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Density of Lipschitz mappings in the class of Sobolev mappings between metric spaces

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Abstract We prove that Lipschitz mappings are dense in the Newtonian–Sobolev classes $N^{1,p}(X, Y)$ of mappings from spaces X supporting p-Poincaré inequalities into a finite Lipschitz polyhedron Y if and only if Y is $[p]$-connected, $\pi_1(Y) = \pi_2(Y) = \cdots = \pi_{[p]}(Y) = 0$, where $[p]$ is the largest integer less than or equal to p.

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1 Introduction

Sobolev mappings between Riemannian manifolds $W^{1,p}(M, N)$ play an important role in the study of geometric variational problems like the theory of harmonic or p-harmonic mappings. Eells and Lemaire [16], raised a question about density of smooth mappings $C^\infty(M, N)$ in $W^{1,p}(M, N)$. Schoen and Uhlenbeck [52,53], answered the question in the negative by proving that the radial projection mapping $x/|x| \in W^{1,p}(B^n, S^{n-1})$, $1 \le p < n$, cannot be approximated by $C^\infty(B^n, S^{n-1})$ mappings when $n - 1 \le p < n$. In the same papers they proved that $C^\infty(M, N)$ is a dense subset of $W^{1,p}(M, N)$ when $p \ge \dim M$. Here and in what follows we assume that M and N are compact smooth Riemannian manifolds, $\partial N = \emptyset$. For a long time it was believed that Bethuel [3], discovered a necessary and sufficient condition for the density of $C^\infty(M, N)$ in $W^{1,p}(M, N)$ when $1 \le p < \dim M$. Namely he claimed that the density holds if and only if $\pi_{[p]}(N) = 0$. This statement made people believe

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that the topology of M plays no role in the problem of the density of Sobolev mappings between manifolds. This, however, turned out to be false: Hang and Lin [29], provided a counterexample to Bethuel's claim by demonstrating that despite the equality $\pi_3(\mathbb{CP}^2) = 0$, $C^\infty(\mathbb{CP}^3, \mathbb{CP}^2)$ is *not* dense in $W^{1,3}(\mathbb{CP}^3, \mathbb{CP}^2)$. Therefore when searching for a necessary and sufficient condition for the density of smooth mappings one has to take into account the topology of both manifolds M and N, or rather the interplay between the topology of M and the topology of N. In a subsequent paper Hang and Lin [30], discovered such a condition. They proved

Proposition 1 *Assume that M and N are compact smooth Riemannian manifolds without boundary. If $1 \leq p < \dim M$, then smooth mappings $C^\infty(M, N)$ are dense in $W^{1,p}(M, N)$ if and only if $\pi_{[p]}(N) = 0$ and M satisfies the $([p] - 1)$-extension property with respect to N.*

(Although this is an outstanding result we name it proposition as we keep the name theorem solely for the new results proved in this paper.) We say that M satisfies the $([p] - 1)$*-extension property with respect to N* if for some smooth triangulation of M, every continuous mapping from the $[p]$ dimensional skeleton $f : M^{[p]} \to N$ has the property that its restriction $f|_{M^{[p]-1}}$ admits an extension to a continuous mapping from M to N.

In view of results of Hang and Lin the claim of Bethuel remains true in its *local* form when the manifold M is replaced by the ball. Indeed, in this case it is not difficult to prove the $([p] - 1)$-extenion property, see Corollary 1.6 in [30]. Prior to the work of Hang and Lin, Hajłasz [21] (cf. [20]) discovered a *global* sufficient condition for the density of smooth mappings.

Proposition 2 *If M and N are compact smooth Riemannian manifolds, $\partial N = \emptyset$, and $\pi_1(N) = \cdots = \pi_{[p]}(N) = 0$, then smooth mappings $C^\infty(M, N)$ are dense in $W^{1,p}(M, N)$.*

This result easily follows from the theorem of Hang and Lin (cf. Corollary 1.6 in [30]) and is weaker than their result, but it has an additional feature that the condition involved is independent of the topology of M.

Other papers related to the problem of approximation of Sobolev mappings between manifolds include [2,4,5,8–10,13,14,17,25,27,28,31,35,36,47,48,55,57,58]; also references cited below.

The theory of Sobolev mappings between manifolds has been extended to the case of Sobolev mappings with values into metric spaces. The first papers on this subject include the work of Ambrosio [1], on limits of classical variational problems and the work of Gromov and Schoen [19], on Sobolev mappings into the Bruhat–Tits buildings, with applications to rigidity questions for discrete groups. Later the theory of Sobolev mappings with values into metric spaces was developed in a more elaborated form by Korevaar and Schoen [42], in their approach to the theory of harmonic mappings into Alexandrov spaces of non-positive curvature. Other papers on Sobolev mappings from a manifold into a metric space include [11,15,37–40,50,54]. Finally analysis on metric spaces, the theory of Carnot–Carathéodory spaces and the theory of quasiconformal mappings between metric spaces led to the theory of

Sobolev mappings between metric spaces [33,34,43,56], among which the theory of Newtonian–Sobolev mappings $N^{1,p}(X, Y)$ is particularly important.

In the study of Sobolev mappings between metric spaces one often assumes that (X, d, μ) is a doubling metric-measure space of finite measure $\mu(X) < \infty$ supporting the p-Poincaré inequality and Y is another metric space. One can always assume that Y is isometrically embedded into some Banach space V (as every metric space Y admits an isometric embedding into $\ell^\infty(Y)$). Next one defines the Newtonian-Sobolev space of V valued mappings $N^{1,p}(X, V)$ and identifies $N^{1,p}(X, Y)$ with a subset of $N^{1,p}(X, V)$ consisting of those mappings $f : X \to V$ for which $f(X) \subset Y$. Since $N^{1,p}(X, V)$ is a Banach space, $N^{1,p}(X, Y)$ inherits the metric structure from $N^{1,p}(X, V)$. One can prove that Lipschitz mappings are dense in $N^{1,p}(X, V)$ and therefore $N^{1,p}(X, Y)$-mappings can be approximated by Lipschitz mappings taking values into V (for more details see Sect. 2). In this setting Heinonen, Koskela, Shanumgalingam and Tyson [34, Remark 6.9], ask: *It is an interesting problem to determine when one can choose the Lipschitz approximation to have values in the target Y. [...] For instance, one can ask to what extent Bethuel's results have analogs for general spaces.*

There are two immediate difficulties in dealing with the approximation of Sobolev mappings between metric spaces. Firstly, the necessary and sufficient condition of Hang and Lin depends on the homotopy properties of continuous mappings from the skeletons of M into N. Therefore one should have possibility to investigate the detailed topological structure of X. On the other hand very little is known about the topological properties of spaces supporting Poincaré inequalities and the known examples show that such spaces can be very bizarre see, e.g. [7,32,44]. Moreover the proof of Hang and Lin employs the CW-structure of both manifolds and, in general, spaces supporting Poincaré inequalities have no such structure.

Secondly, looking from the perspective of geometric analysis one would like to regard metric spaces that are bi-Lipschitz homeomorphic as equivalent. However, a recent example of Hajłasz [24], shows that changing the metric in the target space Y to a bi-Lipschitz equivalent one may cause loss of the density of Lipschitz mappings. More precisely, suppose that Y_1 and Y_2 are bi-Lipschitz homeomorphic metric spaces that are isometrically embedded into Banach spaces $Y_1 \subset V_1$, $Y_2 \subset V_2$. Let $\Phi : Y_1 \to Y_2$ be a bi-Lipschitz homeomorphism. It is easy to see that the mapping $u \overset{\Phi^*}{\mapsto} \Phi \circ u$ induces a one-to-one correspondence between the spaces $\mathrm{Lip}\,(X, Y_1)$ and $\mathrm{Lip}\,(X, Y_2)$ and also a one-to-one correspondence between $N^{1,p}(X, Y_1)$ and $N^{1,p}(X, Y_2)$ for any $1 \le p < \infty$. However, the mapping $\Phi^* : N^{1,p}(X, Y_1) \to N^{1,p}(X, Y_2)$ need not be continuous. This phenomenon can even be seen in the very classical setting of Sobolev spaces [24, Theorem 1.2]. The lack of continuity of Φ^* allows to construct examples as in the following proposition which is the main result of [24].

Proposition 3 *Fix an integer $n \ge 2$. There is a compact and connected set $X \subset \mathbb{R}^{n+2}$ and a global bi-Lipschitz homeomorphism $\Phi : \mathbb{R}^{n+2} \to \mathbb{R}^{n+2}$ with the property that for any closed n-dimensional manifold M Lipschitz mappings* $\mathrm{Lip}\,(M, X)$ *are dense in $W^{1,n}(M, X)$, but Lipschitz mappings* $\mathrm{Lip}\,(M, Y)$ *are not dense in $W^{1,n}(M, Y)$, where $Y = \Phi(X)$.*

The following theorem shows, however, that in some cases the answer to the density problem does not depend on the particular choice of a metric in the target from the class of bi-Lipschitz equivalent metrics.

