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A new characterization of the Sobolev space

by

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Abstract. The purpose of this paper is to provide a new characterization of the Sobolev space $W^{1,1}(\mathbb{R}^n)$. We also show a new proof of the characterization of the Sobolev space $W^{1,p}(\mathbb{R}^n)$, $1 \le p < \infty$, in terms of Poincaré inequalities.

The Sobolev space $W^{1,p}(\mathbb{R}^n)$, $1 \le p < \infty$, consists of functions $u \in L^p(\mathbb{R}^n)$ such that $|\nabla u| \in L^p(\mathbb{R}^n)$. It is a Banach space with respect to the norm

$$\|u\|_{1,p} = \|u\|_p + \|\nabla u\|_p.$$

Let us point out that from the point of view of Banach spaces the structure of various Sobolev type spaces, with the particular emphasis on $W^{1,1}$, has been investigated by A. Pełczyński, M. Wojciechowski and others; see e.g. [4], [40]–[43] and references therein.

The purpose of this paper is to provide a new characterization of the Sobolev space $W^{1,1}(\mathbb{R}^n)$. Here, however, we emphasize future applications in geometric analysis and analysis on metric spaces rather than the theory of Banach spaces. Actually one of the reasons for finding new characterizations of the Sobolev space is the development of analysis on metric spaces; see e.g. [1], [2], [5], [6], [12], [13], [15]–[19], [21], [23], [24], [28]–[31], [38], [47]–[49], [51]. More references will be given later. In order to define a Sobolev type space on a metric-measure space we need a characterization of the space $W^{1,p}(\mathbb{R}^n)$ that does not involve derivatives. One such characterization is given in the following result.

THEOREM 1 ([19]). *$u \in W^{1,p}(\mathbb{R}^n)$, $1 < p < \infty$, if and only if $u \in L^p(\mathbb{R}^n)$ and there exists $0 \le g \in L^p(\mathbb{R}^n)$ such that*

$$|u(x) - u(y)| \le |x - y|(g(x) + g(y)) \quad a.e. \tag{1}$$

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Moreover

$$\|\nabla u\|_p \approx \inf_g \|g\|_p,$$

where the infimum is taken over the class of functions g satisfying (1).

Inequality (1) holds a.e. in the sense that there is a set $E \subset \mathbb{R}^n$ of measure zero such that (1) holds for all $x, y \in \mathbb{R}^n \setminus E$. Writing $A \approx B$ we mean that there is a constant $C \ge 1$ such that $C^{-1}B \le A \le CB$.

Let us note that yet another characterization of the Sobolev space has been obtained recently in [9] and [10].

The above theorem was a point of departure in [19] for the definition of a Sobolev space on an arbitrary metric space equipped with a locally finite Borel measure. For further results involving this approach see e.g. [3], [8], [13], [15], [16], [18], [20]–[22], [24]–[30], [32]–[37], [39], [44], [45], [49], [52], [53].

If $u \in W^{1,p}(\mathbb{R}^n)$, $1 \le p < \infty$, then we have an elementary inequality ([7], [19])

$$(2)\qquad |u(x)-u(y)| \le C(n)|x-y|(M_{2|x-y|}|\nabla u|(x)+M_{2|x-y|}|\nabla u|(y)) \quad \text{a.e.}$$

where

$$M_R g(x) = \sup_{r<R} \fint_{B(x,r)} |g(z)|\,dz$$

is the restricted Hardy–Littlewood maximal function. Here and in what follows the integral average of a function u over a set E is denoted by

$$u_E = \fint_E u\,dx = \frac{1}{|E|}\int_E u\,dx,$$

where $|E|$ denotes Lebesgue measure of E. Moreover C will always stand for a general constant that can change its value even in the same string of estimates. Writing $C = C(n)$ we will emphasize that the constant depends on n only.

