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# APPROXIMATION OF SOBOLEV MAPPINGS

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## 1. INTRODUCTION AND STATEMENT OF RESULT

IN THIS PAPER we assume that $M^m$ and $N^n$ are two smooth, compact, Riemannian manifolds, $\partial N^n = \emptyset$, and additionally we assume that $N^n$ is embedded in $\mathbb{R}^\nu$. For two such manifolds we define

$$W^{1,p}(M^m, N^n) = \{f \in W^{1,p}(M^m, \mathbb{R}^\nu) : f(x) \in N^n \text{ a.e. } x \in M^m\}$$

where $1 \le p < \infty$.

This definition is far from being intrinsic. For an intrinsic definition of $W^{r,p}(M^m, N^n)$ see [1].

In this space, beside the standard topology induced by the norm $\|\cdot\|_{1,p}$, we also have weak topology and weak convergence.

Let $f_k, f \in W^{1,p}(M^m)$, where $1 < p < \infty$. We say that $f_k$ converges to $f$ in weak topology iff $f_k \to f$ in $L^p$ and the set $\{\|\nabla f_k\|_p\}_k$ is bounded. Weak convergence is denoted by $f_k \rightharpoonup f$. It is not difficult to prove that our definition is equivalent to the standard definition of weak convergence in Banach space. We aim to prove the following theorem.

THEOREM 1. If $\pi_1(N^n) = \cdots = \pi_k(N^n) = 0$ ($k$-positive integer) and:

(a) $1 \le p < k + 1$, then $C^\infty(M^m, N^n)$ is dense in $W^{1,p}(M^m, N^n)$ (i.e. in the norm topology);

(b) $p = k + 1$, then $C^\infty(M^m, N^n)$ is sequentially dense for the weak topology in $W^{1,p}(M^m, N^n)$ (i.e. every $W^{1,p}(M^m, N^n)$ mapping is a weak limit of a sequence of smooth mappings).

*Remarks*. (1) Note that usually, weak sequential density and density in the weak topology are two distinct notions. It is not contradictory as weak topology is not metrizable (compare with the remark made at the end of this section).

(2) Note that $k < n$. For if not, the manifold $N^n$ would be contractible, but it is not.

(3) We should recall the theorem of Schoen and Uhlenbeck which, together with the theorem of Bethuel (theorem 3) gives the complete characterization when smooth mappings are dense in $W^{1,p}(M^m, N^n)$ (in the norm topology).

THEOREM 2 [2, 3]. If $p \ge m$ then $C^\infty(M^m, N^n)$ is dense in $W^{1,p}(M^m, N^n)$.

(4) Point (a) in theorem 1 is a special case of a very difficult theorem of Bethuel, but our approach is different from that of Bethuel.

THEOREM 3 [4, theorem 1]. If $1 \le p < m$, then $C^\infty(M^m, N^n)$ is dense in $W^{1,p}(M^m, N^n)$ if and only if $\pi_{[p]}(N^n) = 0$.

The following corollary follows from point (b) of theorem 1.

COROLLARY 1. If $k \ge 2$ is an integer then $C^\infty(M^m, S^k)$ is sequentially dense for the weak topology in $W^{1,k}(M^m, S^k)$. ■

This special case of theorem 1 has already been proved by Bethuel [4]. As far as I know it was the only theorem about the weak sequential density so far.

We should note one more theorem proved by Bethuel in [4]. He proved that if $m > p > 1$ is an integer and $\pi_{[p]}(N^n) \ne 0$, then smooth mappings are dense in $W^{1,p}(M^m, N^n)$ for the weak topology, but he was not able to prove the weak sequential density except the case $N^n = S^p$ (compare with remark 1 made after theorem 1).

## 2. TRIANGULATIONS, SKELETONS AND RETRACTIONS

We introduce some basic notions concerning triangulations, skeletons and discontinuous retractions corresponding to them. We say slightly more than we use in the rest of the paper. In fact the ideas of this paragraph are due to Bojarski [5].

By $\sigma^l$ we denote the standard $l$-dimensional simplex. Let $y \in \operatorname{int} \sigma^l$ and $P_y^l: \sigma^l \setminus \{y\} \to \partial\sigma^l$ be standard retraction–projection longways radii. Obviously $P_y^l \in W^{1,p}(\sigma^l, \partial\sigma^l)$ for $p < l$.

Let $T$ be a finite, smooth triangulation of $N^n$. By $T^l$ we denote the $l$-dimensional skeleton of this triangulation consisting of the simplexes $\{\sigma_i^l\}_{i=1}^{r_l}$ (writing $T^l$ we sometimes will think about a subset of $N^n$ and sometimes about the set $\{\sigma_i^l\}_{i=1}^{r_l}$).

If $y_i^n \in \operatorname{int} \sigma_i^n$ for $i = 1, \dots, r_n$, then we set

$$P^n_{y_1^n, \dots, y_{r_n}^n} = P^n_{y_1^n} \circ \cdots \circ P^n_{y_{r_n}^n} \in W^{1,p}(N^n, T^{n-1}) \qquad \text{for } p < n,$$

where the mappings $P^n_{y_i^n}$ are extended to the whole manifold by the formula $P^n_{y_i^n}(x) = x$ when $x \notin \sigma_i^n$.

This mapping is locally Lipschitz outside the set $W^0 = \{y_1^n, \dots, y_{r_n}^n\}$, where it is not defined.

If $y_i^{n-1} \in \operatorname{int} \sigma_i^{n-1}$ for $i = 1, \dots, r_{n-1}$, then we set

$$P^{n-1}_{y_1^{n-1}, \dots, y_{r_{n-1}}^{n-1}} = P^{n-1}_{y_1^{n-1}} \circ \cdots \circ P^{n-1}_{y_{r_{n-1}}^{n-1}} : T^{n-1} \to T^{n-2},$$

where the mappings $P^{n-1}_{y_i^{n-1}}$ are extended to the whole skeleton as above. Evidently

$$P^{n-1}_{y_1^{n-1}, \dots, y_{r_{n-1}}^{n-1}} \circ P^n_{y_1^n, \dots, y_{r_n}^n} \in W^{1,p}(N^n, T^{n-2})$$

for $p < n - 1$.

