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Sobolev Spaces on an Arbitrary Metric Space

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Abstract. We define Sobolev space $W^{1,p}$ for $1 < p \leqslant \infty$ on an arbitrary metric space with finite diameter and equipped with finite, positive Borel measure. In the Euclidean case it coincides with standard Sobolev space. Several classical imbedding theorems are special cases of general results which hold in the metric case. We apply our results to weighted Sobolev space with Muckenhoupt weight.

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1. Introduction

It is well known that $W^{1,\infty}$ consists of Lipschitz functions and hence we can naturally define $W^{1,\infty}$ on an arbitrary metric space. In the case $1 < p \leqslant \infty$ we find a Lipschitz type characterization of $W^{1,p}$ (Theorem 1) which can be used as a definition of $W^{1,p}$ in the case of metric space as a domain. Since the space $W^{1,p}$ has an integral nature, we have to equip the metric space with a measure.

The imbedding theorems depend on the dimension; so if we want to get imbedding theorems in this general, metric context, we have to introduce a condition describing the dimension. This condition is very simple. It suffices to assume that $\mu(B(x,r)) \geqslant Cr^s$ (see the definition of *s-regularity* in Section 5). If we are concerned with fractal (self-similar) sets then they are *s-regular* with respect to Hausdorff's measure. It is surprising, but the imbedding theorems hold in this general metric context (Theorem 6). As corollaries we obtain classical imbedding theorems and the weighted imbedding theorem of Fabes–Kenig–Serapioni.

The average value of f will be denoted by $f_A = \mu(A)^{-1} \int_A f \, \mathrm{d}\mu = -\!\!\!\!\!\!\int_A f \, \mathrm{d}\mu$. By C we will denote a general constant. It can change its value even in the same proof.

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2. Lipschitz Type Characterization

In this section we give a Lipschitz type characterization of Sobolev space in the Euclidean case (Theorem 1). This will be used in the next section in the definition of Sobolev space on metric space.

We start with recalling some standard definitions:

$$W^{1,p}(\Omega) = \{f \in \mathcal{D}'(\Omega) \mid f \in L^p(\Omega), \nabla f \in L^p(\Omega)\},$$
$$L^{1,p}(\Omega) = \{f \in \mathcal{D}'(\Omega) \mid \nabla f \in L^p(\Omega)\},$$

where $\Omega \subseteq \mathbb{R}^n$ is an open set and $1 \leqslant p \leqslant \infty$.

$W^{1,p}(\Omega)$ is a Banach space when endowed with the norm $\|f\|_{W^{1,p}} = \|f\|_{L^p} + \|\nabla f\|_{L^p}$. $L^{1,p}(\Omega)$ is endowed with a seminorm $\|f\|_{L^{1,p}} = \|\nabla f\|_{L^p}$ (it is not a norm, because it annihilates constant functions).

It is known (see [15, Lemma 7.16]) that for $f \in L^{1,p}(Q)$ (Q-cube in $\mathbb{R}^n$) the following inequality holds a.e.

$$|f(x) - f_Q| \leqslant C \int_Q \frac{|\nabla f(y)|}{|x-y|^{n-1}} \, \mathrm{d}y \tag{1}$$

(hence $L^{1,p}(Q) = W^{1,p}(Q)$).

For any $x, y \in Q$ we can find a subcube $\overline{Q}$ with $x, y \in \overline{Q}$, diam $\overline{Q} \approx |x-y|$. Hence, by (1) and the following inequality of Hedberg ([22], [39, Lemma 2.8.3])

$$\int_{B(x,R)} \frac{|\nabla f(y)|}{|x-y|^{n-1}} \, \mathrm{d}y \leqslant CRM_R|\nabla f|(x),$$

where $M_R g = \sup_{r<R} ⨍_{B(x,r)} |g(y)| \, \mathrm{d}y$, we get

$$\begin{aligned} |f(x) - f(y)| &\leqslant |f(x) - f_{\overline{Q}}| + |f(y) - f_{\overline{Q}}| \\ &\leqslant C|x-y|(M_{|x-y|}|\nabla f|(x) + M_{|x-y|}|\nabla f|(y)). \end{aligned}$$

This inequality is taken from [4] (see also [16], [20], [5], and [6] for generalizations).