If we take $R = \infty$ in the definition of M_R, then we obtain the classical Hardy–Littlewood maximal function

$$Mu(x) = \sup_{r>0} \fint_{B(x,r)} |u|\,.$$

Hence it follows from inequality (2) that

$$|u(x)-u(y)| \le C|x-y|(M|\nabla u|(x) + M|\nabla u|(y)) \quad \text{a.e.}$$

This and the boundedness of the maximal function in L^p for $p > 1$ (see [50]) imply (1) with $g = CM|\nabla u| \in L^p(\mathbb{R}^n)$. The implication in the opposite direction in Theorem 1 follows from the lemma.

LEMMA 2 ([20, Proposition 1]; cf. [28, Remark 5.13]). *If $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $0 \le g \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ satisfy inequality* (1) *a.e., then $\nabla u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ in* *the weak sense and*

$$|\nabla u| \le C(n) g \quad a.e. \tag{3}$$

This lemma is relatively easy and its proof is based on the observation that (1) implies absolute continuity of u on almost all lines parallel to coordinate axes. If we know in addition that $g \in L^p(\mathbb{R}^n)$, then inequality (3) implies that $|\nabla u| \in L^p(\mathbb{R}^n)$, which completes the proof of Theorem 1.

Observe that the only place in the proof of Theorem 1 where the assumption $p > 1$ was employed was the application of the boundedness of the maximal function in L^p. It turns out that the assumption $p > 1$ is essential because Theorem 1 does not hold for $p = 1$. This follows from the next example.

EXAMPLE 3 ([20]). Let $\Omega = (-1/2, 1/2)$ and $u(x) = -x/(|x| \log |x|)$. Then $u \in W^{1,1}(\Omega)$ because $u'(x) = |x|^{-1}(\log |x|)^{-2} \in L^1(\Omega)$. Suppose now that there exists $0 \le g \in L^1(-1/2, 1/2)$ such that (1) holds a.e. Then for a.e. $0 < x < 1/2$ we have $|u(x) - u(-x)| \le 2x(g(x) + g(-x))$ and hence

$$\frac{-2}{\log x} \le 2x(g(x) + g(-x)),$$

which, in turn, yields

$$\int_{-1/2}^{1/2} g(x)\, dx = \int_0^{1/2} (g(x) + g(-x))\, dx \ge \int_0^{1/2} \frac{-dx}{x \log x} = \infty.$$

This contradicts integrability of g. The function u is defined on the interval $(-1/2, 1/2)$ only, but one can extend it to a function in $W^{1,1}(\mathbb{R})$ to fit into the setting of Theorem 1.

The main result of the present paper is the following characterization of $W^{1,1}(\mathbb{R}^n)$.

THEOREM 4. *$u \in W^{1,1}(\mathbb{R}^n)$ if and only if $u \in L^1(\mathbb{R}^n)$ and there exist $0 \le g \in L^1(\mathbb{R}^n)$ and $\sigma \ge 1$ such that*

$$|u(x) - u(y)| \le |x - y|(M_{\sigma|x-y|} g(x) + M_{\sigma|x-y|} g(y)) \quad a.e. \tag{4}$$

Moreover if (4) *holds*, *then* $|\nabla u| \le C(n, \sigma) g$ *a.e.*

The implication from left to right follows from the elementary inequality at (2). It turns out, however, that the implication from right to left is much more difficult than the corresponding one in Theorem 1.

The proof that inequality (4) implies $u \in W^{1,1}(\mathbb{R}^n)$ is split into two steps. In the first step we prove that (4) implies the family of Poincaré type inequalities

$$⨍_B |u - u_B| \le Cr ⨍_{3\sigma B} g \tag{5}$$

for every ball B of any radius r. Here and in what follows, $3\sigma B$ denotes the ball concentric with B and with radius 3σ times that of B. Observe that inequality (5) with $3\sigma B$ replaced by B would readily follow from (1) upon integration over $x, y \in B$. In our situation, however, we cannot integrate (4) because the maximal function of an L^1 function need not be integrable. This is the main difficulty in the proof and, actually, this first step is the main new ingredient in the proof.