This mapping is locally Lipschitz outside the 1-dimensional singularity

$$W^1 = (P^n_{y_1^n, \dots, y_{r_n}^n})^{-1}(\{y_1^{n-1}, \dots, y_{r_{n-1}}^{n-1}\}) \cup W^0$$

(note that this is a disjoint sum as the mapping $P^n_{y_1^n, \dots, y_{r_n}^n}$ is not defined on $W^0$).

Continuing the above construction we obtain the mapping

$$P^{n-k}_{y_1^{n-k}, \dots, y_{r_{n-k}}^{n-k}} \circ \cdots \circ P^n_{y_1^n, \dots, y_{r_n}^n} \in W^{1,p}(N^n, T^{n-k-1}).$$

for $p < n - k$.

This mapping has a $k$-dimensional singularity

$$W^k = (P^{n-k+1}_{\dots} \circ \cdots \circ P^n_{\dots})^{-1}(\{y_1^{n-k}, \dots, y_{r_{n-k}}^{n-k}\}) \cup W^{k-1}.$$

*Remark*. Note that the sets $W^k$ for $k = 0, 1, \dots, n$ are skeletons of dual complex to triangulation $T$. This dual complex is sometimes used in the geometric proof of the Poincaré duality theorem. Note also that $T^{n-k-1}$ is a deformation retract of $N^n \setminus W^k$.

Actually, we have to modify slightly the mappings just described as they are needed for our proof. We will define retractions not onto the skeletons $T^{n-k-1}$, but onto their neighborhoods.

Let $\varphi_\varepsilon$: $[0, 1] \to [0, 1]$, $\varphi_\varepsilon(t) = \max\{t, \varepsilon\}$ ($\varepsilon \in [0, 1]$). We define the mapping $P^l_{y,\varepsilon}$: $\sigma^l \setminus \{y\} \to \sigma^l$ by formula

$$P^l_{y,\varepsilon}(x) = y + (P^l_y(x) - y) \cdot \varphi_\varepsilon\left(\frac{|x - y|}{|P^l_y(x) - y|}\right)$$

(we have used the affine structure on the simplex $\sigma^l$, see Fig. 1).

Note that $P^l_{y,\varepsilon}$ is a retraction from $\sigma^l \setminus \{y\}$ longways radii onto $\sigma^l \setminus \sigma^l_{y,\varepsilon}$, where $\sigma^l_{y,\varepsilon}$ is a homothetic image of the simplex $\sigma^l$ under the homothety with the center $y$ and scale equal to $\varepsilon$. Hence $P^l_{y,\varepsilon}$ is a homotopy between $Id_{\sigma^l \setminus \{y\}}$ and $P^l_y$ when $\varepsilon$ goes from 0 to 1. If $y_i^n \in \operatorname{int} \sigma_i^n$, $i = 1, \dots, r_n$, and $\varepsilon \in [0, 1]$ then we can define analogously as above

$$P^n_{y_1^n, \dots, y_{r_n}^n, \varepsilon} = P^n_{y_1^n, \varepsilon} \circ \cdots \circ P^n_{y_{r_n}^n, \varepsilon} \in W^{1,p}(N^n, N^n)$$

for $p < n$.

Moreover,

$$\|P^n_{y_1^n, \dots, y_{r_n}^n, \varepsilon} - Id_{N^n}\|_{1,p} \underset{\varepsilon \to 0}{\longrightarrow} 0.$$

Now we define analogously retractions onto neighborhoods of lower dimensional skeletons. First we will be concerned with a single simplex $\sigma^n$. By $\{\sigma_i^{n-1}\}_{i=1}^{n+1}$ we denote the family of simplexes from the boundary $\partial\sigma^n$.

Let $y^n \in \operatorname{int} \sigma^n$, $\varepsilon_0 \in [0, 1]$, $y_i^{n-1} \in \operatorname{int} \sigma_i^{n-1}$, $i = 1, \dots, n + 1$, $\varepsilon_1 \in [0, 1]$. Let

$$P^n_{y^n, \varepsilon_0}: \sigma^n \setminus \{y^n\} \to \sigma^n \setminus \sigma^n_{y^n, \varepsilon_0}$$
$$P^{n-1}_{y_i^{n-1}, \varepsilon_1}: \partial\sigma^n \setminus \{y_i^{n-1}\} \to \partial\sigma^n \setminus \sigma^{n-1}_{y_i^{n-1}, \varepsilon_1}$$

be defined as above.

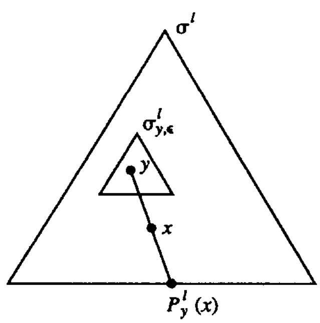


Fig. 1.

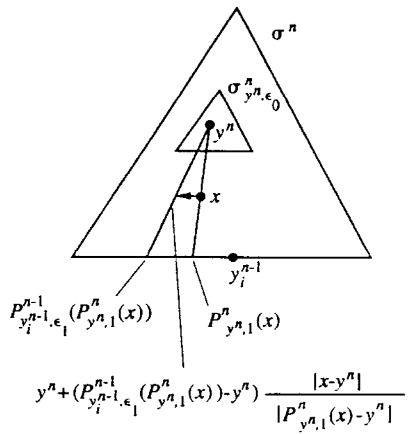


Fig. 2.

To omit huge notations, we will denote by $\mathcal{Y}^l$ the given family of points from interiors of $l$-dimensional simplexes. In our case $\mathcal{Y}^{n-1} = \{y_i^{n-1}\}_{i=1}^{n+1}$.