We say that the domain $\Omega \subseteq \mathbb{R}^n$ has the *extension property* if there exists a bounded linear operator $E\colon W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$, such that for every $u \in W^{1,p}(\Omega)$, $Eu|_\Omega = u$ a.e. An example of such a domain is any bounded domain with Lipschitz boundary. If Ω is a bounded domain with the extension property then $L^{1,p}(\Omega) = W^{1,p}(\Omega)$.

Since the maximal operator is bounded in L^p for $p > 1$, we have that if $f \in L^{1,p}(\Omega)$ where $\Omega = \mathbb{R}^n$ or Ω is a bounded domain with the extension property, then there exists a nonnegative function $g \in L^p(\Omega)$ such that

$$|f(x) - f(y)| \leqslant |x-y|(g(x) + g(y)) \text{ a.e.} \tag{2}$$

Many properties of the Sobolev function can be recovered from the inequality (2) and its generalizations to higher order derivatives (see [4], [5], [6], [16], [20], [18]). Hence an important question arises now. Is there a function f which satisfies (2) and which does not belong to $W^{1,p}$? The following theorem gives a negative answer.

THEOREM 1. *Let f be a measurable function on Ω, where $\Omega = \mathbb{R}^n$ or Ω is a bounded domain with the extension property. Then $f \in L^{1,p}(\Omega)$, $1 < p \leqslant \infty$ if and only if there exists a nonnegative function $g \in L^p(\Omega)$ such that the inequality (2) holds a.e.*

Remark. The above theorem can be stated in much more general form. For example, it follows from our proof that the implication $\Leftarrow$ holds for any domain Ω.

Proof. It remains to prove the implication $\Leftarrow$. We follow the ideas of Calderón [7, Th. 4].

It suffices to show (due to Riesz's representation and Radon–Nikodym's theorems) that there exists a nonnegative function $h \in L^p(\Omega)$ such that

$$\left|\frac{\partial f}{\partial x_i}[\varphi]\right| \stackrel{\text{def}}{=} \left|-\int f\frac{\partial\varphi}{\partial x_i}\right| \leqslant \int|\varphi|h$$

for all $\varphi \in C_0^\infty(\Omega)$. Integrating (2) twice over a ball $B = B(x,\varepsilon) \subset \Omega$, we obtain the inequality

$$\fint_B |f - f_B| \leqslant C\varepsilon\fint_B g$$

Let $\psi \in C_0^\infty(B(1))$, $\int\psi = 1$, $\psi_\varepsilon = \varepsilon^{-n}\psi(x/\varepsilon)$. We have

$$\int f\frac{\partial\varphi}{\partial x_i} = \lim_{\varepsilon\to 0}\int\frac{\partial\varphi}{\partial x_i}(\psi_\varepsilon * f) = \lim_{\varepsilon\to 0} -\int\varphi\left(\frac{\partial\psi_\varepsilon}{\partial x_i} * f\right).$$

Since $\int\frac{\partial\psi_\varepsilon}{\partial x_i} = 0$, we have

$$\left(f * \frac{\partial\psi_\varepsilon}{\partial x_i}\right)(x) = (f - f_{B(x,\varepsilon)}) * \frac{\partial\psi_\varepsilon}{\partial x_i},$$

$$\left|f * \frac{\partial\psi_\varepsilon}{\partial x_i}\right|(x) \leqslant C\int_B|f - f_B|\cdot\varepsilon^{-n-1} \leqslant C\fint_B g.$$

Hence

$$\left|\int f\frac{\partial\varphi}{\partial x_i}\right| \leqslant \int|\varphi|Mg.$$

COROLLARY 1. *If Ω is as above, then* $L^{1,\infty}(\Omega) = \mathrm{Lip}\,(\Omega)$.

Since the above characterization does not involve the notion of derivative it can be used in the definition of Sobolev space on an arbitrary metric space as we do in the next section.

3. Sobolev Spaces on Metric Spaces

DEFINITION. Let (X, d, μ) be a metric space (X, d) with finite diameter $\mathrm{diam}\, X = \sup_{x,y\in X} d(x,y) < \infty$ and a finite positive Borel measure μ. Let $1 < p \leqslant \infty$.