Theorem 4 *Suppose that* (X, d, μ) *is a doubling metric-measure space of finite measure* $\mu(X) < \infty$*, and* Y_1*,* Y_2 *are two bi-Lipschitz homeomorphic metric spaces of finite diameter isometrically embedded into Banach spaces* V_1 *and* V_2 *respectively. Suppose that Lipschitz mappings* $\mathrm{Lip}\,(X, Y_1)$ *are dense in* $N^{1,p}(X, Y_1)$*,* $1 \le p < \infty$*, in the following strong sense: for every* $f \in N^{1,p}(X, Y_1)$ *and every* $\varepsilon > 0$ *there is* $f_\varepsilon \in \mathrm{Lip}\,(X, Y_1)$ *such that* $\mu(\{x : f(x) \neq f_\varepsilon(x)\}) < \varepsilon$ *and* $\|f - f_\varepsilon\|_{1,p} < \varepsilon$*. Then the Lipschitz mappings* $\mathrm{Lip}\,(X, Y_2)$ *are dense in* $N^{1,p}(X, Y_2)$*.*

Let us emphasize that we *do not* assume here that X supports the p-Poincaré inequality.

In view of Lemma 13 it is very natural to expect the strong approximation property as described in Theorem 4 and indeed it is the reason for which the claim of the main result of the paper, Theorem 6, does not depend on the particular bi-Lipschitz metric in the target. On the other hand the strong approximation property cannot be taken for granted as is shown by Proposition 3. In fact the proof of Proposition 3 involves a construction of a Sobolev mapping that can be approximated by Lipschitz mappings, but the approximating mappings must differ from the given Sobolev mapping at almost every point.

We say that the measure μ satisfies the *local lower mass bound with exponent* $Q > 0$ if there exist a constant $C > 0$ such that

$$\frac{\mu(B(x,r))}{\mu(B(x_0,r_0))} \ge C\left(\frac{r}{r_0}\right)^Q \tag{1}$$

whenever $B(x,r) \subset B(x_0,r_0)$ are balls in X. Easy iteration of the doubling condition implies that every doubling measure satisfies the local lower mass bound with the exponent $Q = \log_2 C_d$, but it might happen that some doubling measure satisfies the bound with the exponent Q smaller than the one coming from the doubling constant.

Theorem 4 plays an important role in the proof of the following result which generalizes the density result of Schoen and Uhlenbeck [52,53], Theorem 1.3 from [24] and a result of White [57, Theorem 2].

Theorem 5 *Let* (X, d, μ) *be a metric-measure space supporting the* p*-Poincaré inequality. Assume also that the doubling measure* μ *is finite and satisfies the local lower mass bound* (1)*. If* $p \ge Q$ *and* Y *is a compact metric-doubling space which is bi-Lipschitz homeomorphic to a Lipschitz neighborhood retract of a Banach space, then for every isometric embedding of* Y *into a Banach space Lipschitz mappings* $\mathrm{Lip}\,(X, Y)$ *are dense in* $N^{1,p}(X, Y)$*. Moreover, for every* $f \in N^{1,p}(X, Y)$*, there is* $\varepsilon > 0$ *such that if* $f_1, f_2 \in \mathrm{Lip}\,(X, Y)$ *satisfy* $\|f - f_i\|_{1,p} < \varepsilon$*,* $i = 1, 2$*, then the mappings* f_1 *and* f_2 *are homotopic.*

Note that every finite Lipschitz polyhedron Y satisfies the condition from Theorem 5. The theorem shows, in particular, that under the given assumptions there are well defined homotopy classes in $N^{1,p}(X, Y)$. If $p > Q$, it is easy due to the Sobolev embedding into Hölder continuous functions [26], but the case $p = Q$ is far from being obvious.

In the main result of the paper we overcome the two potential difficulties described above and provide a definite answer to the question of Heinonen, Koskela, Shanmugalingam and Tyson in the case of mappings into finite Lipschitz polyhedra by proving the following

Theorem 6 *Let Y be a finite Lipschitz polyhedron and* $1 \le p < \infty$. *Then the class of Lipschitz mappings* $\mathrm{Lip}\,(X, Y)$ *is dense in* $N^{1,p}(X, Y)$ *for every metric-measure space X of finite measure that supports the p-Poincaré inequality if and only if* $\pi_1(Y) = \pi_2(Y) = \cdots = \pi_{[p]}(Y) = 0$.

Theorem 6 is the main result of the paper. Other new results are Theorems 4 and 5.

The proof of Theorem 6 employs ideas from the proof of Proposition 2, but the two proofs substantially differ in many aspects. Both proofs are based on a similar idea of deformation of the target space, but the construction of the deformation is somewhat different now: in the present construction the deformations are discontinuous—this makes the construction easier (and it is important in our more complicated setting), but the proof of the properties of this new construction requires a completely new argument. Moreover we prove an 'if and only if' result which adds a proof of the necessity. At last, but not least the setting of metric-measure spaces imposes additional difficulties.

The paper is structured as follows. In Sect. 2 we collect basic definitions and results about doubling spaces, spaces supporting Poincaré inequalities and the Newtonian–Sobolev spaces $N^{1,p}$. Section 3 is devoted to Lipschitz polyhedra, so that eventually the statement of Theorem 6 becomes clarified. The final Sects. 4–6 are devoted to the proofs of the results of the paper.

Notation By C we will denote a general constant—it can change its value within the same string of estimates. The average value of a function f over a set E of finite measure will be denoted by

$$f_E = \fint_E f\, d\mu = \mu(E)^{-1} \int_E f\, d\mu.$$

The Hausdorff measure will be denoted by $\mathcal{H}^\ell$. B will denote a ball in a metric space and σB, where $\sigma \ge 1$, the concentric ball with the radius σ times that of B. The characteristic function of a set E will be denoted by χ_E.

2 Spaces supporting Poincaré inequalities and related topics

The theory of upper gradients and spaces supporting p-Poincaré inequalities, as briefly described below, was introduced by Heinonen and Koskela [33], and the theory of Newtonian-Sobolev spaces $N^{1,p}$ by Shanmugalingam [51]. Finally in the approach to $N^{1,p}$ mappings between metric spaces we follow Heinonen et al. [34].

By a *metric-measure* space (X, d, μ) we mean a metric space (X, d) equipped with a Borel-regular measure μ. We say that the measure μ is *doubling* if there is a constant $C_d \geq 1$ (called *doubling constant*) such that

$$\mu(B(x, 2r)) \leq C_d \mu(B(x, r))$$

for very ball $B(x, r) \subset X$. We assume also that $0 < \mu(B(x, r)) < \infty$ on every ball. Till the end of the paper (X, d, μ) will *always* denote a metric-measure space equipped with a finite doubling measure, i.e. $\mu(X) < \infty$.

By a *metric-doubling* space we mean a metric space with the property that there is a constant $M > 0$ such that every ball in the space can be covered by at most M balls of half the radius. It is easily seen that existence of a doubling measure implies the metric-doubling condition.

Following [51] we say that a Borel measurable function $f : X \to \mathbb{R}$ belongs to the *Newtonian–Sobolev space* $N^{1,p}(X)$, $1 \leq p < \infty$, if $f \in L^p(X)$ and there exists another Borel function $0 \leq g \in L^p(X)$ such that

$$|f(\gamma(a)) - f(\gamma(b))| \leq \int_\gamma g \tag{2}$$

for *all* rectifiable curves $\gamma : [a, b] \to X$. The function g is called *upper gradient* of f (see [33]). A collection Γ of rectifiable curves is called *p-exceptional* if there is a Borel function $0 \leq \varrho \in L^p(X)$ such that $\int_\gamma \varrho = +\infty$ for all $\gamma \in \Gamma$. We say that the inequality (2) holds for *p-almost every (p-a.e.)* curve if the collection of rectifiable curves for which the inequality does not hold is p-exceptional. In this case g is called *p-weak upper gradient* of f. It is well known and easy to see that the existence of an upper gradient in L^p is equivalent with the existence of a p-weak upper gradient in L^p and therefore the two approaches can be equivalently used to define the space $N^{1,p}$. This follows from the fact that every p-weak upper gradient can be approximated in the L^p norm by upper gradients that are pointwise greater than or equal to the given p-weaker upper gradient.

If the functions f and g are defined on some open set $\Omega \subset X$ and the inequality (2) is satisfied for curves $\gamma : [a, b] \to \Omega$, then we say that g is an upper (or p-weak upper) gradient of f on Ω.