Now we define the retraction onto the neighborhood of the $(n-2)$-dimensional skeleton of $\sigma^n$ by the formula

$$P_{(\mathcal{Y}^n,\varepsilon_0),(\mathcal{Y}^{n-1},\varepsilon_1)} = \varphi_{n+1}\circ\cdots\circ\varphi_1\circ P^n_{y^n,\varepsilon_0}(x) \tag{1}$$

where

$$\varphi_i(x) = y^n + (P^{n-1}_{y_i^{n-1},\varepsilon_1}(P^n_{y^n,1}(x)) - y^n)\frac{|x-y^n|}{|P^n_{y^n,1}(x)-y^n|}$$

(see Fig. 2). Note that for given $z\in\sigma^n\setminus\{y^n\}$ at most one $\varphi_i$ does not act on $z$ as an identity.

The singularity of the mapping (1) is 1-dimensional. As we have already said, (1) is a retraction onto the neighborhood of the $(n-2)$-skeleton. This neighborhood can be deformationally retracted onto this skeleton. The retraction can be obtained by passing to the limits $\varepsilon_0\to 1$, $\varepsilon_1\to 1$.

Evidently $P_{(\mathcal{Y}^n,\varepsilon_0),(\mathcal{Y}^{n-1},\varepsilon_1)}\in W^{1,p}$ for $p<n-1$.

Extending the above definition to the whole family of $n$-dimensional simplexes $\{\sigma_i^n\}_{i=1}^{r_n} = T^n$, we obtain the retraction onto a neighborhood of $T^{n-2}$

$$P_{(\mathcal{Y}^n,\varepsilon_0),(\mathcal{Y}^{n-1},\varepsilon_1)}\in W^{1,p}(N^n,N^n)\qquad\text{for } p<n-1,$$

where $\mathcal{Y}^n = \{y_1^n,\dots,y_{r_n}^n\}$, $\mathcal{Y}^{n-1} = \{y_1^{n-1},\dots,y_{r_{n-1}}^{n-1}\}$.

This mapping has 1-dimensional singularity $W^1$, which does not depend on the choice of $\varepsilon_0$ and $\varepsilon_1$. Passing to the limits $\varepsilon_0\to 1$, $\varepsilon_1\to 1$, we obtain a deformation retraction of a neighborhood of $T^{n-2}$ onto $T^{n-2}$. Moreover,

$$P_{(\mathcal{Y}^n,1),(\mathcal{Y}^{n-1},1)} = P^{n-1}_{y_1^{n-1},\dots,y_{r_{n-1}}^{n-1}}\circ P^n_{y_1^n,\dots,y_{r_n}^n}.$$

Analogously we can define retractions onto neighborhoods of the lower dimensional skeletons. This leads us to the following theorem.

THEOREM 4 [5, proposition 4]. If $\mathcal{Y}^{n-j} = \{y_1^{n-j}, \ldots, y_{r_{n-j}}^{n-j}\}$, where $y_i^{n-j} \in \operatorname{int} \sigma_i^{n-j}$, and $\varepsilon_j \in [0, 1]$ for $j = 0, 1, \ldots, n - k - 1$ then

(1) $$P = P_{(\mathcal{Y}^n, \varepsilon_0), \ldots, (\mathcal{Y}^{k+1}, \varepsilon_{n-k-1})} \in W^{1,p}(N^n, N^n)$$

for $p < k + 1$.

(2) Mapping $P$ has the $(n - k - 1)$-dimensional singularity $W^{n-k-1}$. It is the $(n - k - 1)$-dimensional skeleton of the dual complex to triangulation $T$. This singularity depends on the choice of $\{y_i^{n-j}\}_{i,j}$, but not on $\{\varepsilon_j\}_j$.

(3) The mapping $P$ retracts $N^n \setminus W^{n-k-1}$ onto the neighborhood of $T^k$. We obtain the deformation retraction of this neighborhood onto $T^k$, when we pass to the limits $\varepsilon_j \to 1$ for $j = 0, \ldots, n - k - 1$.

(4) $\|P - Id_{N^n}\|_{1,p} \to 0$ when $\varepsilon_0, \ldots, \varepsilon_{n-k-1} \to 0$.

(5) The skeletons $W^{n-k-1}$ and $T^k$ are transversal in the sense of Borsuk, i.e. $T^k$ is a deformation retract of $N^n \setminus W^{n-k-1}$ and $W^{n-k-1}$ is a deformation retract of $N^n \setminus T^k$. ■

*Remarks*. (1) The notion of the transversality in the sense of Borsuk was introduced in [6].

(2) We have not used the assumption that $N^n$ is embedded in $\mathbb{R}^\nu$ in this section.

## 3. A LIPSCHITZ MAPPING

Let $N^n \subset \mathbb{R}^\nu$ be a compact submanifold and $T$ be its smooth triangulation.

LEMMA 1. If $\pi_1(N^n) = \cdots = \pi_k(N^n) = 0$, then there exists a Lipschitz mapping $\eta: \mathbb{R}^\nu \to N^n$ such that $\eta_{|T^k} = Id_{T^k}$.

*Proof*. First we prove the following lemma.

LEMMA 2. Under the assumptions of lemma 1 there exists a Lipschitz retraction $\rho: \mathbb{R}^\nu \cup CT^k \to CT^k$, where $CT^k$ denotes the cone over $T^k$ ($CT^k$ is embedded in $\mathbb{R}^{\nu+1}$).