The Sobolev spaces $L^{1,p}(X, d, \mu)$ and $W^{1,p}(X, d, \mu)$ are defined as follows

$$\begin{aligned} L^{1,p}(X,d,\mu) &= \{f\colon X \to \mathbb{R} \mid f \text{ is measurable and } \exists E \subset X,\ \mu(E) = 0 \text{ and} \\ &\qquad \exists g \in L^p(\mu) \text{ such that } |f(x) - f(y)| \leqslant d(x,y)(g(x) + g(y)) \\ &\qquad \text{for all } x,\ y \in X \setminus E\} \end{aligned}$$

$$W^{1,p}(X,d,\mu) = \{f \in L^{1,p}(X,d,\mu) \mid f \in L^p(\mu)\}$$

The space $L^{1,p}(X, d, \mu)$ is equipped with the seminorm $\|f\|_{L^{1,p}} = \inf_g \|g\|_{L^p}$. The space $W^{1,p}(X, d, \mu)$ is equipped with the norm $\|f\|_{W^{1,p}} = \|f\|_{L^p} + \|f\|_{L^{1,p}}$.

THEOREM 2. *If $1 < p < \infty$ then to every $f \in L^{1,p}(X, d, \mu)$ there exists the unique $g \in L^p(\mu)$ which minimizes L^p norm among the functions which can be used in the definition of $\|f\|_{L^{1,p}}$.*

Proof. Standard application of Mazur's lemma (see, e.g. [10]) or Banach–Saks' theorem (see [2]) gives the existence of a minimizer. The uniquess follows from uniform convexity of L^p.

Although the above theorem can be used to get a more elegant form of the definition of the norm $\|\ \|_{W^{1,p}}$, we will not use it in the sequel.

LEMMA 1. $L^{1,p}(X, d, \mu) = W^{1,p}(X, d, \mu)$ *as a set.*

Proof. Fix $y \in X \setminus E$ with $g(y) < \infty$. We have

$$|f(x)| \leqslant |f(x) - f(y)| + |f(y)| \leqslant (\mathrm{diam}\, X)(g(x) + g(y)) + |f(y)| \in L^p(\mu)$$

(with respect to x).

THEOREM 3. *$W^{1,p}(X, d, \mu)$ is a Banach space.*

Proof. Let $\{f_n\}$ be a Cauchy sequence in $W^{1,p}$. Let $f_n \to f$ in L^p. We prove that $f \in W^{1,p}$ and that convergence holds in $W^{1,p}$. Let $\{f_{n_i}\}$ be a subsequence such that $\|f_{n_{i+1}} - f_{n_i}\|_{W^{1,p}} < 2^{-i}$. Hence $f_{n_i} \to f$ a.e. and there exists $g_i \in L^p$ such that

$$|(f_{n_{i+1}} - f_{n_i})(x) - (f_{n_{i+1}} - f_{n_i})(y)| \leqslant d(x,y)(g_i(x) + g_i(y)) \tag{3}$$

and $\|g_i\|_{L^p} < 2^{-i}$. If we set $h = \sum_1^\infty g_i$ then $\|h\|_{L^p} < 1$. It follows directly from (3) that for $j > i$

$$|(f_{n_j} - f_{n_i})(x) - (f_{n_j} - f_{n_i})(y)| \leqslant d(x,y)\left(\sum_{k=i}^{\infty} g_k(x) + \sum_{k=i}^{\infty} g_k(y)\right).$$

Passing to the limit with $j \to \infty$ we get that $f \in L^{1,p} = W^{1,p}$ and $f_{n_i} \to f$ in $L^{1,p}$. It readily follows now that $f_n \to f$ in $W^{1,p}$.

Now we can generalize this result in the spirit of [29, Th.1.1.15].

THEOREM 4. *The norm* $\|\ \|_{W^{1,p}}$ *is equivalent with*

$$\|f\|_{1,p}^* = \|f\|^* + \|f\|_{L^{1,p}}$$

where $\|\ \|^*$ *is any norm, continuous on* $W^{1,p}$.

Remark. In fact it suffices to assume that $\|\ \|^*$ is a continuous seminorm which does not vanish on constant functions. The proof remains the same.
In the proof we need the following Poincaré type inequality

LEMMA 2. *If* $f \in W^{1,p}(X,d,\mu)$ *then* $\|f - f_X\|_{L^p} \leqslant 2(\operatorname{diam} X)\|f\|_{L^{1,p}}$.