In the second step we show that the family of inequalities (5) on every ball B imply that $u \in W^{1,1}(\mathbb{R}^n)$ with $|\nabla u| \le Cg$ a.e. This implication has previously been proved in [15]. Here we provide a new, simpler proof.

Both steps are direct consequences of the following more general results applied to $p = 1$. We will prove the lemmas in the whole generality, i.e. for all $1 \le p < \infty$. This way we will clearly see what kind of new difficulties have to be faced when passing from the case $p > 1$ to $p = 1$.

LEMMA 5. *Let $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, $0 \le g \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$, $1 \le p < \infty$, and $\sigma \ge 1$. Then the inequality*

$$|u(x)-u(y)| \le |x-y|((M_{\sigma|x-y|}g^p(x))^{1/p}+(M_{\sigma|x-y|}g^p(y))^{1/p}) \quad a.e. \tag{6}$$

implies that

$$\fint_B |u-u_B| \le C(n,p,\sigma)r\Big(\fint_{3\sigma B} g^p\Big)^{1/p} \tag{7}$$

for every ball B of any radius r.

For $p > 1$ Lemma 5 was proved in [24] and [29]. The case $p = 1$ turns out to be much more difficult.

LEMMA 6. *Assume that $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, $0 \le g \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$, $1 \le p < \infty$, and $\sigma \ge 1$ are such that*

$$\fint_B |u-u_B| \le r\Big(\fint_{\sigma B} g^p\Big)^{1/p} \tag{8}$$

for every ball B of any radius r. Then $u \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$ and

$$|\nabla u| \le C(n)g \quad a.e.$$

If $u \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$, then by the classical Poincaré inequality [14, p. 142], we have

$$\fint_B |u-u_B| \le C(n)r\fint_B |\nabla u| \le Cr\Big(\fint_{\sigma B} |\nabla u|^p\Big)^{1/p},$$

which shows that we also have the opposite implication in Lemma 6, and thus (8) is a necessary and sufficient condition for u to be in $W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$.

As already mentioned, the case $p = 1$ of Lemma 6 was proved in [15]. The case $1 < p < \infty$ was proved earlier in [36]. If we assume, however, that

$1 \le p < \infty$, and in addition $g \in L^q$ for some $q > p$, then Lemma 6 follows essentially from the work of Calderón [11].

Proof of Lemma 5. All the constants C in the proof will depend on n, p and σ only. First let us sketch the proof for $p > 1$. We will clearly see why this proof cannot be extended to the case $p = 1$.

It follows from (6) that for $x, y \in B$,

$$|u(x) - u(y)| \le |x - y|((M(g^p \chi_{3\sigma B})(x))^{1/p} + (M(g^p \chi_{3\sigma B})(y))^{1/p}) \quad \text{a.e.}$$

Employing the Cavalieri principle we obtain

$$\begin{aligned}
\fint_B |u - u_B| &\le \fint_B \fint_B |u(x) - u(y)|\, dx\, dy \\
&\le 4r \fint_B (M(g^p \chi_{3\sigma B})(x))^{1/p}\, dx \\
&= 4r|B|^{-1} \int_0^\infty |\{x \in B;\ (M(g^p \chi_{3\sigma B})(x))^{1/p} > t\}|\, dt \\
&= 4r|B|^{-1} \Big(\int_0^{t_0} + \int_{t_0}^\infty \Big).
\end{aligned}$$

For $0 < t \le t_0$ we estimate the integrand by the measure of the ball B, and for $t > t_0$ we estimate it by the weak type estimate for the maximal function (see [50]). This gives

$$\begin{aligned}
\fint_B |u - u_B| &\le 4r|B|^{-1} \Big(\int_0^{t_0} |B|\, dt + \int_{t_0}^\infty \Big(\frac{C(n)}{t^p} \int_{3\sigma B} g^p \Big) dt \Big) \\
&= 4r|B|^{-1} \Big(t_0|B| + C(n,p) t_0^{1-p} \int_{3\sigma B} g^p \Big).
\end{aligned}$$

Now taking $t_0 = (|B|^{-1} \int_{3\sigma B} g^p)^{1/p}$ yields the result. Observe that the assumption $p > 1$ was employed to integrate t^{-p} from t_0 to ∞.