*Proof*. The existence of continuous retraction follows from the fact that $T^k$ is a Lipschitz neighborhood retract and $CT^k$ is contractible. We divide $\mathbb{R}^\nu$ into small cubes using the lattice points. On the cubes lying near $T^k$, we define $\rho$ as a Lipschitz neighborhood retraction onto $T^k$. On the remaining cubes we define retraction on skeletons using the fact that $CT^k$ is contractible. This way we have defined a continuous retraction which, in addition, is Lipschitz in a neighborhood of $T^k$. We can improve this retraction to a Lipschitz one because $CT^k \subset \mathbb{R}^{\nu+1}$ is, up to a biLipschitz equivalence, piecewise linear and, hence, is a Lipschitz neighborhood retract. Now it suffices to smooth mapping $\rho$ (we do not improve $\rho$ in a neighborhood of $T^k$, as in this neighborhood $\rho$ is Lipschitz) and compose this improved mapping with the Lipschitz neighborhood retraction onto $CT^k$. ■

We are now in a position to finish the proof of lemma 1.

By lemma 2, it suffices to prove the existence of Lipschitz retraction $\lambda: N^n \cup CT^k \to N^n$ ($N^n \cup CT^k \subset \mathbb{R}^\nu \cup CT^k$). Indeed, given $\lambda$, we will define $\eta$ by the formula $\eta = \lambda \circ \rho$. The existence of continuous retraction follows from the fact that $N^n$ is $k$-connected and $CT^k$ consists of $(k + 1)$-dimensional simplexes. Then, smoothing argument ensures the existence of the Lipschitz retraction. ■

## 4. MORE LIPSCHITZ MAPPINGS

We will use the notation from Section 2, but we will make one restrictive assumption. We assume that the center of retraction $P^l_{x,\varepsilon}\colon \sigma^l\setminus\{x\} \to \sigma^l\setminus\sigma^l_{x,\varepsilon}$, i.e. the point $x$, is far enough from the boundary $\partial\sigma^l$. Namely, we assume that $x \in \sigma^l_{m,1/2}$, where $m$ denotes the baricenter of the simplex $\sigma^l$. We make this assumption to have a uniform estimation of the Lipschitz constants of the mappings we will define soon.

Let $\mathcal{Y}^{n-j} = \{y_1^{n-j}, \ldots, y_{r_{n-j}}^{n-j}\}$, $j = 0, \ldots, n-k-1$. Moreover, assume that $y_{r_i}^{n-j}$ are taken in accordance with our restrictive assumption. Let $\varepsilon \in [0, 1]$. The image of the mapping $P_\varepsilon = P_{(\mathcal{Y}^n,\varepsilon),\ldots,(\mathcal{Y}^{k+1},\varepsilon)}$ is a neighborhood of $T^k$. We denote this neighborhood by $\mathcal{U}_\varepsilon T^k$ (this neighborhood depends on the choice of $\mathcal{Y}^n, \ldots, \mathcal{Y}^{k+1}$, but for the sake of simplicity we do not point it out in the notation).

The interior of $N^n\setminus\mathcal{U}_\varepsilon T^k$ will be denoted by $\mathcal{O}_\varepsilon T^k$, and the common boundary of $\mathcal{U}_\varepsilon T^k$ and $\mathcal{O}_\varepsilon T^k$ will be denoted by $Bd_\varepsilon T^k$. We will work now with fixed $\mathcal{Y}^n, \ldots, \mathcal{Y}^{k+1}$, but we will change $\varepsilon$ (see Fig. 3).

We can represent $N^n$ as a disjoint sum

$$N^n = T^k \cup W^{n-k-1} \cup \bigcup_{\varepsilon\in(0,1)} Bd_\varepsilon T^k. \tag{2}$$

If $\varepsilon_1 \le \varepsilon_2$, then $P_{\varepsilon_2}$ restricted to $Bd_{\varepsilon_1} T^k$ defines the Lipschitz mapping

$$P_{\varepsilon_2}\colon Bd_{\varepsilon_1} T^k \to Bd_{\varepsilon_2} T^k.$$

Therefore, every continuous function $\varphi\colon [0, 1] \to [0, 1]$ such that $\varphi(\varepsilon) \ge \varepsilon$, $\varphi(0) = 0$, $\varphi(1) = 1$ defines the continuous mapping

$$P_\varphi\colon N^n \to N^n$$

by the formula

$$P_{\varphi|T^k} = Id_{T^k};$$

$$P_{\varphi|W^{n-k-1}} = Id_{W^{n-k-1}};$$

$$P_{\varphi|Bd_\varepsilon T^k} = P_{\varphi(\varepsilon)}\colon Bd_\varepsilon T^k \to Bd_{\varphi(\varepsilon)} T^k.$$

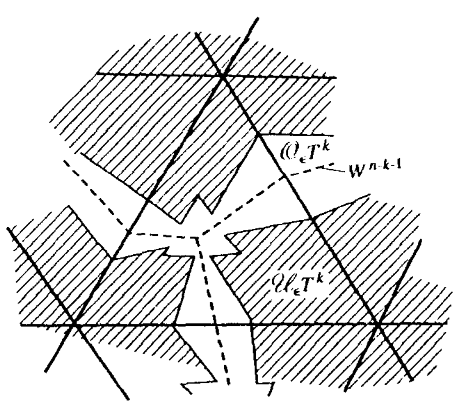


Fig. 3.

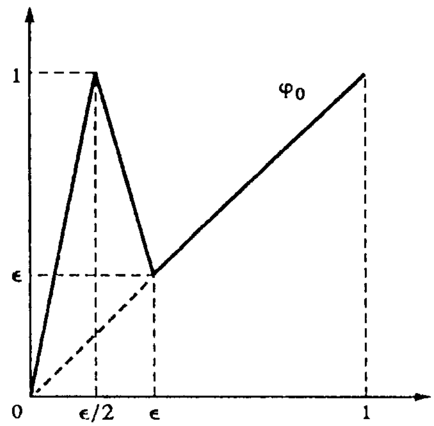


Fig. 4.

Now we will define a rather complicated Lipschitz mapping $\eta_\varepsilon$ from $\mathbb{R}^\nu$ onto $N^n$ (generalizing the construction from Section 3). To clarify the idea, we will do it in a few steps.

*Step* 1. Let $\varphi_0$ be defined by its graph (see Fig. 4).