Proof. Let g be such a function from the definition of $\|f\|_{L^{1,p}}$ that $\|g\|_{L^p} \leqslant (1+\varepsilon)\|f\|_{L^{1,p}}$. We have

$$|f(x) - f(y)| \leqslant \mathrm{d}(x,y)(g(x) + g(y)) \leqslant (\operatorname{diam} X)(g(x) + g(y))$$

hence

$$\begin{aligned} \left|f(x) - \fint_X f\,\mathrm{d}\mu\right| &\leqslant \fint_X |f(x) - f(y)|\,\mathrm{d}\mu(y) \leqslant (\operatorname{diam} X)\left(g(x) + \fint_X g\,\mathrm{d}\mu\right) \\ &\leqslant (\operatorname{diam} X)\left(g(x) + \left(\fint_X g^p\,\mathrm{d}\mu\right)^{1/p}\right), \\ \|f - f_X\|_{L^p} &\leqslant 2(\operatorname{diam} X)\|g\|_{L^p} \leqslant 2(1+\varepsilon)(\operatorname{diam} X)\|f\|_{L^{1,p}}. \end{aligned}$$

Proof of Theorem 4. It suffices to prove that $W^{1,p}$ is complete with respect to the norm $\|\ \|_{1,p}^*$. The rest will follow from Banach's theorem.

Let $\{f_n\}$ be a Cauchy sequence for $\|\ \|^*_{1,p}$. Hence it is a Cauchy sequence in $L^{1,p}$ and as follows from Lemma 2, $\{f_n-(f_n)_X\}$ is a Cauchy sequence in $W^{1,p}$. Denote its limit by f. Now since $\|\ \|^*$ is continuous on $W^{1,p}$ we get that $\{f_n-(f_n)_X\}$ (as well as $\{f_n\}$) is a Cauchy sequence in $\|\ \|^*$. Hence the sequence $\{(f_n)_X\}$ is convergent to a certain constant C. Now it is evident that $f_n\to f+C\in W^{1,p}$ in $\|\ \|^*_{1,p}$.

We know that $\|\ \|_{L^{1,p}}$ is only a seminorm, but evidently it induces the norm on a factorspace $\dot{L}^{1,p}=L^{1,p}/\{\text{constant funct.}\}$. The following result is in the spirit of [29, Th.1.1.13].

COROLLARY 2. *$\dot{L}^{1,p}$ with a norm induced from $L^{1,p}$ is a Banach space.*

Proof. Let $\{[f_n]\}$ be a Cauchy sequence in $\dot{L}^{1,p}$, where $f_n\in L^{1,p}$ is a representative of the class $[f_n]\in\dot{L}^{1,p}$ such that $\int_X f_n\,d\mu=0$. Now the corollary follows, because according to Lemma 2, $\{f_n\}$ is a Cauchy sequence in $W^{1,p}$.

4. Approximation

In this section we prove a theorem in the spirit of [27] about the density of Lipschitz functions in $W^{1,p}(X,d,\mu)$.

THEOREM 5. *If $f\in W^{1,p}(X,d,\mu)$ where $1<p<\infty$ then to every $\varepsilon>0$ there exists a Lipschitz function h such that*

1. $\mu(\{x\mid f(x)\neq h(x)\})<\varepsilon$
2. $\|f-h\|_{W^{1,p}}<\varepsilon$

Remarks. This type of approximation of Sobolev functions, both in norm and in the 'Lusin's sense' (by an analogy with Lusin's theorem) has already been obtained by Liu [27] (compare with [8, Th.13], [37]).

In fact, Liu has proved a stronger result, generalizing a theorem of Calderón and Zygmund [8, Th. 13]. He got an approximation in $W^{m,p}$ by C^m functions. However, the above result (Theorem 5) is relevant for many applications in P.D.E. and Variational Calculus (see, e.g. [1], [14], [17], [18], [26], [4], [28]).

The theorem of Liu has been extended in [30], [38], [39], [5], [6].

Proof. Let $E_\lambda=\{x\mid |f(x)|\leqslant\lambda \text{ and } g(x)\leqslant\lambda\}$ where g is taken from the definition of $\|f\|_{L^{1,p}}$. Since f, $g\in L^p$ then $\lambda^p\mu(X\setminus E_\lambda)\overset{\lambda\to\infty}{\longrightarrow}0$. Function $f|_{E_\lambda}$ is Lipschitz with the constant 2λ. We can extend $f|_{E_\lambda}$ to the Lipschitz function $\overline{f}$ on X with the same constant (see, e.g. [33, Th. 5.1], [12, Sec. 2.10.4]). Now we can slightly modify this extension by putting

$$f_\lambda=(\operatorname{sgn}\overline{f})\min(|\overline{f}|,\lambda).$$

Evidently, f_λ is Lipschitz with the same constant 2λ, $f_\lambda|_{E_\lambda} = f|_{E_\lambda}$ and $|f_\lambda| \leqslant \lambda$. Moreover,

$$\mu(\{x \mid f(x) \neq f_\lambda(x)\}) \leqslant \mu(X \setminus E_\lambda) \overset{\lambda\to\infty}{\longrightarrow} 0.$$