The space $N^{1,p}(X)$ is equipped with the norm

$$\|f\|_{1,p} = \|f\|_p + \inf_g \|g\|_p ,$$

where the infimum is taken over all p-weak upper gradients (or equivalently over all upper gradients) of f. Actually, to be more precise we need to identify functions that differ on the set of *p-capacity* zero, as otherwise $\|\cdot\|_{1,p}$ is a seminorm only, see [6,51]. The following result was proved in [51] (see [23] for a different proof).

Proposition 7 *$N^{1,p}(X)$ is a Banach space.*

In the case of a Riemannian manifold the $N^{1,p}$ space is isometrically equivalent to the classical Sobolev space.

Proposition 8 *If X is a Riemannian manifold and $1 \leq p < \infty$, then $N^{1,p}(X) = W^{1,p}(X)$. Moreover for every $f \in N^{1,p}(X)$, $|\nabla f|$ is a p-weak upper gradient and it is minimal in the sense that, if $0 \leq g \in L^p(X)$ is another p-weak upper gradient of f, then $g \geq |\nabla f|$ a.e. Therefore the spaces $N^{1,p}(X)$ and $W^{1,p}(X)$ are isometric.*

For the proof see [51] and also [23].

In the case of metric-measure spaces there are several approaches to Sobolev spaces ([12,18,22,26]), but in a reasonable generality, for example in the case in which the space X supports the p-Poincaré inequality (as defined below) most of the approaches are equivalent to $N^{1,p}(X)$. This was established through the work of many authors, see the survey paper [23], and [41] for important recent results.

Let (X, d, μ) be a metric-measure space and (Y, d_Y) another metric space. The space $L^p(X, Y)$ is defined as a class of all mappings $F : X \to Y$ that satisfy the following three properties: (1) F is *essentially separably valued*, i.e. $F(X\setminus Z)$ is a separable subset of Y for some $Z \subset X$ with $\mu(Z) = 0$; (2) $x \mapsto d_Y(F(x), y)$ is measurable for every $y \in Y$; (3) $x \mapsto d_Y(F(x), y_0) \in L^p(X)$ for some $y_0 \in Y$ (and hence for all $y_0 \in Y$ because $\mu(X) < \infty$).

Following [34] and also [50] we say that the mapping $F \in L^p(X, Y)$ belongs to the *Newtonian–Sobolev class of mappings* $N^{1,p}(X, Y)$ if there is a Borel function $0 \leq g \in L^p(X)$ such that

$$d_Y(F(\gamma(a)), F(\gamma(b))) \leq \int_\gamma g \tag{3}$$

for every rectifiable curve $\gamma : [a, b] \to X$.

This construction defines $N^{1,p}(X, Y)$ as a set but it does not provide any metric. To obtain a metric structure we use an isometric embedding of Y into some Banach space V. It is always possible as every metric space Y admits an isometric embedding into $\ell^\infty(Y)$, the space of bounded functions on Y.

Since any Banach space V is a metric space, the above construction can be used to define the Banach space valued Newtonian–Sobolev mappings $N^{1,p}(X, V)$. Namely $F \in N^{1,p}(X, V)$ if $F \in L^p(X, V)$ and there is a Borel measurable function $0 \leq g \in L^p(X)$ such that

$$\|F(\gamma(a)) - F(\gamma(b))\| \leq \int_\gamma g$$

for every rectifiable $\gamma : [a, b] \to X$. Such a g is called an upper gradient of F. Equivalently we could require the existence of a p-weak upper gradient of F. One can prove that the space $N^{1,p}(X, V)$ is a Banach space with respect to the norm

$$\|F\|_{1,p} = \|F\|_p + \inf_g \|g\|_p,$$

where the infimum is taken over the class of all (p-weak) upper gradients of F.

Now for $Y \subset V$ we define[1]

$$N^{1,p}(X, Y) = \{F \in N^{1,p}(X, V) : F(X) \subset Y\}. \tag{4}$$

Obviously as a set this is the same class as the one defined by (3). The advantage of the approach based on (4) is that now we can use the linear structure in the target; this approach proved to be very useful in the study of quasiconformal mappings between metric spaces [34]. Moreover with this definition the space $N^{1,p}(X, Y)$ inherits the metric structure from the Banach space $N^{1,p}(X, V)$.

The following elementary lemma shows that upper gradients of $N^{1,p}$ functions have a localization property; the proof is quite standard and left to the reader.

Lemma 9 *If* $0 \le g \in L^p(X)$ *is an upper gradient of* $F \in N^{1,p}(X, V)$ *and* F *is constant on a closed set* $E \subset X$*, then the function* $h = g\chi_{X\setminus E}$ *is an upper gradient of* F*.*

For a Lipschitz function h on a metric space (Y, d_Y) we define the *lower Lipschitz constant* at $x \in Y$ by

$$\operatorname{lip} h(x) = \liminf_{r\to 0} \frac{L(x, h, r)}{r},$$

where

$$L(x, h, r) = \sup\{|h(y) - h(x)| : d_Y(x, y) \le r\}.$$

The following result is a version of the chain rule.

Lemma 10 *Let* (X, d, μ) *be a metric-measure space and* (Y, d_Y) *another metric space. If* $f \in N^{1,p}(X, Y)$*,* $1 \le p < \infty$*,* $0 \le g \in L^p(X)$ *is an upper gradient of* f *in the sense of* (3) *and* $h \in Lip(Y)$*, then the function* $(\operatorname{lip} h) \circ f \cdot g$*, i.e. the function*

$$x \mapsto g(x)(\operatorname{lip} h)(f(x)),$$

is a p*-weak upper gradient of* $h \circ f$*.*

Proof The proof is rather standard, so we sketch it only leaving details to the reader (cf. Lemmas 6.7 and 7.6 in [23]). For a rectifiable curve γ in X, $\widetilde{\gamma}$ and $\ell(\gamma)$ will denote the arc-length parametrization and the length of the curve respectively. For a p-a.e. rectifiable curve γ

$$\int_\gamma g = \int_0^{\ell(\gamma)} g(\widetilde{\gamma}(\tau))\, d\tau < \infty.$$

[1] The definition in [34] is slightly different as it allows for an exceptional set of capacity zero. Since we can modify functions in $N^{1,p}$ on sets of capacity zero by [51, Lemma 3.6], the two definitions are equivalent.

For such γ, $\tau \mapsto (h \circ f \circ \widetilde{\gamma})(\tau)$ is absolutely continuous and thus

$$|(h \circ f)(\widetilde{\gamma}(\ell(\gamma))) - (h \circ f)(\widetilde{\gamma}(0))| \leq \int_0^{\ell(\gamma)} \left| \frac{d}{dt} h(f(\widetilde{\gamma}(\tau))) \right| d\tau.$$

Routine calculation involving the Lebesgue differentiation theorem yields

$$\left| \frac{d}{dt} h(f(\widetilde{\gamma}(\tau))) \right| \leq (\operatorname{lip} h)(f(\widetilde{\gamma}(\tau)) g(\widetilde{\gamma}(\tau)).$$

for a.e. $\tau \in [0, \ell(\gamma)]$. This completes the proof. $\square$

In order to have a rich theory of Sobolev spaces one needs to narrow the class of metric-measure spaces X. Following [33] we say that the space (X, d, μ) *supports the* p*-Poincaré inequality* for some $1 \leq p < \infty$ if μ is a doubling measure and there exist constants $C_P > 0$ and $\sigma \geq 1$ such that for every ball $B \subset X$, for every integrable function $f \in L^1(\sigma B)$ and for every $0 \leq g \in L^p(\sigma B)$ being a p-weak upper gradient of f on σB the inequality

$$\fint_B |f - f_B| \, d\mu \leq C_P (\operatorname{diam} B) \left(\fint_{\sigma B} g^p \, d\mu \right)^{1/p} \tag{5}$$

is satisfied.

It follows from the Hölder inequality that, if the space X supports the p-Poincaré inequality, then it supports the q-Poincaré inequality for every $q \geq p$.

The following result was proved in [34].

Proposition 11 *Suppose that the space (X, d, μ) supports the p-Poincaré inequality for some $1 \leq p < \infty$, as described above. Then, for every Banach space V, the pair (X, V) supports the p-Poincaré inequality in the following sense: there is a constant $C > 0$ such that, for every ball $B \subset X$, for every $F \in L^1(6\sigma B, V)$ and for every $0 \leq g \in L^p(6\sigma B)$ being a p-weak upper gradient of F on $6\sigma B$, the inequality*

$$\fint_B \|F - F_B\| \, d\mu \leq C (\operatorname{diam} B) \left(\fint_{6\sigma B} g^p \, d\mu \right)^{1/p}.$$

is satisfied.