Then $P_{\varphi_0}$ defines the Lipschitz mapping with the following properties:

(1) $P_{\varphi_0|\mathcal{U}_\varepsilon T^k} = Id_{\mathcal{U}_\varepsilon T^k}$,

(2) $P_{\varphi_0}(Bd_{\varepsilon/2}T^k) = T^k$,

(3) $P_{\varphi_0}(\mathcal{O}_{\varepsilon/2}T^k) = N^n$,

(4) $\mathrm{Lip}(P_{\varphi_0}) \le C\varepsilon^{-1}$,

(Lip—Lipschitz constant), where this estimation of the Lipschitz constant is uniform with respect to the choice of $\mathcal{Y}^n, \ldots, \mathcal{Y}^{k+1}$ (remember about the restrictive assumption).

*Step* 2. Let $\varepsilon < 2/3$. We extend now $P_{\varphi_0}$ to a mapping defined on a tubular neighborhood of $N^n$. Consider the family of functions $\varphi_\delta$ for $\delta \in [0, 1 - \varepsilon]$ defined by their graphs (see Fig. 5). (Note that if $\delta = 0$, then $\varphi_\delta$ coincides with $\varphi_0$ defined in step 1. Moreover, $P_{\varphi_{1-\varepsilon}} = Id_{N^n}$.) If $\lambda > 0$ is sufficiently small, then

$$\bar{\mathcal{V}} = \{x \in \mathbb{R}^\nu : \mathrm{dist}(x, N^n) \le \lambda\}$$

is a tubular neighborhood of $N^n$, and the nearest point projection $\pi : \bar{\mathcal{V}} \to N^n$ is smooth. Now we define the Lipschitz mapping

$$Q : \bar{\mathcal{V}} \to N^n$$

as follows

$$Q(x) = P_{\varphi_{((1-\varepsilon)/\lambda)\,\mathrm{dist}(x, N^n)}}(\pi(x)).$$

The mapping $Q$ has the following properties:

(1) $Q_{|N^n} = P_{\varphi_0}$,

(2) $Q_{|\partial\bar{\mathcal{V}}} = \pi_{|\partial\bar{\mathcal{V}}}$,

(3) $\mathrm{Lip}(Q) \le C\varepsilon^{-1}$ (for some constant $C$ independent on the choice of $\mathcal{Y}^n, \ldots, \mathcal{Y}^{k+1}$).

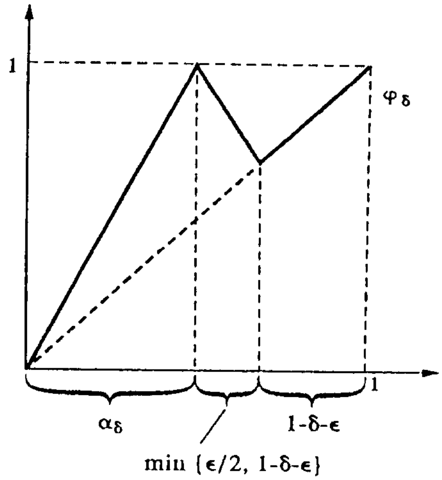


Fig. 5.

Properties (1) and (2) are obvious. We will prove (3). This follows from the observation that $Q$ is Lipschitz in the horizontal (i.e. parallel to $N^n$) direction as well as in the vertical one, and in both cases the Lipschitz constant is bounded by $C\varepsilon^{-1}$.

*Step* 3. Finally, in this step we define the Lipschitz mapping (depending on the choice of $\mathcal{Y}^n, \ldots, \mathcal{Y}^{k+1}$ and $\varepsilon \in [0, 1]$)

$$\eta_\varepsilon : \mathbb{R}^\nu \to N^n$$

with the following properties

(a) $\eta_{\varepsilon|\mathcal{U}_\varepsilon T^k} = Id_{\mathcal{U}_\varepsilon T^k}$,

(b) $\mathrm{Lip}(\eta_\varepsilon) \leq C\varepsilon^{-1}$ (for some constant $C$ independent of the choice of $\mathcal{Y}^n, \ldots, \mathcal{Y}^{k+1}$).

*Remark*. Note that the mapping $\eta_\varepsilon$ is similar to the mapping $\eta$ defined in Section 3. In fact one can prove, following the same ideas as in Section 3, that there exists a Lipschitz mapping satisfying (a). I hope that one can prove that the mapping obtained in a similar way as in Section 3 admits the estimation (b). This may simplify our considerations.

First we have to define a special subset $\bar{G} \subset \bar{\mathcal{V}}$. The representation (2) defines the function

$$\gamma : N^n \to [0, 1].$$

Namely, $\gamma$ sends a point $x \in N^n$ to an index of film to which $x$ belongs ($T^k$ is endowed with an index 1 and $W^{n-k-1}$ with an index 0). Now we define the set $\bar{G}$ as follows

$$\bar{G} = \left\{ x \in \bar{\mathcal{V}} : \gamma(\pi(x)) \geq \alpha_\delta, \text{ where } \alpha_\delta \text{ is defined in Fig. 5 with } \delta = \frac{1-\varepsilon}{\lambda} \operatorname{dist}(x, N^n) \right\}.$$

The nearest point projection $\pi : \bar{\mathcal{V}} \to N^n$ admits a smooth extension $\pi : \mathbb{R}^\nu \to \mathbb{R}^\nu$, which is equal to identity outside the slightly bigger tubular neighborhood than $\bar{\mathcal{V}}$ (hence $\pi$ is

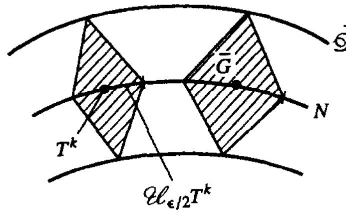


Fig. 6.