Now assume that $p = 1$. We will employ some ideas from the proof of the Sobolev embedding theorem in [19]. Fix a ball B. Replacing u by $u - b$, where b is any constant, will not affect inequalities (6) and (7). Hence by subtracting a suitable constant from u we can assume that $\operatorname{ess\,inf}_E |u| = 0$, where $E \subset B$ is any set of positive Lebesgue measure. The set E will be chosen later.

For $x, y \in B$ we have

$$|u(x) - u(y)| \le |x - y|(Mh(x) + Mh(y)), \tag{9}$$

where $h = g\chi_{3\sigma B}$. Now it suffices to prove that

$$(10) \qquad \fint_B |u - u_B| \le Cr \fint_{3\sigma B} h.$$

If $h = 0$ a.e. then u is constant in B and hence (10) follows. Thus we can assume that $h > 0$ on a set of positive measure and hence $\int_{3\sigma B} h > 0$. Moreover we can assume that

$$(11) \qquad h \ge \frac{1}{2} \fint_{3\sigma B} h > 0 \text{ on } 3\sigma B \quad \text{and} \quad h = 0 \text{ on } \mathbb{R}^n \setminus 3\sigma B,$$

otherwise we replace h by

$$h + \Big(\fint_{3\sigma B} h \Big) \chi_{3\sigma B}.$$

We will prove that any integrable function h with properties (9) and (11) satisfies (10) as well. For $k \in \mathbb{Z}$ let

$$E_k = \{x \in B;\ Mh(x) \le 2^k\} \quad \text{and} \quad a_k = \sup_{E_k} |u|.$$

Since

$$(12) \qquad \int_B |u - u_B| \le 2 \int_B |u| \le 2 \sum_{k=-\infty}^{\infty} a_k |E_k \setminus E_{k-1}|$$

we need to find good estimates for a_k in order to estimate the left hand side of (10).

From $E_k \subseteq E_{k+1}$ it follows that $a_k \le a_{k+1}$. In order to estimate the growth of a_k we have to estimate a_k in terms of a_{k-1} first. By (9) the function u restricted to E_k is 2^{k+1}-Lipschitz. Hence for $x \in E_k$ and $y \in E_{k-1}$ we have

$$(13) \qquad |u(x)| \le |u(x) - u(y)| + |u(y)| \le 2^{k+1}|x - y| + a_{k-1}.$$

To obtain a good estimate for the right hand side we have to show that for a given $x \in E_k$ there exists $y \in E_{k-1}$ with a relatively small distance to x, $|x - y|$. Choose $x \in E_k$ arbitrarily and observe that

$$(14) \qquad |B(x,r) \cap B| \ge \omega_n (r/2)^n \quad \text{for } r \le \operatorname{diam} B.$$

Here ω_n denotes the volume of the unit ball. Assume that $|E_{k-1}| > 0$. Choose $r < \operatorname{diam} B$ such that

$$(15) \qquad \omega_n (r/2)^n > |B \setminus E_{k-1}|.$$

Then (14) and (15) imply that there exists $y \in E_{k-1}$ such that $|x - y| < r$. Since the lower bound for r satisfying (15) is $2\omega_n^{-1/n}|B \setminus E_{k-1}|^{1/n}$ we conclude from (13) upon taking the supremum over $x \in E_k$ that

$$a_k \le a_{k-1} + C 2^k |B \setminus E_{k-1}|^{1/n}.$$

Invoking the weak type estimate for the maximal function (see [50]), we obtain

$$|B \setminus E_{k-1}| = |\{x \in B;\, Mh(x) > 2^{k-1}\}| \leq \frac{C}{2^k} \fint_{3\sigma B} h\,,$$

and hence

$$a_k \leq a_{k-1} + C2^{k(1-1/n)}\Big(\fint_{3\sigma B} h\Big)^{1/n}.$$

Assume now that $n \geq 2$. The case $n = 1$ can be treated in a similar way, we leave the details to the reader.