It follows from the proof that, if $F \in N^{1,p}(X, V)$ and $0 \leq g \in L^p(X)$ is a p-weak upper gradient of F, then the following pointwise inequality is satisfied

$$\|F(x) - F(y)\| \leq C d(x, y) \left((\mathcal{M} g^p(x))^{1/p} + (\mathcal{M} g^p(y))^{1/p} \right) \quad \text{a.e.},$$

where $\mathcal{M}h(x) = \sup_{r>0} \fint_{B(x,r)} |h|\, d\mu$ is the Hardy–Littlewood maximal function. Therefore F restricted to the set $E_t = \{x : \mathcal{M}g^p \le t^p\}$ is Lipschitz continuous with the Lipschitz constant Ct.

We will need the following recent remarkable and unexpected result of Lee and Naor [46, Theorem 1.6].

Proposition 12 *Let X be a metric-doubling space, Y another metric space such that $X \subset Y$ and V a Banach space. Then every L-Lipschitz mapping $f : X \to V$ admits an extension to a $C(\log M)L$-Lipschitz mapping $\tilde{f} : Y \to V$, where $C > 0$ is a universal constant that does not depend on X, Y and V and M is the constant from the definition of the metric-doubling space.*

The following result is a generalization of [34, Theorem 6.7] to a general Banach space target.

Lemma 13 *Suppose the space (X, d, μ) supports the p-Poincaré inequality for some $1 \le p < \infty$ and V is a Banach space. If $F \in N^{1,p}(X, V)$, then for every $\varepsilon > 0$ there is a Lipschitz mapping $G \in \operatorname{Lip}(X, V)$ such that $\mu\{x : F(x) \ne G(x)\} < \varepsilon$ and $\|F - G\|_{1,p} < \varepsilon$.*

Proof Elementary estimates based on the triangle inequality show that the retraction $\pi_R : V \to \overline{B}(0, R) \subset V$, defined by $\pi_R(x) = x$ if $\|x\| \le R$ and $\pi_R(x) = Rx/\|x\|$ if $\|x\| > R$ is Lipschitz continuous with the Lipschitz constant 2. Hence for $F \in N^{1,p}(X, V)$, $\pi_R \circ F \to F$ is $N^{1,p}$ as $R \to \infty$ and therefore we can assume that F is bounded, i.e. values of F belong to some ball $B(0, R)$. Let E_t be defined as before. Then by Proposition 12 $F|_{E_t} : E_t \to V$ admits an extension to a Lipschitz mapping $F_t : X \to V$ with the Lipschitz constant bounded by $C't$. Of course we can assume that each of the mappings F_t takes values into $B(0, R)$ (as otherwise we compose it with π_R), i.e. the mappings F_t are uniformly bounded. Since the mapping F_t differs from F on a set $X \setminus E_t$ which is small in the sense that $t^p \mu(X \setminus E_t) \to 0$ as $t \to \infty$, it easily follows that $\|F - F_t\|_{1,p} \to 0$ as $t \to \infty$. □

Remark Instead of using the difficult Proposition 12 we could employ a Whitney type extension to define F_t. Proposition 12 will, however, be unavoidable in the proof of Theorem 5.

3 Lipschitz polyhedra

In the first part of the section we collect standard definitions and results concerning finite polyhedra. More details can be found, e.g. in Chapter 1 of [49]. Some of the definitions given here are slightly different from what the reader would find in books. This is because we wanted to make the definitions more adequate to our situation. Then we prove some technical results needed in the proof of Theorem 6.

By a *simplicial complex* we mean a finite collection K of simplexes in some Euclidean space $\mathbb{R}^\nu$ such that:

1. If $\sigma \in K$ and τ is a face of σ, then $\tau \in K$;

2. If $\sigma, \tau \in K$, then either $\sigma \cap \tau = \emptyset$ or $\sigma \cap \tau$ is a common face of σ and τ.

A subset of K which is itself a complex will be called a *subcomplex*.

The set $|K| = \bigcup_{\sigma \in K} \sigma$ will be called a *rectilinear polyhedron* and K will be called a *triangulation* of $|K|$.

The dimension of K is the supremum of dimensions of simplexes in K. It coincides with the Hausdorff dimension of $|K|$. We say that $|K|$ is a *k-homogeneous rectilinear polyhedron* if $|K|$ is a union of k-dimensional simplexes.

By a *Lipschitz polyhedron* we mean any metric space which is bi-Lipschitz homeomorphic to a rectilinear polyhedron. Similarly we define *k-homogeneous Lipschitz polyhedron*.

A class of examples of spaces supporting the p-Poincaré inequalities is provided by the following result.

Lemma 14 *Let $X = |K|$ be an n-homogeneous Lipschitz polyhedron. Suppose that k is the largest integer with the property that, for every $x, y \in X$, there is a sequence of n-dimensional simplexes $\sigma_1, \ldots, \sigma_\ell \in K$ such that $x \in \sigma_1$, $y \in \sigma_\ell$ and $\sigma_i \cap \sigma_{i+1}$ is a simplex of dimension at least k. If $k < n-1$, then the space X equipped with the Hausdorff measure $\mathcal{H}^n$ supports the p-Poincaré inequality if and only if $p > n-k$. If $k = n-1$, then the space supports the 1-Poincaré inequality.*

Proof The case $k < n-1$, $p > n-k$ follows easily from [33, Theorem 6.15]. The case $k = n-1$, $p = 1$ is proved in [57, Section 2]; White proves a weaker version of the Poincaré inequality, but his argument works in our case as well. □

If $X \subset \mathbb{R}^\nu$ is a set and $x_0 \in \mathbb{R}^{\nu+1} \setminus \mathbb{R}^\nu$, then we define the *cone* over X with vertex x_0, denoted by $\mathcal{C}(X)$, as the union of all segments that join x_0 with points in X. Clearly $\mathcal{C}(X)$ is a rectilinear polyhedron if X is a rectilinear polyhedron. Triangulation of X induces in a natural way a triangulation of $\mathcal{C}(X)$.

As a direct application of Lemma 14 we obtain

Lemma 15 *Let $S^k \subset \mathbb{R}^{k+1}$ be a sphere and $B^{n-k-1} \subset \mathbb{R}^{n-k-1}$ be a ball, where $1 \le k \le n-1$. Then the space $\mathcal{C}(S^k \times B^{n-k-1}) \subset \mathbb{R}^{n+1}$ equipped with the Euclidean metric and the Hausdorff measure $\mathcal{H}^n$ supports the 1-Poincaré inequality.*

If the dimension of the simplicial complex is n, then, for $\ell \le n$, K^ℓ will denote a subcomplex of K that consists of all simplexes in K of dimension less than or equal to ℓ.

We say that a rectilinear polyhedron X is a subpolyhedron of a rectilinear polyhedron Y if $X \subset Y$ (cf. definition on page 6 and Theorem 1.4.4. in [49]).

For the following lemma, see Theorem 1.4.4. in [49].

Lemma 16 *If X and Y are rectilinear polyhedra such that $X \subset Y$, then there is a triangulation of X which is a subcomplex of a triangulation of Y.*

The following result is a well known Relative Simplicial Approximation Theorem. For a proof see [49, Theorem 1.6.11].

Lemma 17 *Let X, Y and Z be rectilinear polyhedra with $X \subset Y$ and let $f : Y \to Z$ be a continuous mapping such that $f|_X$ is piecewise linear. Then, given $\varepsilon > 0$, there is a piecewise linear mapping $g : Y \to Z$ such that $f|_X = g|_X$ and $\|f - g\|_\infty < \varepsilon$.*

The following result will be employed in the proof of the main result, Theorem 6.

Lemma 18 *Let $|K| \subset \mathbb{R}^\nu$ be a rectilinear polyhedron. Assume that $\pi_1(|K|) = \pi_2(|K|) = \cdots = \pi_k(|K|) = 0$. Let Z be a polyhedral neighborhood of $|K^k|$ in $|K|$ such that $|K^k|$ is a deformation retract of Z. Then there is a Lipschitz mapping $\eta : \mathbb{R}^\nu \to |K|$ such that $\eta|_Z = \mathrm{id}_Z$.*

Remark $|K^k|$ is a deformation retract of Z in the sense that there is a continuous mapping $H : Z \times [0, 1] \to Z$ such that $H(\cdot, 0) = \mathrm{id}_Z$ and $H(\cdot, 1) : Z \to |K^k|$ is a retraction.