It suffices to prove that $f_\lambda \overset{\lambda\to\infty}{\longrightarrow} f$ in $W^{1,p}$. We have

$$\begin{aligned} \|f - f_\lambda\|_{L^p}^p &= \int_{X\setminus E_\lambda} |f - f_\lambda|^p \,\mathrm{d}\mu \\ &\leqslant 2^{p-1}\left(\int_{X\setminus E_\lambda} |f|^p \,\mathrm{d}\mu + \lambda^p \mu(X \setminus E_\lambda)\right) \overset{\lambda\to\infty}{\longrightarrow} 0. \end{aligned}$$

Now the 'gradient' estimations. Let

$$g_\lambda = \begin{cases} 0 & \text{for } x \in E_\lambda, \\ g(x) + 3\lambda & \text{for } x \in X \setminus E_\lambda. \end{cases}$$

One can directly check that

$$|(f - f_\lambda)(x) - (f - f_\lambda)(y)| \leqslant d(x,y)(g_\lambda(x) + g_\lambda(y)).$$

Now the theorem follows since evidently $\|g_\lambda\|_{L^p} \overset{\lambda\to\infty}{\longrightarrow} 0$.

5. Imbedding Theorems

In this section we denote by C (C', $\overline{C}$, C_1, C_2, ...) a general constant which depends only on p, s and b (s and b are introduced below) and hence is independent of X, d and μ. The constant C can change even in the same proof.

The classical imbedding theorems depend on the dimension, so we have to introduce the condition describing the 'dimension' in the general metric context.

DEFINITION. We say that the space (X, d, μ) is *s-regular* ($s > 0$) with respect to μ if $\mu(X) < \infty$, $\operatorname{diam} X < \infty$ and there exists a constant b such that for all $x \in X$ and all $r \leqslant \operatorname{diam} X$

$$\mu(B(x,r)) \geqslant b \cdot r^s.$$

We say that the metric space is *s-regular* if it is *s-regular* with respect to Hausdorff's measure H^s.

The very important case is when we deal with X being a subset of $\mathbb{R}^n$. We say that a bounded set $X \subset \mathbb{R}^n$ is *strictly s-regular* if there exists a positive Borel measure μ such that

$$C_1 r^s \leqslant \mu(B(x,r) \cap X) \leqslant C_2 r^s$$

for all $x \in X$ and all $r \leqslant \operatorname{diam} X$. (This is a minor modification of the definition of *s-sets* given in [25]). It is not difficult to prove (see [25]) that in the case of *strictly s-regular* sets we can take as a canonical measure μ the Hausdorff measure H^s.

Many natural fractal sets are *s-regular*. For example, the standard ternary Cantor set is $\log_3 2$*-regular*. It is also possible to construct fractal sets which are *s-regular* for some integer s, and purely unrectifiable (see [33], [31] or [12] for definition); an example of planar Cantor set (Sierpiński's type gasket) which is *strictly 1-regular*, but purely unrectifiable, can be found in [31, p. 34].

Many important domains and their boundaries which are used in the theory of Sobolev spaces are *strictly regular* (see [25] for details).

The structure of *strictly s-regular* sets is very rich and it inherits many properties from the Euclidean space, therefore it was possible to define Hardy and Besov type spaces on such sets, as was done in [25] and in the anterior papers of the same authors. However, we want to emphasize that our approach is independent, of a different nature and much more general.

THEOREM 6. *Let $f \in W^{1,p}(X, d, \mu)$ where (X, d, μ) is s-regular.*

1. *If $p < s$ then $f \in L^{p^*}(\mu)$ where $p^* = \frac{sp}{s-p}$. Moreover*

$$\|f\|_{L^{p^*}} \leqslant C((\operatorname{diam} X)^{-1}\|f\|_{L^p} + \|f\|_{L^{1,p}})$$

and hence

$$\|f - f_X\|_{L^{p^*}} \leqslant C'\|f\|_{L^{1,p}}.$$

2. *If $p = s$ then there exist constants C_1 and C_2 such that*

$$⨍_X \exp\left(C_1 \frac{\mu(X)^{1/s}}{\operatorname{diam} X} \frac{|f - f_X|}{\|f\|_{L^{1,s}(\mu)}}\right) \mathrm{d}\mu \leqslant C_2.$$