Iterating this inequality yields

$$\begin{aligned} a_k &\leq a_{k_0} + C\Big(\sum_{i=k_0+1}^{k} 2^{i(1-1/n)}\Big)\Big(\fint_{3\sigma B} h\Big)^{1/n} \\ &\leq a_{k_0} + C2^{k(1-1/n)}\Big(\fint_{3\sigma B} h\Big)^{1/n} \end{aligned} \tag{16}$$

for $k > k_0$, provided $|E_{k_0}| > 0$. For $k \leq k_0$ we will use the estimate $a_k \leq a_{k_0}$. Choose k_0 such that

$$|E_{k_0-1}| < |B|/2 \leq |E_{k_0}|. \tag{17}$$

Such a k_0 exists because $E_k = \emptyset$ for sufficiently small k, due to the lower bound (11) for h, and $|E_k| \to |B|$ as $k \to \infty$. Since $E_{k_0} \neq \emptyset$, there exists $x \in B$ such that

$$\frac{1}{2}\fint_{3\sigma B} h \leq Mh(x) \leq 2^{k_0}. \tag{18}$$

The left inequality at (18) follows from (11). On the other hand the left inequality at (17) along with the weak type estimates for the maximal function implies

$$\frac{|B|}{2} < |B \setminus E_{k_0-1}| = |\{x \in B;\, Mh(x) > 2^{k_0-1}\}| \leq \frac{C}{2^{k_0}} \int_{3\sigma B} h. \tag{19}$$

The two inequalities (18) and (19) yield

$$2^{k_0} \approx \fint_{3\sigma B} h. \tag{20}$$

As mentioned at the beginning of the proof, we can assume that $\operatorname{ess\,inf}_{E_{k_0}} |u| = 0$. Since u is 2^{k_0+1}-Lipschitz on E_{k_0} we have

$$a_{k_0} \leq 2^{k_0+1} \operatorname{diam} B \leq C \operatorname{diam} B \fint_{3\sigma B} h\,. \tag{21}$$

Now (12) together with the estimates (16), (20) and (21) implies

$$
\begin{aligned}
\int_B |u-u_B| &\le 2\sum_{k=-\infty}^{\infty} a_k |E_k\setminus E_{k-1}| \\
&\le 2\Big(\sum_{k=-\infty}^{k_0} a_{k_0}|E_k\setminus E_{k-1}| \\
&\qquad + \sum_{k=k_0+1}^{\infty}\Big(a_{k_0} + C2^{k(1-1/n)}\Big(\int_{3\sigma B} h\Big)^{1/n}\Big)|E_k\setminus E_{k-1}|\Big) \\
&\le 2\Big(\sum_{k=-\infty}^{\infty} a_{k_0}|E_k\setminus E_{k-1}| + C\Big(\int_{3\sigma B} h\Big)^{1/n}\sum_{k=k_0+1}^{\infty} 2^{k(1-1/n)}|B\setminus E_{k-1}|\Big) \\
&\le C\operatorname{diam} B\Big(\fint_{3\sigma B} h\Big)|B| + C\Big(\int_{3\sigma B} h\Big)^{1/n}\sum_{k=k_0+1}^{\infty} 2^{k(1-1/n)}2^{-k}\int_{3\sigma B} h \\
&\le C\operatorname{diam} B\int_{3\sigma B} h,
\end{aligned}
$$

which completes the proof of Lemma 5.

Proof of Lemma 6. At the beginning we will follow the argument of Calderón [11, Theorem 4] (cf. [19, proof of Theorem 1]); then, however, we have to use different ideas because Calderón's argument relies on the $L^{q/p}$ integrability of the maximal function of g^p under the additional assumption that $g\in L^q$ for some $q>p$.

All the constants C in the proof will depend on n only.

