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Sobolev Mappings, Co-Area Formula and Related Topics

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ABSTRACT. We generalize the classical area and co-area formulas to the setting of Sobolev mappings. In one of the versions of the co-area formula that we obtain, the integral-geometric measure is involved. The proof is based on a Sard type theorem for Borel mappings between Euclidean spaces which is of independent interest. We apply our results to minimizing harmonic mappings.

Key words and phrases: area formula, co-area formula, countably rectifiable sets, Hausdorff measure, integral-geometric measure, Sobolev mappings, harmonic mappings.

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1. Introduction

The area and co-area formulas are the most general versions of the change of variables formula in the integral. Namely the transformation of variables is made by an arbitrary Lipschitz mapping $f\colon \mathbb{R}^n \to \mathbb{R}^m$ in place of a diffeomorphism. Here we neither assume that f is one-to-one nor that $n = m$.

The paper is devoted to study of various problems related to area and co-area formulas in the case in which the transformation $f\colon \mathbb{R}^n \to \mathbb{R}^m$ belongs to the Sobolev space. This is a weaker condition than being Lipschitz continuous. The core for new results will be a careful study of the structure of the preimage $f^{-1}(y)$ for a generic point $y \in \mathbb{R}^m$. This will play a crucial role in all the proofs in Section 5. Actually the main results of the paper are gathered in Section 5.

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The paper is organized as follows. In Section 2 we recall classical results from the geometric measure theory. We discuss the area and co-area formulas, countably rectifiable sets and the integral geometric measure. Although the content of the section is classical we state all the results very carefully.

In Section 3 we recall the definition of the Sobolev space and generalize both the area and the co-area formulas to the case in which the transformation belongs to the Sobolev space. These results seem to be folklore, although it is difficult to find a reference for them.

In Section 4 we recall the notion of capacity and the p-quasicontinuous representative of a Sobolev function. Then we show a simple direct proof (avoiding the use of p-quasicontinuity) of the fact that a $W^{1,p}$ function admits a representative which is continuous on almost all affine planes of the dimension less than p.

In the last Section 5 we study the structure of the preimage of a point $f^{-1}(y)$ when f is a p-quasicontinuous representative of a Sobolev mapping $f\colon \mathbb{R}^n \to \mathbb{R}^m$. This leads to a Sard type theorem for Sobolev maps and then to a version of the co-area formula involving the integral-geometric measure. At the end of the section we provide an application of the co-area formula to study of minimizing harmonic maps. This will give a slightly different proof of the fact that the radial projection $x/|x|\colon B^3 \to S^2$ is a minimizing harmonic map.

We made some efforts to make the exposition self-contained and so available to young researchers and graduate students.

NOTATION. Notation in the paper is fairly standard. The k-dimensional Hausdorff measure will be denoted by $\mathcal{H}^k$. In all the results, even if not explicitly stated, the Hausdorff measure will be considered for integer dimensions only. The Hausdorff dimension of a set E will be denoted by $\dim_{\mathcal{H}} E$.

Since the Lebesgue measure in $\mathbb{R}^n$ coincides with $\mathcal{H}^n$, [13], [64], we will use the Hausdorff measure notation to denote the Lebesgue measure. However we will also denote the Lebesgue measure of a set E by $|E|$.

Symbols M^n and N^m will denote Riemannian manifolds of dimensions n and m respectively. $\mathbb{RP}^2$ will denote the projective space which can be obtained from the sphere S^2 by identification of the antipodal points. The characteristic function of a set E will be denoted by χ_E. $\mathrm{Lip}\,(f)$ will denote the Lipschitz constant of f. By A^T we will denote the transpose of the matrix A. C will denote a general constant that can change its value in the same string of estimates. Writing $C(n,p)$ we will indicate that the constant depends on n and p only.

Any element in the L^p space or in the Sobolev space $W^{1,p}$ (defined later) is the equivalence class of functions which differ on a set of measure zero. Thus when we will say that there exists a representative of a Sobolev function with some properties we will mean a particular function from the equivalence class. Choosing correct representative of a Sobolev function will be crucial for our arguments.