Proof Since $Z \subset \mathbb{R}^\nu$, the cone $\mathcal{C}(Z)$ is contained in $\mathbb{R}^{\nu+1}$. We can assume that the distance of the vertex x_0 of the cone to $\mathbb{R}^\nu$ equals 2. Thus every point in $\mathcal{C}(Z)$ can be denoted by $[z, t]$, where $z \in Z$ and $t \in [0, 2]$. Namely $[z, t]$ is the point of the segment that joints z to the vertex x_0 whose distance from $\mathbb{R}^\nu$ equals t. Observe that $[z, 2] = x_0$ for all $z \in Z$.

In the first step we construct a continuous retraction

$$\rho : \mathbb{R}^\nu \cup \mathcal{C}(Z) \to \mathcal{C}(Z).$$

Let Q be a large cube in $\mathbb{R}^\nu$ such that $|K| \subset \operatorname{Int} Q$. It follows from Lemma 16 that there is a triangulation of Z which is a subcomplex of a triangulation L of Q, $|L| = Q$.

We define ρ to be the identity in $\mathcal{C}(Z)$, and we set $\rho(\mathbb{R}^\nu \setminus \operatorname{Int} Q) = \{x_0\}$.

We still have to define ρ in $\operatorname{Int}(Q \setminus Z)$. In this step we use contractibility of $\mathcal{C}(Z)$ which implies vanishing of all homotopy groups of $\mathcal{C}(Z)$. First we define $\rho(x) = x_0$ whenever x is a vertex of L in $\operatorname{Int}(Q \setminus Z)$. Next we extend ρ to one dimensional edges in L that are contained in $\operatorname{Int}(Q \setminus Z)$ employing arcwise connectedness of $\mathcal{C}(Z)$. Then we extend ρ to interior of triangles in $\operatorname{Int}(Q \setminus Z)$ from their boundaries using the fact that $\pi_1(\mathcal{C}(Z)) = 0$. Similarly we can extend ρ to interiors of three dimensional simplexes in $\operatorname{Int} Q \setminus Z$ using $\pi_2(\mathcal{C}(Z)) = 0$, etc. This completes the construction of ρ.

Using the fact that $|K|$ is k-connected and the fact that $\mathcal{C}(|K^k|)$ consists of simplexes whose dimension does not exceed $k + 1$, we construct a continuous retraction

$$\gamma : \mathcal{C}(|K^k|) \cup |K| \to |K|, \qquad \gamma|_{|K|} = \mathrm{id}_{|K|}.$$

Namely we set $\gamma|_{|K|} = \mathrm{id}_{|K|}$, we define $\gamma(x_0)$ arbitrarily and extend γ to higher dimensional simplexes in $\mathcal{C}(|K^k|)$ using the k-connectivity of $|K|$, similarly as in the construction of ρ.

Now we define a continuous retraction

$$\lambda : \mathcal{C}(Z) \cup |K| \to |K|$$

by the formula

$$\lambda([z,t]) = \begin{cases} H(z,t) & \text{if } 0 \le t \le 1, \\ \gamma([H(z,1), 2(t-1)]) & \text{if } 1 \le t \le 2, \end{cases}$$

where H is defined as before. The mapping $\tilde{\eta}$ defined as a composition of mappings

$$\mathbb{R}^\nu \overset{i_1}{\subset} \mathbb{R}^\nu \cup \mathcal{C}(Z) \overset{\rho}{\to} \mathcal{C}(Z) \overset{i_2}{\subset} \mathcal{C}(Z) \cup |K| \overset{\lambda}{\to} |K|$$

i.e. $\tilde{\eta} = \lambda \circ i_2 \circ \rho \circ i_1 : \mathbb{R}^\nu \to |K|$ is a continuous mapping such that $\tilde{\eta}|_Z = \mathrm{id}_Z$. Observe that $\tilde{\eta}|_{\partial Q \cup Z}$ is piecewise linear. Hence it follows from Lemma 17 that there is a piecewise linear mapping $\eta : Q \to |K|$ which coincides with $\tilde{\eta}$ on $\partial Q \cup Z$. Now it is clear that the mapping η extended to $\mathbb{R}^\nu$ by $\eta(\mathbb{R}^\nu \setminus Q) = \gamma(x_0)$ has all desired properties. The proof is complete. □

4 Proof of Theorem 4

Let $\Phi : Y_1 \to Y_2$ be a bi-Lipschitz homeomorphism and $u \in N^{1,p}(X, Y_2)$. Then $\Phi^{-1} \circ u \in N^{1,p}(X, Y_1)$. By the assumptions of the theorem, for every $\varepsilon > 0$ there exists $f_\varepsilon \in \mathrm{Lip}\,(X, Y_1)$ such that $\| f_\varepsilon - \Phi^{-1} \circ u\|_{1,p} < \varepsilon$ and $\mu(\Omega_\varepsilon) < \varepsilon$ where $\Omega_\varepsilon = \{x : f_\varepsilon(x) \neq \Phi^{-1}(u(x))\}$. It suffices to prove that $\Phi \circ f_\varepsilon \to u$ in $N^{1,p}(X, V_2)$ as $\varepsilon \to 0$. Since the space Y_2 has finite diameter, the mappings into Y_2 are uniformly bounded as mappings into V_2 and hence $\Phi \circ f_\varepsilon \to u$ in $L^p(X, V_2)$ as a consequence of the a.e. convergence. Thus we need to take care of the upper gradient estimates only. Let $0 \le g \in L^p(X)$ and $0 \le g_\varepsilon \in L^p(X)$ be upper gradients of $\Phi^{-1} \circ u$ and $(f_\varepsilon - \Phi^{-1} \circ u)$ respectively such that $\|g_\varepsilon\|_p < \varepsilon$. Then $g + g_\varepsilon$ is an upper gradient of f_ε. If L is the Lipschitz constant of Φ, then Lg and $L(g + g_\varepsilon)$ are upper gradients of u and $\Phi \circ f_\varepsilon$ respectively and hence $L(2g + g_\varepsilon)$ is an upper gradient of $u - \Phi \circ f_\varepsilon$. Now we choose a closed set $F_\varepsilon \subset X \setminus \Omega_\varepsilon$ such that $\mu(X \setminus F_\varepsilon) < 2\varepsilon$. Lemma 9 yields that $h_\varepsilon = L(2g + g_\varepsilon)\chi_{X \setminus F_\varepsilon}$ is an upper gradient of $u - \Phi \circ f_\varepsilon$ and, since $\|h_\varepsilon\|_p \to 0$ as $\varepsilon \to 0$, the claim follows. The proof is complete. □

5 Proof of Theorem 5

Let $\Phi : Y \to Y_1$ be a bi-Lipschitz homeomorphism, where $Y_1 \subset V_1$ is a Lipschitz neighborhood retract of the Banach space V_1, i.e. there is a Lipschitz retraction $\pi_1 : V_1 \supset U_1 \to Y_1$ of some neighborhood U_1 of Y_1 onto Y_1. Let $Y \subset V$ be an isometric embedding into some Banach space V. Then Y is a Lipschitz neighborhood retract of V. Indeed, according to Proposition 12, Φ admits an extension to a Lipschitz mapping $\widetilde{\Phi} : V \to V_1$ and hence $\pi = \Phi^{-1} \circ \pi_1 \circ \widetilde{\Phi} : U \to Y$, where $U = (\widetilde{\Phi})^{-1}(U_1)$ is a Lipschitz retraction.

Now we prove the density. By Lemma 13 we can approximate $F \in N^{1,p}(X, Y)$ by Lipschitz mappings $F_t \in \mathrm{Lip}\,(X, V)$ which coincide with F outside sets $X \setminus E_t$ of arbitrarily small measure. If we could prove that mappings F_t take values into the neighborhood U, then the same argument as in the proof of Theorem 4 would imply that the composed map $\pi \circ F_t$ approximates F. If $p > Q$, this follows from the Sobolev embedding into Hölder continuous functions [26], and if $p = Q$, we need to follow the argument from the proof of [24, Theorem 1.3]. Although [24, Theorem 1.3] concerns classical Sobolev spaces, the argument is pretty much the same and it is a routine procedure to translate it to the setting of metric spaces. Therefore we will sketch it only. The set $X \setminus E_t$ is open. Since the space X is doubling there is a family of Whitney balls in $X \setminus E_t$ and associated Lipschitz partition of unity. Denote it by $\{B_i\}_{i \in I}$ and $\{\varphi_i\}_{i \in I}$ respectively. Now we define

$$H_t = \begin{cases} F(x) & \text{for } x \in E_t, \\ \sum_{i \in I} \varphi_i(x) F_{B_i} & \text{for } x \in X \setminus E_t. \end{cases}$$

Estimates as in the proof of [24, Theorem 1.3] imply that dist $(H_t(x), Y)$ is bounded from above by

$$(\operatorname{diam} B) \left(\fint_{\sigma B} g^Q \, d\mu \right)^{1/Q}, \tag{6}$$

where B is a ball of small radius centered at x and $0 \le g \in L^Q(X)$ is an upper gradient of F. Now it follows from the lower mass bound that the right hand side of (6) goes to zero with the radius of the ball.