Let $\psi\in C_0^\infty(B^n(0,1))$ with $\psi\ge 0$ and $\int\psi = 1$ be a generating mollifier. As usual we set $\psi_\varepsilon(x) = \varepsilon^{-n}\psi(x/\varepsilon)$ and consider a smooth approximation of u defined by $u*\psi_\varepsilon$.

The distributional derivative $\partial u/\partial x_i$, $i=1,\dots,n$, is a functional on $C_0^\infty(\mathbb{R}^n)$ defined by

$$
\frac{\partial u}{\partial x_i}[\varphi] := -\int u\,\frac{\partial\varphi}{\partial x_i}\qquad \text{for } \varphi\in C_0^\infty(\mathbb{R}^n).
$$

Note that

$$
\frac{\partial u}{\partial x_i}[\varphi] = -\lim_{\varepsilon\to 0}\int (u*\psi_\varepsilon)\,\frac{\partial\varphi}{\partial x_i} = \lim_{\varepsilon\to 0}\int\Big(u*\frac{\partial\psi_\varepsilon}{\partial x_i}\Big)\varphi. \tag{22}
$$

Since $\int\partial\psi_\varepsilon/\partial x_i = 0$ we conclude that

$$
\Big(u*\frac{\partial\psi_\varepsilon}{\partial x_i}\Big)(x) = (u-u_{B(x,\varepsilon)})*\frac{\partial\psi_\varepsilon}{\partial x_i}(x)
$$

and hence

$$\left|u * \frac{\partial \psi_\varepsilon}{\partial x_i}\right|(x) \le C \int_{B(x,\varepsilon)} |u(y) - u_{B(x,r)}|\, dy \cdot \varepsilon^{-n-1} \le C\Big(\fint_{B(x,\sigma\varepsilon)} g^p\Big)^{1/p}.$$

For a compact set $K \subset \mathbb{R}^n$ we have

$$\begin{aligned}(23)\qquad \int_K \left|u * \frac{\partial \psi_\varepsilon}{\partial x_i}\right|^p &\le C^p \int_K \fint_{B(x,\sigma\varepsilon)} g^p(y)\, dy\, dx \\ &= C^p \omega_n^{-1} (\sigma\varepsilon)^{-n} \int_{K_{\sigma\varepsilon}} g^p(y) \int_K \chi_{B(y,\sigma\varepsilon)}(x)\, dx\, dy \\ &\le C^p \int_{K_{\sigma\varepsilon}} g^p(y)\, dy,\end{aligned}$$

where $K_{\sigma\varepsilon}$ is the set of points in $\mathbb{R}^n$ with distance to K less than $\sigma\varepsilon$. Now (22), (23) and Hölder's inequality yield

$$\begin{aligned}(24)\qquad \left|\frac{\partial u}{\partial x_i}[\varphi]\right| &\le \liminf_{\varepsilon \to 0} \Big(\int |\varphi|^{p'}\Big)^{1/p'} \Big(\int_{\operatorname{supp}\varphi} \left|u * \frac{\partial \psi_\varepsilon}{\partial x_i}\right|^p\Big)^{1/p} \\ &\le C\Big(\int |\varphi|^{p'}\Big)^{1/p'} \Big(\int_{\operatorname{supp}\varphi} g^p\Big)^{1/p},\end{aligned}$$

where $1/p + 1/p' = 1$ with $p' = \infty$ if $p = 1$.

Assume for the time being that $1 < p < \infty$. Fix a ball B. Then (24) applied to $\varphi \in C_0^\infty(B)$ implies that

$$(25)\qquad \varphi \mapsto \frac{\partial u}{\partial x_i}[\varphi]$$

extends to a continuous functional in $(L^{p'}(B))^* = L^p(B)$. Hence $\partial u / \partial x_i \in L^p(B)$ and

$$(26)\qquad \Big(\int_B \left|\frac{\partial u}{\partial x_i}\right|^p\Big)^{1/p} \le C\Big(\int_B g^p\Big)^{1/p}.$$