3. *If $p > s$ then*

$$\|f - f_X\|_{L^\infty} \leqslant C\mu(X)^{\frac{1}{s}-\frac{1}{p}}\|f\|_{L^{1,p}}$$

and hence

$$|f(x) - f(y)| \leqslant C\mu(X)^{\frac{1}{s}-\frac{1}{p}}\|f\|_{L^{1,p}} \ a.e.$$

Remarks. (1) generalizes the classical Sobolev imbedding theorem. In the Euclidean case (2) is just the inequality of John and Nirenberg (see [24], [34]) applied to $W^{1,n}(Q^n) \subset \mathrm{BMO}$. (3) In the Euclidean case this inequality leads to Hölder continuity. It is not difficult to add some conditions on (X, d, μ) which will lead to continuity or even Hölder continuity of f in the metric case.

Proof. Let $g \geqslant 0$ be a function as in the definition of $\|f\|_{L^{1,p}}$, such that $\|g\|_{L^p} \approx \|f\|_{L^{1,p}}$ (writting $u \approx v$ we mean that $C^{-1}v \leqslant u \leqslant Cv$ for some general constant C).

Let $E_k = \{x \mid g(x) \leqslant 2^k\}$, $k \in Z$. We have

$$\infty > \int_X g^p \,\mathrm{d}\mu \approx \sum_{k=-\infty}^{\infty} 2^{kp}\mu(E_k \setminus E_{k-1}). \tag{4}$$

Let $a_k = \sup_{E_k} |f|$. We will estimate a_k in terms of a_{k-1}. Let $x \in E_k$. If we take the ball $B(x,r)$ with the radius $r = b^{-1/s}\mu(X \setminus E_{k-1})^{1/s}$ then, by the *s-regularity* property, $\mu(B(x,r)) \geqslant \mu(X \setminus E_{k-1})$, hence there exists $\overline{x} \in B(x,r) \cap E_{k-1}$. Now, bearing in mind that $f|_{E_k}$ is Lipschitz with constant 2^{k+1}, we get

$$\begin{aligned} |f(x)| &\leqslant |f(x) - f(\overline{x})| + |f(\overline{x})| \leqslant d(x,\overline{x})2^{k+1} + a_{k-1} \\ &\leqslant C\mu(X \setminus E_{k-1})^{1/s}2^{k+1} + a_{k-1}. \end{aligned}$$

Hence Chebyschev's inequality

$$\mu(X \setminus E_{k-1}) \leqslant \frac{C}{2^{kp}}\|g\|^p_{L^p}$$

leads to

$$a_k \leqslant C2^{k(1-\frac{p}{s})}\|g\|^{p/s}_{L^p} + a_{k-1}. \tag{5}$$

It follows easily from this inequality that f is respectively p^*, exponentially, or L^∞ integrable, but we want to obtain precise estimates, so more work is required.

We can assume that $g > 0$ everywhere (for if not it suffices to replace g by $g + \|g\|_{L^p}$; after this substitution the condition $\|g\|_{L^p} \approx \|f\|_{L^{1,p}}$ still holds).

Let $b_k = \inf_{E_k} |f|$. Evidently $b_k \leqslant \|f\|_{L^p}\mu(E_k)^{-1/p}$. Since $g > 0$ everywhere, then there exists k_0 such that $\mu(E_{k_0-1}) < \mu(X)/2$ and $\mu(E_{k_0}) \geqslant \mu(X)/2$, so $\mu(X \setminus E_{k_0-1}) > \mu(X)/2$. By Chebyschev's inequality we have

$$\|g\|^p_{L^p} \geqslant 2^{(k_0-1)p}\mu(X \setminus E_{k_0-1}),$$

hence

$$2^{k_0} \leqslant C\|g\|_{L^p}\mu(X)^{-1/p}. \tag{6}$$

Since $f|_{E_k}$ is Lipschitz with the constant 2^{k+1} then $a_k \leqslant b_k + 2^{k+1}\operatorname{diam} X$. Hence

$$\begin{aligned} a_{k_0} &\leqslant \|f\|_{L^p}\mu(E_{k_0})^{-1/p} + C(\operatorname{diam} X)\|g\|_{L^p}\mu(X)^{-1/p} \\ &\leqslant C'\mu(X)^{-1/p}(\|f\|_{L^p} + (\operatorname{diam} X)\|g\|_{L^p}). \end{aligned} \tag{7}$$