Lipschitz). Now we are in a position to define the mapping $\eta_\varepsilon$,

$$\eta_\varepsilon(x) = \begin{cases} Q(x) & \text{for } x \in \bar{G}, \\ \eta \circ Q(x) & \text{for } x \in \bar{\mathcal{V}} \setminus \bar{G}, \\ \eta \circ \pi(x) & \text{for } x \in \mathbb{R}^\nu \setminus \bar{\mathcal{V}}. \end{cases}$$

We know that $Q$ maps the boundary of $\bar{G}$ onto $T^k$, $\eta$ acts on $T^k$ as the identity, and $Q$ coincides with $\pi$ on $\partial\bar{\mathcal{V}}$. Therefore the mapping $\eta_\varepsilon$ is continuous and hence Lipschitz.

Since the mappings $\eta$ and $\pi$ are Lipschitz with the Lipschitz constant independent of $\varepsilon$ and the Lipschitz constant of $Q$ admits the estimation $\mathrm{Lip}(Q) \leq C\varepsilon^{-1}$, then the Lipschitz constant of $\eta_\varepsilon$ admits the desired estimation (b).

## 5. PROOF OF THEOREM 1

In this section (as in Section 4) $C$ will denote the general constant and in different formulas it may denote different constants.

This proof extends some ideas of [7, theorem 1] and of the last part of [8].

Let $f \in W^{1,p}(M^m, N^n)$. We have to prove that:

(a) if $p < k + 1$, then $f$ can be approximated by $C^\infty(M^m, N^n)$ mappings in the norm topology;

(b) if $p = k + 1$, then there exists a sequence of $C^\infty(M^m, N^n)$ mappings convergent to $f$ in $W^{1,p}$-weak topology.

We will prove simultaneously (a) and (b). First we have to introduce some notation concerning the triangulation $T$ of $N^n$ and related retractions. Let us consider the mapping defined by the formula

$$P_{(\mathcal{Y}^l, \varepsilon)} = P_{(\mathcal{Y}^n, 0), \dots, (\mathcal{Y}^{l+1}, 0), (\mathcal{Y}^l, \varepsilon)}.$$

Let

$$\mathcal{Q}_\varepsilon T^{l-1} = \mathrm{int}(N^n \setminus P_{(\mathcal{Y}^l, \varepsilon)}(N^n \setminus W^{n-l})).$$

The set $\mathcal{Q}_\varepsilon T^{l-1}$ depends on the choice of $\mathcal{Y}^n, \dots, \mathcal{Y}^l$ and $\varepsilon$. There exists a constant $C > 0$ such that the maximal number $k_n(\varepsilon)$ of sets $\mathcal{Y}^n$ (chosen in accordance with the restrictive assumption from Section 4) such that the corresponding sets $\mathcal{Q}_{2\varepsilon} T^{n-1}$ are pairwise disjoint is not less than $C\varepsilon^{-n}$. Assume that the set $\mathcal{Y}^n$ is fixed. Now there exists a constant $C$ (independent of the choice of $\mathcal{Y}^n$) such that the maximal number $k_{n-1}(\varepsilon)$ of sets $\mathcal{Y}^{n-1}$ (restrictive assumption) with the corresponding sets $\mathcal{Q}_{2\varepsilon} T^{n-2}$ pairwise disjoint ($\mathcal{Q}_{2\varepsilon} T^{n-2}$ has no common points with $\mathcal{Y}^n$) is not less than $C\varepsilon^{-(n-1)}$. Analogously, we can define the numbers $k_l(\varepsilon) \geq C\varepsilon^{-l}$.

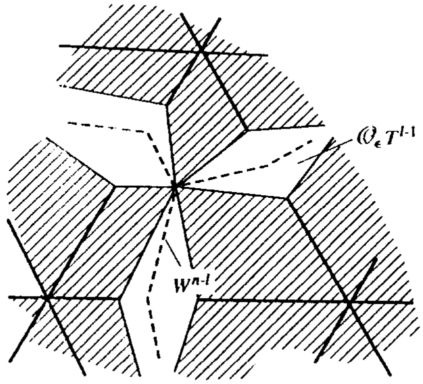


Fig. 7.

Let $\mathcal{Y}_1^n, \ldots, \mathcal{Y}_{k_n(\varepsilon)}^n$ be a family (of sets $\mathcal{Y}^n$) such that the corresponding sets

$$\mathcal{Q}_{2\varepsilon,1} T^{n-1}, \ldots, \mathcal{Q}_{2\varepsilon, k_n(\varepsilon)} T^{n-1}$$

are pairwise disjoint. We have

$$\int_{\bigcup_{i=1}^{k_n(\varepsilon)} f^{-1}(\mathcal{Q}_{2\varepsilon,i} T^{n-1})} (|f|^p + |df|^p) = \sum_{i=1}^{k_n(\varepsilon)} \int_{f^{-1}(\mathcal{Q}_{2\varepsilon,i} T^{n-1})} (|f|^p + |df|^p) \le \|f\|_{1,p}^p.$$

Hence there exists $j \in \{1, \ldots, k_n(\varepsilon)\}$ such that

$$\int_{f^{-1}(\mathcal{Q}_{2\varepsilon,j} T^{n-1})} (|f|^p + |df|^p) \le \frac{1}{k_n(\varepsilon)} \|f\|_{1,p}^p \le C^{-1}\varepsilon^n \|f\|_{1,p}^p.$$

Fix the set $\mathcal{Y}_j^n$. Analogously there exists the set $\mathcal{Y}^{n-1}$ such that

$$\int_{f^{-1}(\mathcal{Q}_{2\varepsilon} T^{n-2})} (|f|^p + |df|^p) \le C^{-1}\varepsilon^{n-1} \|f\|_{1,p}^p.$$

Fix the set $\mathcal{Y}^{n-1}$. Now there exists the set $\mathcal{Y}^{n-2}$ .... Finally we find the set $\mathcal{Y}^{k+1}$ such that

$$\int_{f^{-1}(\mathcal{Q}_{2\varepsilon} T^{k})} (|f|^p + |df|^p) \le C^{-1}\varepsilon^{k+1} \|f\|_{1,p}^p.$$

Hence the following trivial fact

$$\mathcal{O}_{2\varepsilon} T^k = \bigcup_{i=1}^{n-k} \mathcal{Q}_{2\varepsilon} T^{n-i}$$

implies that there exist the constants $C'$, $C > 0$ such that

$$\int_{f^{-1}(\mathcal{O}_{2\varepsilon} T^{k})} (|f|^p + |df|^p) \le C'(\varepsilon^n + \varepsilon^{n-1} + \cdots + \varepsilon^{k+1}) \|f\|_{1,p}^p \le C\varepsilon^{k+1} \|f\|_{1,p}^p. \tag{3}$$

(If $\mathcal{Y}^n, \ldots, \mathcal{Y}^{k+1}$ are chosen as above.)

