20).

Let $E'_s = E_s \cap (Q \setminus U^s)$. Corollary 1 implies that $(D^\alpha f|_{E'_s})_{|\alpha|\le m}$ satisfies the assumptions of Whitney's extension theorem ([W], [M], Th. 3.2). Obviously,

$$|Q \setminus E'_s|s^p \to 0 \quad \text{as } s \to \infty. \tag{18}$$

Moreover,

$$B_{k-m,p}(Q \setminus E'_s) \to 0 \quad \text{as } s \to \infty.$$

This follows easily from the definition of the E'_s and from Theorem 7.

Now we estimate Whitney's norm of f on E'_s (see [M], Section 2.3 for notations):

$$\|f\|_m^{E'_s} = \sup_{\substack{|\alpha|\le m\\ x\in E'_s}} |D^\alpha f(x)| + \sup_{\substack{|\alpha|\le m\\ x,y\in E'_s\\ x\ne y}} \frac{|D^\alpha f(y) - T_x^{m-|\alpha|}D^\alpha f(y)|}{|x-y|^{m-|\alpha|}}.$$

We have $|D^\alpha f| \le s$ in E'_s (because $|D^\alpha f| \le M(D^\alpha f)$). Moreover, as follows from Corollary 1,

$$\sup \frac{|D^\alpha f(y) - T_x^{m-|\alpha|} D^\alpha f(y)|}{|x-y|^{m-|\alpha|}} \le C \sup_{E'_s} M^\sharp |\nabla^m f| \le C \sup_{E'_s} 2M|\nabla^m f| \le C' s\,.$$

Hence

$$\|f\|_m^{E'_s} \le Cs\,.$$

If $\bar{f}$ is a Whitney extension of a function $f \in \mathcal{E}^m(E'_s)$, then

$$\sup_{\substack{x\in Q\\ |\alpha|\le m}} |D^\alpha \bar{f}(x)| \le C\|f\|_m^{E'_s}$$

(see [M], Th. 3.5). Hence

$$|D^\alpha \bar{f}(x)| \le Cs \tag{17}$$

in Q for all $|\alpha| \le m$. The formula which defines $\bar{f}$ (see [M], the beginning of the proof of Th. 3.2) and the fact that $\operatorname{supp} f \subset \frac{1}{10}Q$ implies that $\bar{f}(x) = 0$ in $\mathbb{R}^n \setminus Q$ (for all sufficiently large s). Thus

$$\|f - \bar{f}\|_{m,p}^p = \sum_{|\alpha|\le m} \int_{Q\setminus E'_s} |D^\alpha f - D^\alpha \bar{f}|^p \le 2^{p-1} \sum_{|\alpha|\le m} \Big(\int_{Q\setminus E'_s} |D^\alpha f|^p + \int_{Q\setminus E'_s} |D^\alpha \bar{f}|^p \Big) \xrightarrow[s\to\infty]{} 0\,.$$

This convergence follows from two facts:

1. $\int_{Q\setminus E'_s} |D^\alpha f|^p \to 0$, because $|Q \setminus E'_s| \to 0$,
2. $\int_{Q\setminus E'_s} |D^\alpha \bar{f}|^p \le (Cs)^p |Q \setminus E'_s| \to 0$ (see (17)).

The general case can be reduced to the case with compact support by a standard partition of unity argument as in Meyers–Serrin's theorem ([MS], [Ma], Th. 1.1.5/1, [H3], Th. 1).

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