Case $p < s$: It follows from (5) and monotonicity of a_k that

$$a_k \leqslant C\|g\|^{p/s}_{L^p}\left(\sum_{i=-\infty}^{k} 2^{i(1-\frac{p}{s})}\right) + a_{k_0} \leqslant C'\|g\|^{p/s}_{L^p}2^{k(1-\frac{p}{s})} + a_{k_0},$$

Hence

$$\int_X |f|^{p^*}\,\mathrm{d}\mu \;\leqslant\; \sum_{k=-\infty}^{\infty} a_k^{p^*}\mu(E_k\setminus E_{k-1})$$

$$\leqslant\; C\left(\|g\|^{p^*p/s}\left(\sum_{k=-\infty}^{\infty} 2^{kp}\mu(E_k\setminus E_{k-1})\right) + a_{k_0}^{p^*}\mu(X)\right). \qquad (8)$$

Since by *s-regularity* $\mu(X)\geqslant b(\operatorname{diam} X)^s$, then it follows easily from (8), (7) and (4) that

$$\|f\|_{L^{p^*}} \leqslant C((\operatorname{diam} X)^{-1}\|f\|_{L^p} + \|g\|_{L^p})$$

The inequality for $\|f - f_X\|_{L^{p^*}}$ follows easily from that for $\|f\|_{L^{p^*}}$ and from Lemma 2.

Case $p = s$: By the rescaling argument we can assume that $\mu(X) = 1$ and $\operatorname{diam} X = 1$. Now it suffices to prove that $\int_X \exp(C_1|f|)\,\mathrm{d}\mu \leqslant C_2$, where $f\in W^{1,s}$ is such that $\int_X f\,\mathrm{d}\mu = 0$, $\|f\|_{L^{1,s}} = 1$. It follows from (5) that $a_k \leqslant \overline{C}k + a_0$ for $k\geqslant 0$ (because $\|g\|_{L^s}\approx 1$). Let C_1 be a constant such that $\exp(C_1\overline{C}) = 2^s$. We have

$$\int_X \exp(C_1|f|)\,\mathrm{d}\mu \;=\; \int_{E_0} + \int_{X\setminus E_0};$$

$$\int_{E_0} \;\leqslant\; \mathrm{e}^{C_1a_0}\mu(X) \;=\; \mathrm{e}^{C_1a_0},$$

$$\int_{X\setminus E_0} \;\leqslant\; \sum_{k=1}^{\infty} \mathrm{e}^{C_1(\overline{C}k+a_0)}\mu(E_k\setminus E_{k-1})$$

$$=\; \mathrm{e}^{C_1a_0}\sum_{k=1}^{\infty} 2^{ks}\mu(E_k\setminus E_{k-1}) \;\leqslant\; C\,\mathrm{e}^{C_1a_0}$$

Hence it suffices to prove that a_0 is bounded by an universal constant (which depends on s and b only).

If $k_0\geqslant 0$ then as it follows from (7) and Lemma 2

$$a_0 \leqslant a_{k_0} \leqslant C.$$

If $k_0 < 0$ then it follows from the definition of k_0 that $\mu(E_0)\geqslant\mu(E_{k_0})\geqslant \mu(X)/2 = 1/2$, hence

$$a_0 \leqslant b_0 + 2\operatorname{diam} X \leqslant \|f\|_{L^s}\mu(E_0)^{-1/s} + 2\operatorname{diam} X < C.$$

Case $p > s$: We can assume that $\int_X f\,\mathrm{d}\mu = 0$. It follows from (5) and (6) that

$$a_k \;\leqslant\; C\|g\|_{L^p}^{p/s}\left(\sum_{i=k_0+1}^{\infty} 2^{i(1-\frac{p}{s})}\right) + a_{k_0} \;\leqslant\; C'\mu(X)^{\frac{1}{s}-\frac{1}{p}}\|g\|_{L^p} + a_{k_0}$$

for all $k \in Z$. Now by (7), Lemma 2 and *s-regularity* we have

$$a_{k_0} \leqslant C\mu(X)^{-1/p}(\operatorname{diam} X)\|g\|_{L^p} \leqslant C'\mu(X)^{\frac{1}{s}-\frac{1}{p}}\|g\|_{L^p}.$$

This implies the first inequality. The second inequality follows from the first one and from the observation that

$$|f(x) - f(y)| \leqslant |f(x) - f_X| + |f(y) - f_X|.$$

Evidently the classisal Sobolev imbedding theorem is a direct consequence of Theorem 6. Now we apply this theorem to the case of weighted Sobolev spaces (with Muckenhoupt weights) and obtain the imbedding theorom of Fabes-Kenig-Serapioni [11] (see also [9], [23]).

