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A SARD TYPE THEOREM FOR BOREL MAPPINGS

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We find a condition for a Borel mapping $f : \mathbb{R}^m \to \mathbb{R}^n$ which implies that the Hausdorff dimension of $f^{-1}(y)$ is less than or equal to $m - n$ for almost all $y \in \mathbb{R}^n$.

1. Introduction and statement of result. Sard's theorem ([14], [13], [15]) implies that if $f \in C^\infty(\mathbb{R}^m, \mathbb{R}^n)$, then for almost all $y \in \mathbb{R}^n$, $f^{-1}(y)$ is a smooth manifold of dimension $m - n$. In general this type of result is no longer true for continuous mappings. Peano curve type examples show that the set $f^{-1}(y)$ can have Hausdorff dimension greater than $m - n$ for all $y \in \mathbb{R}^n$. We will be concerned here with Borel mappings $f : \mathbb{R}^m \to \mathbb{R}^n$, without any smoothness assumptions, hence we cannot expect that $f^{-1}(y)$ is a manifold. However, we can ask what conditions (different than smoothness conditions) f should satisfy in order to get the estimate $\dim_{\mathrm{H}}(f^{-1}(y)) \le m - n$ for almost all $y \in \mathbb{R}^n$ ($\dim_{\mathrm{H}}$ stands for the Hausdorff dimension).

The Eilenberg inequality ([3], [4], [2], [5, Theorem 2.10.25]) implies that the above estimate holds a.e. provided f is Lipschitz. We will generalize this result.

This article grew out from the author's interest in the theory of Sobolev mappings. Related Sard type theorems for Sobolev mappings are presented in [7], [6] and in a slightly different form in [1]. Namely, a Sobolev mapping satisfies a similar assumption to that of Theorem 1 below. Hence by the method presented in the proof of Theorem 1 we get a similar estimate of the Hausdorff dimension of $f^{-1}(y)$, where f is a suitable Sobolev mapping. This estimate is of crucial importance for the validity of the so-called *co-area* formula (see [7]). It is also important in the context of nonlinear elasticity (see [6]). It seems that the method of the proof of Theorem 1 is more important than the theorem itself. This is because we can use the same method to produce a large class of theorems by slightly modifying the assumptions and the claim.

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Let H^k denote Hausdorff measure. We aim to prove the following

THEOREM 1. *Let* $f:\mathbb{R}^m\to\mathbb{R}^n$, *where* $n\le m$, *be a Borel mapping. Assume that there exists a constant* C *such that*

$$H^n(f(A))\le CH^n(A) \tag{1}$$

for all Borel subsets $A\subseteq\mathbb{R}^m$. *Then* $\dim_{\mathrm{H}}(f^{-1}(y))\le m-n$ *for almost all* $y\in\mathbb{R}^n$.

Remarks. 1) Note that if A is a Borel set, then $f(A)$ is measurable—see Proposition 2 below.

2) The condition (1) is satisfied by Lipschitz mappings.

3) In fact, we will prove a slightly stronger result—see the remark at the end of the paper.

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2. Basic notion. Before we proceed to the proof, let us recall some basic notion and results which will be used in the sequel.

Let $0<k<m$ be integers. By $O^*(m,k)$ we denote the manifold of all orthogonal projections from $\mathbb{R}^m$ onto k-dimensional linear subspaces. $O^*(m,k)$ is equipped with the unique invariant measure $\vartheta^*_{m,k}$ normalized to have total mass 1. As usual by $N(f|A,y)$ we denote the Banach indicatrix, i.e. the number of points in $f^{-1}(y)\cap A$. The *integral-geometric measure* is defined for Borel sets by

$$I^k(B)=\frac{1}{\beta(m,k)}\int\limits_{p\in O^*(m,k)}\int\limits_{y\in\mathrm{image}\,p}N(p|B,y)\,dH^k(y)\,d\vartheta^*_{m,k}(p)$$

($\beta(m,k)$ is a normalizing coefficient).

We will use the following theorem of Mattila.

PROPOSITION 1 ([12, Theorem 4.7]). *If* $E\subseteq\mathbb{R}^m$ *is such that* $I^k(E)<\infty$, *then* $\dim_{\mathrm{H}}(E)\le k$.

The well known theorem of Lusin and Sierpiński ([5, Theorem 2.2.13], [8, Lemma 39.2, Theorem 94], [9], [10], [11, p. 44]) states that if $A\subseteq\mathbb{R}^{k+l}$ is a Borel subset and $\pi_k:\mathbb{R}^{k+l}\to\mathbb{R}^k$ is the standard projection, then the set $\pi_k(A)$ is H^k-measurable. It is also known that not only projections, but an arbitrary Borel image of a Borel set is measurable (see [9, Chap. 3.38.III, Th. 2; Chap. 3.38.IV, Remark 1]). However, for the reader's convenience we present a "one-line" proof of this fact.