The proof of the homotopy between f_1 and f_2 is similar to that of [57, Theorem 2] and left to the reader. The proof is complete. □

6 Proof of Theorem 6

First we show necessity of the vanishing of the homotopy groups. Assume that $\pi_k(Y) \neq 0$ for some $1 \le k \le [p]$. We have to construct a space X supporting the p-Poincaré inequality with the property that Lipschitz mappings are not dense in $N^{1,p}(X, Y)$. Let $n > p$ be an integer. According to Lemma 15 the space $X = \mathcal{C}(S^k \times B^{n-k-1})$ supports the 1-Poincaré inequality and therefore the p-Poincaré inequality. Denote points of the cone by

$$[s, b, t], \text{ where } s \in S^k, b \in B^{n-k-1}, \text{ and } t \in [0, 1].$$

This time the height of the cone is 1. For $b \in B^{n-k-1}$ and $t \in [0, 1)$ by $[S^k, b, t]$ we will denote the corresponding section of the cone. Obviously the section is homothetic to S^k.

Denote the vertex of the cone $\mathcal{C}(S^k \times B^{n-k-1})$ by x_0. Then $\mathcal{C}(S^k \times B^{n-k-1}) \setminus \{x_0\}$ is a smooth (non-compact) submanifold of $\mathbb{R}^{n+1}$ (with boundary) and hence the class of $N^{1,p}$ Sobolev functions coincides with the class of $W^{1,p}$ functions on $\mathcal{C}(S^k \times B^{n-k-1}) \setminus \{x_0\}$ (Proposition 8). It is easy to see that $C_p(\{x_0\}) = 0$ for $1 \le p < n$, where C_p is the capacity associated with $N^{1,p}$. Hence, according to Lemma 3.6 in

[51], the p-modulus of the family of rectifiable curves in $\mathcal{C}(S^k \times B^{n-k-1})$ passing through $\{x_0\}$ is zero and thus

$$\begin{aligned} N^{1,p}(\mathcal{C}(S^k \times B^{n-k-1})) &= N^{1,p}(\mathcal{C}(S^k \times B^{n-k-1})\setminus\{x_0\})) \\ &= W^{1,p}(\mathcal{C}(S^k \times B^{n-k-1})\setminus\{x_0\})), \end{aligned}$$

provided $1 \le p < n$. A direct estimate shows that the mapping

$$P : \mathcal{C}(S^k \times B^{n-k-1}) \to S^k, \qquad P([s, b, t]) = s.$$

belongs to $W^{1,p}(\mathcal{C}(S^k \times B^{n-k-1})\setminus\{x_0\})$ for all $1 \le p < n$. Hence

Lemma 19 *$P \in N^{1,p}(\mathcal{C}(S^k \times B^{n-k-1}), S^k)$ for $1 \le p < n$.*

Let $\varphi : S^k \to Y$ be a Lipschitz representative of a nontrivial element in the homotopy group $\pi_k(Y)$. We claim that the mapping $f = \varphi \circ P : \mathcal{C}(S^k \times B^{n-k-1}) \to Y$ cannot be approximated by continuous (and hence Lipschitz) mappings. We argue by contradiction. Assume that $f_i : \mathcal{C}(S^k \times B^{n-k-1}) \to Y$ is a sequence of continuous mappings converging to f in the norm of $N^{1,p}$. Applying Fubini's theorem we obtain that for some subsequence (still denoted by f_i) for a.e. $t \in [0, 1)$ and a.e. $b \in B^{n-k-1}$

$$f_i|_{[S^k,b,t]} \to f|_{[S^k,b,t]}$$

in the norm of $N^{1,p}(S^k)$. Now it follows from Theorem 5 that f_i and f are homotopic for sufficiently large i. This however is not possible because all the mappings $f_i|_{S^k\times\{b\}\times\{t\}}$ are contractible to a point, while $f|_{S^k\times\{b\}\times\{t\}}$ is not. This completes the proof of the necessity of the homotopy condition.

We are left with the proof of the density assuming the homotopy condition for Y. Therefore we assume that the space X supports the p-Poincaré inequality.

In the proof of the theorem we will need the following technical lemma.

Lemma 20 *Let X be a space that supports the p-Poincaré inequality, $1 \le p < \infty$, and let $Y \subset V$ be a metric space of bounded diameter that is isometrically embedded into a Banach space V. Assume that a compact subset $Z \subset Y$ has the property that there is a Lipschitz mapping $\eta : V \to Y$ such that $\eta|_Z = \mathrm{id}_Z$. Then any Sobolev mapping $f \in N^{1,p}(X, Y)$ such that $f(X) \subset Z$ can be approximated by Lipschitz mappings* Lip (X, Y) *in the $N^{1,p}$ norm.*

Proof According to Lemma 13 there is a sequence of Lipschitz mappings $f_k : X \to V$ such that $\mu(\{f_k \ne f\}) \to 0$ and $\|f - f_k\|_{1,p} \to 0$ as $k \to \infty$. Then $\eta \circ f_k \to f$ in the norm of $N^{1,p}$ by the same argument as in the proof of Theorem 4. We leave the details to the reader. □

According to Theorem 4 it suffices to assume that Y is a rectilinear polyhedron in the Euclidean space, $Y \subset \mathbb{R}^\nu$, because we will prove that in this case Lipschitz mappings are dense in the strong sense described in Theorem 4.

Lemmas 18 and 20 show that in order to prove the density of Lipschitz mappings in $N^{1,p}(X, Y)$ it suffices to prove that every mapping in $N^{1,p}(X, Y)$ can be approximated by mappings into a polyhedral neighborhood Z of $Y^{[p]}$ such that $Y^{[p]}$ is a deformation retract of Z. This will be achieved by making "small holes" in the image around the dual skeleton of $Y^{[p]}$. This gives an idea of what we are going to do next.

Let σ^ℓ be an ℓ-dimensional simplex. If $y \in \sigma^\ell$, then every point $x \in \sigma^\ell$ can be represented in the form $x = y + t(z - y)$ for some $z \in \partial\sigma^\ell$ and $t \in [0, 1]$. Given $y \in \sigma^\ell$ and $t \in (0, 1]$ we define

$$\sigma^\ell_{y,t} = \{y + s(z - y) : \ z \in \partial\sigma^\ell \text{ and } 0 \le s \le t\}.$$

This is a simplex obtained from σ^ℓ by dilation. In the case y is the barycenter of σ^ℓ, we simply write σ^ℓ_t. For $y \in \sigma^\ell_{1/2}$ and $t \in (0, 1]$ we define the mapping

$$P_{y,t} : \sigma^\ell \to \sigma^\ell$$

by the formula

$$P_{y,t}(y + s(z - y)) = \begin{cases} z & \text{if } t \le s \le 1, \\ y + \frac{s}{t}(z - y) & \text{if } 0 \le s \le t. \end{cases}$$

Geometrically speaking $P_{y,t}$ is the retraction of the neighborhood $\sigma^\ell \setminus \sigma^\ell_{y,t}$ of $\partial\sigma^\ell$ onto $\partial\sigma^\ell$ along the rays emerging from y and dilation of $\sigma^\ell_{y,t}$ onto σ^ℓ. The mappings $P_{y,t}$ are Lipschitz continuous with the Lipschitz constant Ct^{-1}, where the same constant C can be chosen for all $y \in \sigma^\ell_{1/2}$. The pointwise limit of $P_{y,t}$ as $t \to 0$ is $P_y : \sigma^\ell \setminus \{y\} \to \partial\sigma^\ell$, $P_y(y + t(z - y)) = z$ for all $z \in \partial\sigma^\ell$ and $t \in (0, 1]$. P_y considered as a mapping from σ^ℓ is discontinuous at y.

For every $y \in \sigma^\ell_{1/2}$ and $t \in (0, 1)$ we have the estimate for the lower Lipschitz constant of $P_{y,t}$

$$\operatorname{lip} P_{y,t}(x) \le \begin{cases} C \operatorname{diam} \sigma^\ell / |y - x| & \text{if } x \in \sigma^\ell \setminus \sigma^\ell_{y,t}, \\ Ct^{-1} & \text{if } x \in \sigma^\ell_{y,t}. \end{cases}$$

If there are more simplexes and the above construction is applied to one of them, we point out the simplex in question by writing $P^{\sigma^\ell}_{y,t}$ and $P^{\sigma^\ell}_y$.