COROLLARY 3. *If $\omega \in A_p$ $(1 < p < \infty)$ then there exists $\delta > 0$ such that for $1 \leqslant k \leqslant n/(n-1) + \delta$ the following inequality*

$$\left(\frac{1}{\omega(B_R)}\int_{B_R} |u(x) - u_{\omega,R}|^{kp}\omega(x)\,\mathrm{d}x\right)^{1/kp}$$
$$\leqslant CR\left(\frac{1}{\omega(B_R)}\int_{B_R} |\nabla u|^{p}\omega(x)\,\mathrm{d}x\right)^{1/p} \tag{9}$$

holds for all $u \in C^\infty(B_R)$, where $\omega(E) = \int_E \omega\,\mathrm{d}x$ and $u_{\omega,R} = \frac{1}{\omega(B_R)}\int_{B_R} u\omega\,\mathrm{d}x$. Moreover, if $u \in C_0^\infty(B_R)$ then the above inequality holds with $|u(x) - u_{\omega,R}|^{kp}$ replaced by $|u(x)|^{kp}$. The constant C depends on ω, n, p and δ.

Remark. See [34, Chapter 9] or [13, Chapter 4] for definition and basic properties of the Muckenhoupt weights A_p.

Proof. Since by the Muckenhoupt theorem the maximal operator is bounded in $L^p(\omega)$ (see [34, Th. 4.1, p. 233] or [13, Th. 2.8, p. 400]), the weighted Sobolev space $W^{1,p}$ on a ball B_R with respect to the weight ω is just the Sobolev space in the sense of Section 3. It follows from the open end property (see [13, Th. 2.6, p. 399] or [34]) that there exists $\varepsilon > 0$ such that $\omega \in A_{p-\varepsilon}$. Now the strong doubling condition (see [34, Th. 2.4(iv), p. 226] or [13, Th. 2.9, p. 400]) implies that if $B_R = B(0,1)$ and $\omega(B(0,1)) = 1$, then the ball with measure $\omega(x)dx$ is $n(p-\varepsilon)$*-regular*, with constant 'b' depending on $A_{p-\varepsilon}$ constant of ω only. Hence the inequality (9) follows in this special case from Theorem 6. Note that inequality (9) is invariant under multiplying ω by a constant, hence it holds in $B_R = B(0,1)$ without assumption that $\omega(B(0,1)) = 1$. The inequality for an arbitrary R follows from the rescaling argument (since $\omega_R(x) = \omega(x/R)$ is also a $A_{p-\varepsilon}$ weight with the same $A_{p-\varepsilon}$ constant). Finally, the inequality for $u \in C_0^\infty$ follows from Theorem 6, because such u satisfies the Poincaré type inequality.

Note also that we can apply the other cases of Theorem 6 and extend Corollary 3. Since it is evident how to state the result we will not go into details.

6. Final Remarks

One can extend the above theory to the case of an unbounded metric space. Since the constant in the Sobolev inequality (Theorem 6.1) does not depend on diam X and $\mu(X)$, then this inequality extends to the case of an unbounded metric space.

Many domains $\Omega \subset \mathbb{R}^n$ can be regarded as metric spaces, with the geodesic or related metric. Hence one can use the above method to obtain the (global) imbedding theorems in domains.

Recently many papers deal with the Sobolev mappings from a domain $\Omega \subset \mathbb{R}^n$ into a compact manifold or with the Sobolev mappings between two compact manifolds (see, e.g. [3], [4], [14], [17], [18], [19], [21], [32], [36] and references in these papers; that list is very far from being complete). The definition which is usually employed depends on the imbedding of the image-manifold. However, it is possible to define the space of Sobolev mappings between two metric spaces. Namely, let (X, d, μ) be as in the definition of $W^{1,p}(X, d, \mu)$ and let (Y, ρ) be a metric space. We say that $f \in W^{1,p}(X, Y)$ if and only if $\varphi \circ f \in W^{1,p}(X, d, \mu)$ for all $\varphi \in \mathrm{Lip}\,(Y)$. If X and Y are two compact manifolds then it is not difficult to prove that the space $W^{1,p}(X, Y)$ coincides with that from the cited papers (see [17] or [18] for details).

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