PROPOSITION 2. *If* $f:\mathbb{R}^m\to\mathbb{R}^n$ *is a Borel mapping, and* $A\subseteq\mathbb{R}^m$ *is a Borel subset, then* $f(A)$ *is measurable.*

Proof. Let $\pi_n : \mathbb{R}^{m+n} \to \mathbb{R}^n$ and $\pi_m : \mathbb{R}^{m+n} \to \mathbb{R}^m$ denote the standard projections. Let $G_f \subseteq \mathbb{R}^{m+n}$ denote the graph of f. Since f is Borel, G_f is a Borel set and hence by Lusin–Sierpiński's theorem $f(A) = \pi_n(G_f \cap \pi_m^{-1}(A))$ is measurable.

3. Proof of Theorem 1. Since the problem is local it suffices to consider the mapping f restricted to a ball B^m.

Step 1: $m = n$. For each $p \in \mathbb{N}$, let $H_p = \{A_{i,p}\}_{i=1}^p$ be a decomposition of B^n into Borel sets (i.e. $B^n = \bigcup_{i=1}^p A_{i,p}$, $A_{i,p} \cap A_{j,p} = \emptyset$ for $i \neq j$) such that each $A_{i,p} \in H_p$ is a union of elements from H_{p+1} and $\sup_i \operatorname{diam} A_{i,p} \to 0$ as $p \to \infty$. Evidently

$$N(f|B^n, y) = \lim_{p\to\infty} \sum_{i=1}^{p} \chi_{f(A_{i,p})}(y)$$

(χ_E stands for the characteristic function of E). Note that the above limit exists, since the sequence is non-decreasing. It follows directly from Proposition 2 (or, in a more elementary way, from (1)) that $N(f|B^n, \cdot)$ is a measurable function. Hence by (1) we obtain

$$\begin{aligned} (2) \qquad \int_{\mathbb{R}^n} N(f|B^n, y)\, dH^n(y) &= \lim_{p\to\infty} \sum_{i=1}^{p} \int_{\mathbb{R}^n} \chi_{f(A_{i,p})}(y)\, dH^n(y) \\ &= \lim_{p\to\infty} \sum_{i=1}^{p} H^n(f(A_{i,p})) \\ &\le C \lim_{p\to\infty} \sum_{i=1}^{p} H^n(A_{i,p}) = CH^n(B^n). \end{aligned}$$

This inequality completes the proof.

Step 2: General case. Fix $p \in O^*(m, m-n)$. Let $B^n_{p,x} = p^{-1}(x) \cap B^m$ for $x \in \operatorname{image} p$. Assume for the moment that we have proved the measurability of $N(f|B^n_{p,x}, y)$ as a function of p, x and y. Applying (2) to $f|_{B^n_{p,x}}$ we have

$$\int_{\mathbb{R}^n} N(f|B^n_{p,x}, y)\, dH^n(y) \le CH^n(B^n_{p,x}).$$

Integrating both sides over $p \in O^*(m, m-n)$ and $x \in \operatorname{image} p$ we get

$$\int_{\mathbb{R}^n} \int_{p\in O^*(m,m-n)} \int_{x\in \operatorname{image} p} N(f|B^n_{p,x}, y)\, dH^{m-n}(x)\, d\vartheta^*_{m,m-n}(p)\, dH^n(y) \le C_1.$$

Since $N(f|B^n_{p,x}, y) = N(p|f^{-1}(y), x)$, by Fubini's theorem we get $I^{m-n}(f^{-1}(y)) < \infty$ for H^n-almost all $y \in \mathbb{R}^n$. Now the theorem follows from Proposition 1.

To complete our argument it remains to prove the measurability of the function $N(f|B^n_{p,x}, y)$. We first describe precisely its domain of definition. Set

$$\pi : B^m \times \mathbb{R}^n \times O^*(m, m-n) \longrightarrow B^m \times \mathbb{R}^n \times O^*(m, m-n),$$
$$(x, y, p) \overset{\pi}{\longmapsto} (p(x), y, p).$$

Note that the image W of π is a manifold equipped with the measure $H^{m-n} \otimes H^n \otimes \vartheta^*_{m,m-n}$. This is the domain of definition of the considered function N. Let G_A denote the graph of $f|_A : A \to \mathbb{R}^n$, where $A \subseteq B^m$ is a Borel subset. We have $G_A \times O^*(m, m-n) \subseteq B^m \times \mathbb{R}^n \times O^*(m, m-n)$. Let $S_A = \pi(G_A \times O^*(m, m-n))$. Since G_A is a Borel set, being the graph of a Borel mapping, it follows from Proposition 2 that S_A is a measurable subset of W. Let $B^m = \bigcup_{i=1}^p A_{i,p}$ be a Borel decomposition as in Step 1. Now it is clear that

$$N : W \to \mathbb{N}_0 \cup \{\infty\}, \qquad N(f|B^n_{p,x}, y) = \lim_{p\to\infty} \sum_{i=1}^{p} \chi_{S_{A_{i,p}}}.$$

Hence N is measurable.

Remark. In fact, we have proved a slightly stronger result than stated in Theorem 1. Namely, it suffices to assume that (1) holds only for Borel subsets contained in n-dimensional affine subspaces of $\mathbb{R}^m$ and as a result we conclude that for almost all y the set $f^{-1}(y)$ has locally finite I^{m-n} measure (i.e. $I^{m-n}(f^{-1}(y) \cap K) < \infty$ for all bounded sets $K \subseteq \mathbb{R}^m$).

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