Let $\dim Y = \ell$. If $p \ge \ell$, then Y is a Lipschitz retract of $\mathbb{R}^\nu$ (Lemma 18) and hence the density of Lipschitz mappings follows from Lemma 20. Accordingly, we can assume that $p < \ell$. Let σ^ℓ be an ℓ-dimensional simplex in the triangulation of Y. For $y \in \sigma^\ell_{1/2}$ and $\varepsilon, t \in (0, 1)$ we consider the mapping

$$Q_{y,\varepsilon,t} := P^{\sigma^\ell_{y,\varepsilon}}_{y,t} : \sigma^\ell_{y,\varepsilon} \to \sigma^\ell_{y,\varepsilon} \tag{7}$$

and extend it to the entire Y by the identity, i.e. $Q_{y,\varepsilon,t}(z) = z$ for $z \in Y \setminus \sigma^\ell_{y,\varepsilon}$. Clearly $Q_{y,\varepsilon,t} \in \operatorname{Lip}(Y, Y)$. Pointwise limit of $Q_{y,\varepsilon,t}$ as $t \to 0$ is the retraction

$$Q_{y,\varepsilon} : Y \setminus \{y\} \to Y \setminus \sigma^\ell_{y,\varepsilon}$$

along the rays in $\sigma^\ell_{y,\varepsilon}$ onto the boundary $\partial\sigma^\ell_{y,\varepsilon}$ and the identity in $Y\setminus\sigma^\ell_{y,\varepsilon}$. The estimate of the lower Lipschitz constant of $P_{y,\varepsilon}$ readily gives

$$\operatorname{lip} Q_{y,\varepsilon,t}(z) \le \begin{cases} C\frac{\varepsilon \operatorname{diam}\sigma^\ell}{|y-z|} & \text{if } z \in \sigma^\ell_{y,\varepsilon}\setminus\sigma^\ell_{y,\varepsilon t} \\ Ct^{-1} & \text{if } z \in \sigma^\ell_{y,\varepsilon t}, \\ 1 & \text{if } z \in Y\setminus\sigma^\ell_{y,\varepsilon}, \end{cases} \tag{8}$$

and hence

$$\operatorname{lip} Q_{y,\varepsilon,t}(z) \le R_{y,\varepsilon}(z) = \begin{cases} C\frac{\varepsilon \operatorname{diam}\sigma^\ell}{|y-z|} & \text{if } z \in \sigma^\ell_{y,\varepsilon}\setminus\{y\}, \\ 1 & \text{if } z \in Y\setminus\sigma^\ell_{y,\varepsilon}, \end{cases}$$

for all $t \in (0,1)$ with the same constant C for all simplexes σ^ℓ in the triangulation of Y and all $y \in \sigma^\ell_{1/2}$.

Let $f \in N^{1,p}(X,Y)$ and let $0 \le g \in L^p(X)$ be an upper gradient of f. Fix σ^ℓ in the triangulation of Y and $\varepsilon, t \in (0,1)$. Clearly $Q_{y,\varepsilon,t}\circ f \in N^{1,p}(X,Y)$ for every $y \in \sigma^\ell_{1/2}$.

The mapping $Q_{y,\varepsilon,t}\circ f$ differs from f on the set $f^{-1}(\sigma^\ell_{y,\varepsilon})$. We want to find $y \in \sigma^\ell_{1/2}$ such that the $N^{1,p}$ norm of $Q_{y,\varepsilon,t}\circ f$ on that set $f^{-1}(\sigma^\ell_{y,\varepsilon})$ is relatively small.

According to Lemma 10, $G_{y,\varepsilon}(x) = R_{y,\varepsilon}(f(x))g(x)$ is a common p-weak upper gradient of the functions $Q_{y,\varepsilon,t}\circ f$ for all $t \in (0,1)$. We have

$$\begin{aligned} &\fint_{\sigma^\ell_{1/2}} \int_{f^{-1}(\sigma^\ell_{y,\varepsilon})} G^p_{y,\varepsilon}(x)\, d\mu(x)\, d\mathcal{H}^\ell(y) \\ &\quad = (\mathcal{H}^\ell(\sigma^\ell_{1/2}))^{-1} \int_X g^p(x) \int_{\{y\in\sigma^\ell_{1/2}:\, f(x)\in\sigma^\ell_{y,\varepsilon}\}} R^p_{y,\varepsilon}(f(x))\, d\mathcal{H}^\ell(y)\, d\mu(x) \\ &\quad \le C(\operatorname{diam}\sigma^\ell)^{-\ell} \int_X g^p(x) \int_{\{y\in\sigma^\ell_{1/2}:\, f(x)\in\sigma^\ell_{y,\varepsilon}\}} \left(\frac{\varepsilon \operatorname{diam}\sigma^\ell}{|y-f(x)|}\right)^p d\mathcal{H}^\ell(y)\, d\mu(x). \end{aligned}$$

Let $B^\ell(0,r)$ be the ball in $\mathbb{R}^\ell$ whose volume equals

$$\mathcal{H}^\ell\left(\left\{y \in \sigma^\ell_{1/2} :\ f(x) \in \sigma^\ell_{y,\varepsilon}\right\}\right).$$

Then $r \le C\varepsilon \operatorname{diam}\sigma^\ell$ and hence

$$\begin{aligned} \int_{\{y\in\sigma^\ell_{1/2}:f(x)\in\sigma^\ell_{y,\varepsilon}\}} \left(\frac{\varepsilon \operatorname{diam}\sigma^\ell}{|y-f(x)|}\right)^p d\mathcal{H}^\ell(y) &\le (\varepsilon \operatorname{diam}\sigma^\ell)^p \int_{B^\ell(0,r)} \frac{d\mathcal{H}^\ell(y)}{|y|^p} \\ &\le C(\varepsilon \operatorname{diam}\sigma^\ell)^\ell. \end{aligned}$$

The above estimates yield

$$\fint_{\sigma^\ell_{1/2}} \int_{f^{-1}(\sigma^\ell_{y,\varepsilon})} G^p_{y,\varepsilon}(x)\,d\mu(x)\,d\mathcal{H}^\ell(y) \leq C\varepsilon^\ell \int_X g^p(x)\,d\mu(x).$$

Thus there is a subset $A_\varepsilon \subset \sigma^\ell_{1/2}$ such that $\mathcal{H}^\ell(A_\varepsilon) \geq \mathcal{H}^\ell(\sigma^\ell_{1/2})/2$ and

$$\int_{f^{-1}(\sigma^\ell_{y,\varepsilon})} G^p_{y,\varepsilon}(x)\,d\mu(x) \leq C\varepsilon^\ell \int_X g^p(x)\,d\mu(x)$$

for all $y \in A_\varepsilon$.

Fix $y \in A_\varepsilon$ such that $\mu(f^{-1}(y)) = 0$. Observe that for $i \leq j$

$$\{Q_{y,\varepsilon,1/i} \circ f \neq Q_{y,\varepsilon,1/j} \circ f\} \subset f^{-1}(\sigma^\ell_{y,\varepsilon/i})$$

and $\{f^{-1}(\sigma^\ell_{y,\varepsilon/i})\}_i$ is a decreasing sequence of sets with the intersection of measure zero

$$\bigcap_{i=1}^\infty f^{-1}(\sigma^\ell_{y,\varepsilon/i}) = f^{-1}(y).$$

Since all mappings $\{Q_{y,\varepsilon,1/i} \circ f\}_i$ have common p-weak upper gradient $G_{y,\varepsilon} \in L^p$, it easily follows from Lemma 9 that they form a Cauchy sequence in $N^{1,p}$. Accordingly, the pointwise convergence

$$Q_{y,\varepsilon,1/i} \circ f \to Q_{y,\varepsilon} \circ f \quad (\text{in } X \setminus f^{-1}(y) \text{ and hence a.e.})$$

is also a convergence in the $N^{1,p}$ norm and $G_{y,\varepsilon}$ is a p-weak upper gradient of $Q_{y,\varepsilon} \circ f$ (cf. [23, Lemma 7.8], [51, Lemma 4.11]). Now since the measures of the sets

$$\{Q_{y,\varepsilon} \circ f \neq f\} \subset f^{-1}(\sigma^\ell_{y,\varepsilon})$$

and the functions $G_{y,\varepsilon}$ are monotonically decreasing to 0 when ε is decreasing to 0, it follows that

$$\|f - Q_{y,\varepsilon} \circ f\|_{1,p} \to 0 \quad \text{as } \varepsilon \to 0.$$