Let us assume that for every $\varepsilon \in [0, 1]$ the sets $\mathcal{Y}_\varepsilon^n, \ldots, \mathcal{Y}_\varepsilon^{k+1}$ are chosen in such a way that (3) holds. Moreover, we can assume that

$$|f^{-1}(\mathcal{O}_{2\varepsilon} T^k)| \underset{\varepsilon \to 0}{\longrightarrow} 0 \tag{4}$$

($|\cdot|$ denotes Lebesgue measure).

Let $\eta_\varepsilon\colon \mathbb{R}^\nu \to N^n$ be the mapping defined as in Section 4 with respect to the sets $\mathcal{Y}_\varepsilon^n, \ldots, \mathcal{Y}_\varepsilon^{k+1}$.

LEMMA 3. (a) If $p < k + 1$, then $\eta_\varepsilon f \underset{\varepsilon\to 0}{\longrightarrow} f$ in $W^{1,p}$.
(b) If $p = k + 1$, then $\eta_\varepsilon f \underset{\varepsilon\to 0}{\rightharpoonup} f$ in $W^{1,p}$ (i.e. weakly).

*Proof.* Since $\eta_\varepsilon f$ differs from $f$ on a set of arbitrary small measure (compare (4)) and all the mappings are bounded we have that $\eta_\varepsilon f \to f$ in $L^p$ when $\varepsilon \to 0$. We estimate the gradient.

$$\begin{aligned}\int_{M^m} |d(\eta_\varepsilon f - f)|^p &= \begin{pmatrix} \eta_\varepsilon f(x) = f(x) \text{ for } x \in f^{-1}(\mathcal{U}_\varepsilon T^k) \text{ hence} \\ d(\eta_\varepsilon f)(x) = df(x) \text{ for a.e. } x \in f^{-1}(\mathcal{U}_\varepsilon T^k) \end{pmatrix} \\ &= \int_{f^{-1}(\mathcal{O}_\varepsilon T^k)} |d(\eta_\varepsilon f) - df|^p \\ &\le C\left(\varepsilon^{-p}\int_{f^{-1}(\mathcal{O}_\varepsilon T^k)} |df|^p + \int_{f^{-1}(\mathcal{O}_\varepsilon T^k)} |df|^p\right).\end{aligned}$$

The second integral converges to 0 with $\varepsilon \to 0$ because of (4). We estimate the first integral using the inequality (3)

$$\varepsilon^{-p}\int_{f^{-1}(\mathcal{O}_\varepsilon T^k)} |df|^p \le C\varepsilon^{(k+1)-p}\|f\|_{1,p}^p.$$

(a) If $p < k + 1$, then this integral converges to zero.
(b) If $p = k + 1$, then this integral is bounded. ■

It suffices to prove that there exists the sequence of Lipschitz mappings $f_\varepsilon \in \mathrm{Lip}(M^m, N^n)$ such that:
(a) if $p < k + 1$, then $\|f_\varepsilon - \eta_\varepsilon f\|_{1,p} \underset{\varepsilon\to 0}{\longrightarrow} 0$;
(b) if $p = k + 1$, then $f_\varepsilon \to f$ in $L^p$ and $\|d(f_\varepsilon - \eta_\varepsilon f)\|_p < C$ where $C$ is a constant independent of $\varepsilon$.
(Note that the existence of Lipschitz approximation implies the existence of smooth one—by a standard argument—convolution approximation composed with nearest point projection, see e.g. [8, lemma 2].)

We know that there exists a sequence of smooth mappings $f_l \in C^\infty(M^m, \mathbb{R}^\nu)$ such that $f_l \to f$ in $W^{1,p}$ and a.e.

Let $\pi\colon \mathbb{R}^\nu \to \mathbb{R}^\nu$ be a smooth extension of the nearest point projection from the tubular neighborhood onto $N^n$—as in Section 4.

Since $f_l$ converges to $f$ in measure and since $|f^{-1}(\mathcal{O}_{2\varepsilon} T^k)| \to 0$, then for every $\varepsilon > 0$ we can choose an index $l(\varepsilon)$ such that

$$|(\pi \circ f_{l(\varepsilon)})^{-1}(\mathbb{R}^\nu \setminus \mathcal{U}_\varepsilon T^k)| \underset{\varepsilon\to 0}{\longrightarrow} 0. \tag{5}$$

A large part of the values of $f$ lies in $\mathcal{U}_{2\varepsilon} T^k = N^n \setminus \mathcal{O}_{2\varepsilon} T^k$ (more and more with $\varepsilon \to 0$), hence if $l$ is large enough then almost the same part of the values of $f_l$ lies in a neighborhood (in $\mathbb{R}^\nu$) of $\mathcal{U}_{2\varepsilon} T^k$, hence we can choose $l(\varepsilon)$ large enough to satisfy (5) (because the projection of the neighborhood of $\mathcal{U}_{2\varepsilon} T^k$ is a subset of $\mathcal{U}_\varepsilon T^k$).

Evidently $\pi \circ f_l \xrightarrow[l\to\infty]{} \pi \circ f = f$ in $W^{1,p}$ (as composition with smooth mappings is continuous in $W^{1,p}$ (see e.g. [1–3; 7, p. 63])).