Inequality (26) applied to balls that converge to Lebesgue points readily shows that

$$\left|\frac{\partial u}{\partial x_i}\right| \le Cg \quad \text{a.e.}$$

In the case $p = 1$ we have

$$(27)\qquad \left|\frac{\partial u}{\partial x_i}[\varphi]\right| \le C\|\varphi\|_\infty \int_{\operatorname{supp}\varphi} g.$$

Hence (25) extends to a continuous linear functional on $C_0(\mathbb{R}^n)$, the space of continuous functions vanishing at infinity. Thus according to the Riesz representation theorem [46],

$$\frac{\partial u}{\partial x_i}[\varphi] = \int \varphi\, d\mu$$

for some signed Radon measure μ. We will show that μ is absolutely continuous with respect to the Lebesgue measure. By contradiction assume that there is a compact set K of Lebesgue measure zero and such that $\mu(K) \neq 0$. Let $\varphi_i \in C_0^\infty(K_{1/i})$ with $0 \leq \varphi_i \leq 1$ and $\varphi_i|_K \equiv 1$ be a decreasing sequence of functions. Here as before $K_{1/i}$ stands for the $1/i$-neighborhood of K. We have

$$0 < |\mu(K)| \leftarrow \left|\int \varphi_i\, d\mu\right| \leq C\|\varphi_i\|_\infty \int_{K_{1/i}} g \to 0$$

as $i \to \infty$, which is a contradiction. Thus according to the Radon–Nikodym theorem and the definition of the weak derivative we have

$$d\mu = \frac{\partial u}{\partial x_i}\, dx, \qquad \frac{\partial u}{\partial x_i} \in L^1_{\mathrm{loc}}\,.$$

Hence inequality (27) implies that

$$\left|\int \varphi \frac{\partial u}{\partial x_i}\right| \leq C\|\varphi\|_\infty \int_B g$$

for every $\varphi \in C_0^\infty(B)$. Since the sign function $\mathrm{sgn}(\partial u/\partial x_i)$ can be approximated in L^1 by $\varphi \in C_0^\infty(B)$ with $\|\varphi\|_\infty \leq 1$ we conclude that

$$\int_B \left|\frac{\partial u}{\partial x_i}\right| \leq C \int_B g$$

for every ball B. Then the argument with Lebesgue points yields

$$\left|\frac{\partial u}{\partial x_i}\right| \leq Cg \quad \text{a.e.}$$

This completes the proof of Lemma 6 and hence that of Theorem 4.

If we put all the results together we obtain the following theorem as a direct consequence.

THEOREM 7. *For $u \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, the following conditions are equivalent*:

(i) $u \in W^{1,p}(\mathbb{R}^n)$.
(ii) *There exists $0 \leq g \in L^p(\mathbb{R}^n)$ such that*

$$⨍_B |u - u_B| \leq r ⨍_B g$$

for every ball B of any radius r.

(iii) *There exist* $0 \le g \in L^p(\mathbb{R}^n)$, $1 \le q \le p$ *and* $\sigma \ge 1$ *such that*

$$\fint_B |u - u_B| \le r\Big(\fint_{\sigma B} g^q\Big)^{1/q}$$

for every ball B *of any radius* r.

(iv) *There exist* $0 \le g \in L^p(\mathbb{R}^n)$, $1 \le q \le p$ *and* $\sigma \ge 1$ *such that*

$$|u(x) - u(y)| \le |x-y|\big((M_{\sigma|x-y|}g^q(x))^{1/q} + (M_{\sigma|x-y|}g^q(y))^{1/q}\big) \quad a.e.$$

Moreover each of the inequalities in (ii)–(iv) *implies that*

$$|\nabla u| \le Cg \quad a.e.$$

Observe that Theorem 1 is not included here. Theorem 7 extends to the case of a regular domain replacing $\mathbb{R}^n$. Part of the implications extend even to the more general setting of metric spaces equipped with a doubling measure (see [15], [22], [21], [24], [29], [36]). For a direct proof of the implication (iii)⇒(iv) see [24, Theorem 3.2].

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