Denote the k-dimensional simplexes in the triangulation of Y, $0 \leq k \leq \ell$, by $\{\sigma^{k,i}\}_{i=1}^{r_k}$. We can repeat the above construction in every ℓ-dimensional simplex $\sigma^{\ell,i}$ and find corresponding points $y^\ell_i \in \sigma^{\ell,i}_{1/2}$. Next we define

$$Q_{y^\ell_1,\ldots,y^\ell_{r_\ell};\varepsilon} \circ f := Q_{y^\ell_1,\varepsilon} \circ \cdots \circ Q_{y^\ell_{r_\ell},\varepsilon} \circ f \in N^{1,p}.$$

We have

$$Q_{y_1^\ell,\dots,y_{r_\ell}^\ell;\varepsilon}\circ f\to f\quad \text{in } N^{1,p} \text{ as } \varepsilon\to 0.$$

Roughly speaking $Q_{y_1^\ell,\dots,y_{r_\ell}^\ell;\varepsilon}\circ f$ is a deformation of f obtained from f by making small holes in every ℓ-dimensional simplex in the target Y. Therefore

$$Q_{y_1^\ell,\dots,y_{r_\ell}^\ell;\varepsilon}\circ f : X\to Z=Y\setminus\bigcup_{i=1}^{r_\ell}\sigma^{\ell,i}_{y_i^\ell,\varepsilon}.$$

Observe that $Y^{\ell-1}$ is a deformation retract of Z. Indeed, in each set $\sigma^{\ell,i}\setminus\sigma^{\ell,i}_{y_i^\ell,\varepsilon}$ the deformation retraction is along rays emerging from y_i^ℓ. Thus, if $p\geq \ell-1$, the theorem follows from Lemmas 18 and 20 as indicated earlier. Therefore we can assume that $p<\ell-1$.

Assume that the points $y_i^\ell\in\sigma^{\ell,i}$, $i=1,2,\dots,r_\ell$, are chosen as above and fixed. We will prove that for every $\varepsilon'\in(0,1)$ the mapping $f'=Q_{y_1^\ell,\dots,y_{r_\ell}^\ell;\varepsilon'}\circ f$ can be approximated by mappings taking values into a polyhedral neighborhood Z' of $Y^{\ell-2}$ such that $Y^{\ell-2}$ is a deformation retract of Z'. (As before, Lemmas 18 and 20 will prove the theorem in the case in which $p\geq\ell-2$; if $p<\ell-2$, we will have to continue the construction). To obtain a deformation of f' to a mapping with values into Z' we need to repeat the previous construction by making holes corresponding to isolated points in the $(\ell-1)$-dimensional simplexes. This is, however, more involved. Indeed, each point chosen in the $(\ell-1)$-dimensional simplex being part of the boundary of $\sigma^{\ell,i}$ will form a singularity in $\sigma^{\ell,i}\setminus\sigma^{\ell,i}_{y_i^\ell,\varepsilon}$ being the segment on the ray emerging from y_i^ℓ and passing through the given point on the boundary.

More precisely, let $\sigma^{\ell-1}$ be a simplex in the $(\ell-1)$-dimensional skeleton of Y. There are two cases. The easy one is when $\sigma^{\ell-1}$ is *not* a face of any ℓ-dimensional simplex in the triangulation of Y. In this case, exactly as before, we find $y\in\sigma^{\ell-1}_{1/2}$ such that the retraction along rays

$$Q_{y,\varepsilon}:\sigma^{\ell-1}_{y,\varepsilon}\setminus\{y\}\to\partial\sigma^{\ell-1}_{y,\varepsilon}$$

extended to $Y\setminus\{y\}$ as the identity in $Y\setminus\sigma^{\ell-1}_{y,\varepsilon}$ has the property that

$$Q_{y,\varepsilon}\circ f'\to f'\quad\text{in } N^{1,p} \text{ as } \varepsilon\to 0.$$

Now assume that $\sigma^{\ell-1}$ is a face of at least one ℓ-dimensional simplex, say $\sigma^{\ell-1}\subset\partial\sigma^{\ell,i}$. With each $y\in\sigma^{\ell-1}_{1/2}$ we associate a segment

$$I_{y,i}=\{y_i^\ell+s(y-y^{\ell_i}):\ \varepsilon'\leq s\leq 1\}\subset\sigma^{\ell,i}\setminus\sigma^{\ell,i}_{y_i^\ell,\varepsilon'}.$$

Given $y \in \sigma^{\ell-1}_{1/2}$ and $\varepsilon, t \in (0,1)$ we define the mapping

$$Q_{y,\varepsilon,t} : \sigma^{\ell-1}_{y,\varepsilon} \to \sigma^{\ell-1}_{y,\varepsilon}$$

as an $(\ell-1)$-dimensional analog of the mapping (7). We extend it to $\partial\sigma^{\ell,i}$ as the identity in $\partial\sigma^{\ell,i}\backslash\sigma^{\ell-1}_{y,\varepsilon}$. Note that every point in $\sigma^{\ell,i}\backslash\sigma^{\ell,i}_{y^\ell_i,\varepsilon'}$ is of the form

$$y^\ell_i + s(z - y^\ell_i) \quad \text{for some } z \in \partial\sigma^{\ell,i} \text{ and } \varepsilon' \le s \le 1.$$

Now we extend $Q_{y,\varepsilon,t}$ from the boundary of $\sigma^{\ell,i}$ to $\sigma^{\ell,i}\backslash\sigma^{\ell,i}_{y^\ell_i,\varepsilon'}$ by the formula

$$\widetilde{Q}_{y,\varepsilon,t}(y^\ell_i + s(z - y^\ell_i)) = y^\ell_i + s(Q_{y,\varepsilon,t}(z) - y^\ell_i).$$

If we fix y and ε and let $t \to 0$, the above mapping will converge to

$$\widetilde{Q}_{y,\varepsilon} : (\sigma^{\ell,i}\backslash\sigma^{\ell,i}_{y^\ell_i,\varepsilon'})\backslash I_{y,i} \to \sigma^{\ell,i}\backslash\sigma^{\ell,i}_{y^\ell_i,\varepsilon'}$$

with the segment $I_{y,i}$ as the singularity set. Given $y \in \sigma^{\ell-1}_{1/2}$, we define the mapping $\widetilde{Q}_{y,\varepsilon,t}$ in each simplex $\sigma^{\ell,i}$ that has $\sigma^{\ell-1}$ as a face and then we extend it to the remaining part of Y as identity. The resulting mapping $\widetilde{Q}_{y,\varepsilon,t} : Y \to Y$ is Lipschitz continuous with the estimate

$$\operatorname{lip} \widetilde{Q}_{y,\varepsilon,t}(y^\ell_i + s(z - y^\ell_i)) \le \begin{cases} C\frac{\varepsilon \operatorname{diam} \sigma^{\ell-1}}{|y-z|} & \text{if } z \in \sigma^{\ell-1}_{y,\varepsilon}\backslash\sigma^{\ell-1}_{y,\varepsilon t}, \\ Ct^{-1} & \text{if } z \in \sigma^{\ell-1}_{y,\varepsilon t}, \\ 1 & \text{if } z \in \partial\sigma^{\ell,i}\backslash\sigma^{\ell-1}_{y,\varepsilon}. \end{cases}$$

Therefore $\widetilde{Q}_{y,\varepsilon,t} \circ f' \in N^{1,p}(X,Y)$ and similarly as before we can use the Fubini theorem to select $y \in \sigma^{\ell-1}_{1/2}$ such that

$$\widetilde{Q}_{y,\varepsilon,t} \circ f' \to \widetilde{Q}_{y,\varepsilon} \circ f' \quad \text{in } N^{1,p} \text{ as } t \to 0$$

and

$$\widetilde{Q}_{y,\varepsilon} \circ f' \to f' \quad \text{in } N^{1,p} \text{ as } \varepsilon \to 0.$$

Applying the above construction to every $(\ell-1)$-dimensional simplex in the triangulation of Y we find $y^{\ell-1}_i \in \sigma^{\ell-1,i}$, $i = 1, 2, \dots r_{\ell-1}$ such that

$$Q_{y^{\ell-1}_1,\dots,y^{\ell-1}_{r_{\ell-1}};\varepsilon} \circ f' = \widetilde{Q}_{y^{\ell-1}_1,\varepsilon} \circ \cdots \circ \widetilde{Q}_{y^{\ell-1}_{r_{\ell-1}},\varepsilon} \circ f' \to f' \quad \text{as } \varepsilon \to 0.$$

It is easily seen that each of the above mappings map X into a polyhedral neighborhood Z' of $Y^{\ell-2}$ such that $Y^{\ell-2}$ is a deformation retract od Z'. As already explained, it proves the theorem in the case in which $p \geq \ell - 2$. If $p < \ell - 2$, we have to continue the above procedure (as long as needed) by making "holes" corresponding to points in the lower dimensional simplexes. We leave details to the reader. The proof is complete. $\square$

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