Hence we can assume enlarging $l(\varepsilon)$ (in the case of need) that

$$\int_{M^m} |\mathrm{d}(\pi \circ f_{l(\varepsilon)}) - \mathrm{d}f|^p < \varepsilon^{p+1}. \tag{6}$$

Since $\eta_\varepsilon \circ \pi \circ f_{l(\varepsilon)}$ differs from $\pi \circ f_{l(\varepsilon)}$ on a set of measures convergent to 0 with $\varepsilon \to 0$ (by (5) and the definition of $\eta_\varepsilon$) then $\eta_\varepsilon \circ \pi \circ f_{l(\varepsilon)} \to f$ in $L^p$ (as all the mappings are bounded).

Now we estimate the gradient. Let $g_\varepsilon = \pi \circ f_{l(\varepsilon)}$ and $A_\varepsilon = g_\varepsilon^{-1}(\mathfrak{U}_\varepsilon T^k)$ (this is a completion of the set described in (5)). We prove that $f_\varepsilon = \eta_\varepsilon \circ g_\varepsilon$ are desired Lipschitz mappings. We have

$$\int_{M^m} |\mathrm{d}(\eta_\varepsilon g_\varepsilon - \eta_\varepsilon f)|^p = \int_{A_\varepsilon} + \int_{M^m \setminus A_\varepsilon} = I_1 + I_2.$$

The map $\eta_\varepsilon$ acts as an identity on $g_\varepsilon(A_\varepsilon)$, so $\mathrm{d}(\eta_\varepsilon g_\varepsilon) = \mathrm{d}g_\varepsilon$ a.e. in $A_\varepsilon$, hence

$$I_1 = \int_{A_\varepsilon} |\mathrm{d}(g_\varepsilon - \eta_\varepsilon f)|^p \le C\left(\int_{A_\varepsilon} |\mathrm{d}(g_\varepsilon - f)|^p + \int_{A_\varepsilon} |\mathrm{d}(f - \eta_\varepsilon f)|^p\right).$$

The first integral on the right-hand side converges to zero. Estimation of the second integral follows directly from lemma 3:

(a) if $p < k + 1$, then it converges to zero;
(b) if $p = k + 1$, then it is bounded.

$$\begin{aligned} I_2 &= \int_{M^m \setminus A_\varepsilon} |\mathrm{d}\eta_\varepsilon(g_\varepsilon) \circ \mathrm{d}g_\varepsilon - \mathrm{d}\eta_\varepsilon(f) \circ \mathrm{d}f|^p \\ &\le 2^{p-1}\left(\int_{M^m \setminus A_\varepsilon} |\mathrm{d}\eta_\varepsilon(g_\varepsilon) \circ \mathrm{d}g_\varepsilon - \mathrm{d}\eta_\varepsilon(g_\varepsilon) \circ \mathrm{d}f|^p \right. \\ &\quad \left. + \int_{M^m \setminus A_\varepsilon} |\mathrm{d}\eta_\varepsilon(g_\varepsilon) \circ \mathrm{d}f - \mathrm{d}\eta_\varepsilon(f) \circ \mathrm{d}f|^p\right) \\ &\le C\left(\varepsilon^{-p} \int_{M^m \setminus A_\varepsilon} |\mathrm{d}g_\varepsilon - \mathrm{d}f|^p + 2\varepsilon^{-p} \int_{M^m \setminus A_\varepsilon} |\mathrm{d}f|^p\right). \end{aligned}$$

The first integral converges to zero as it follows from (6). Now it remains to estimate the second integral.

$$\begin{aligned} \varepsilon^{-p} \int_{M^m \setminus A_\varepsilon} |\mathrm{d}f|^p &\le \varepsilon^{-p} \int_{f^{-1}(\mathcal{O}_{2\varepsilon} T^k)} + \varepsilon^{-p} \int_{g_\varepsilon^{-1}(\overline{\mathcal{O}_\varepsilon T^k}) \setminus f^{-1}(\mathcal{O}_{2\varepsilon} T^k)} + \varepsilon^{-p} \int_{g_\varepsilon^{-1}(\mathbb{R}^p \setminus N^n)} \\ &= J_1 + J_2 + J_3. \end{aligned}$$

$$J_1 = \varepsilon^{-p} \int_{f^{-1}(\mathcal{O}_{2\varepsilon} T^k)} |\mathrm{d}f|^p \le \text{(by (3))} \le C\varepsilon^{(k+1)-p} \|f\|_{1,p}^p.$$

(a) If $p < k + 1$, then $J_1 \xrightarrow[\varepsilon \to 0]{} 0$.
(b) If $p = k + 1$, then $J_1$ is bounded.

Now we estimate $J_3$. The map $f_l$ converges to $f$ in measure, hence

$$|(\pi \circ f_l)^{-1}(\mathbb{R}^\nu \setminus N^n)| \underset{l\to\infty}{\longrightarrow} 0,$$

thus enlarging (if necessary) $l(\varepsilon)$ in the definition of $g_\varepsilon = \pi \circ f_{l(\varepsilon)}$ we can assume that $J_3 < \varepsilon \underset{\varepsilon\to 0}{\longrightarrow} 0$.

Estimation of $J_2$. Since $\pi \circ f_l$ converges to $f$ in measure and the distance between $\overline{\mathcal{O}_\varepsilon T^k}$ and $\overline{\mathcal{U}_{2\varepsilon} T^k} = N^n \setminus \mathcal{O}_{2\varepsilon} T^k$ is positive then

$$|(\pi \circ f_l)^{-1}(\overline{\mathcal{O}_\varepsilon T^k}) \cap f^{-1}(\overline{\mathcal{U}_{2\varepsilon} T^k})| \underset{l\to\infty}{\longrightarrow} 0.$$

Thus enlarging (if necessary) $l(\varepsilon)$ in a definition of $g_\varepsilon$ we can assume that $J_2 < \varepsilon \underset{\varepsilon\to 0}{\longrightarrow} 0$. ■

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