2. Some Geometric Measure Theory

For material related to that in the section see [11], [13], [14], [18], [46], [58].

Area and Co-area formulas. The classical change of variables formula states that if $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism and $g\colon \mathbb{R}^n \to \mathbb{R}$ a measurable function, then

$$\int_{\mathbb{R}^n} g(x)|Jf(x)|\, dx = \int_{\mathbb{R}^n} g(f^{-1}(x))\, dx\,, \tag{1}$$

where $|Jf(x)| = |\det Df(x)|$.

Area and co-area formulas provide generalizations of the above theorem to the most general case in which the transformation f is neither one-to-one nor between the spaces of the same dimension. The statement of these generalizations is quite complicated so we need some preparations first.

Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be a Lipschitz mapping. We need define the Jacobian of f.

If $n = m$ and f is differentiable at a point x, then $|Jf(x)|$ has a simple geometric interpretation. Take any ball centered at x. Then $Df(x)(B)$ is an ellipsoid and $|Jf(x)|$ equals to the ratio of the volume of the ellipsoid to the volume of the ball

$$|Jf(x)| = \frac{\mathcal{H}^n(Df(x)(B))}{\mathcal{H}^n(B)}\,. \tag{2}$$

Observe that by the Rademacher theorem f is differentiable a.e. and hence $|Jf|$ is defined a.e., (see e.g. [13], [64] for the Rademacher theorem).

We will use the above geometric interpretation to define the absolute value of the Jacobian in the case in which f is a mapping between spaces of different dimensions.

Case $n \leq m$. If B is a ball centered at x, then $Df(x)(B)$ is an n-dimensional ellipsoid lying in the m-dimensional space. In this case we still define $|Jf|$ by the same formula (2). Employing the polar decomposition of the linear mapping $Df(x)$, one can easily prove that (see [13, Section 3.2])

$$|Jf| = \sqrt{\det(Df)^T(Df)}\,, \tag{3}$$

and actually it is more convenient to regard (3) as a definition and (2) as a geometric interpretation.

We said that $Df(x)(B)$ is an n-dimensional ellipsoid. However it can be an ellipsoid of a lower dimension (when $\operatorname{rank} Df(x) < n$). If this is the case we obviously have $|Jf(x)| = 0$.

Observe that (3) is exactly what one does when writing the volume form on an n-dimensional submanifold M^n of $\mathbb{R}^m$ in local coordinates $f\colon \mathbb{R}^n \to M^n \subset \mathbb{R}^m$.

Note that we do not define the Jacobian but its absolute value. If $m > n$, then there is no reasonable way to define Jf for a general Lipschitz mapping $f\colon \mathbb{R}^n \to \mathbb{R}^m$ because there is no continuous orientation of all n-dimensional affine planes in the m-dimensional space.

Case $n \geq m$. If B is a ball centered at x, then $Df(x)(B)$ is the m-dimensional ellipsoid. If $m < n$, then definition (2) would give always zero, and this is not exactly what we would like to have.

Assume that the rank of $Df(x)$ is maximal i.e. it equals m. Then $Df(x)(B)$ is a nondegenerate m-dimensional ellipsoid. The kernel $\ker Df(x)$ is an $(n-m)$-dimensional linear subspace and the mapping $Df(x)$ is a composition of two mappings. First we take the orthogonal projection of $\mathbb{R}^n$ onto the m-dimensional space $(\ker Df(x))^\perp$ and then we compose it with a nondegenerate linear mapping from $(\ker Df(x))^\perp$ to $\mathbb{R}^m$. Now we define $|Jf|$ as the absolute value of the determinant of this mapping between m-dimensional spaces. In other words the ellipsoid $Df(x)(B)$ is the image of the m-dimensional ball $B \cap (\ker Df(x))^\perp$ and

$$|Jf(x)| = \frac{\mathcal{H}^m(Df(x)(B))}{\mathcal{H}^m(B \cap (\ker Df(x))^\perp)}\,.$$

If $\operatorname{rank} Df(x) < m$, then we set $|Jf(x)| = 0$. Although this geometric picture is quite complicated there is a simple formula for $|Jf|$ which follows from the polar decomposition of the linear mapping Df,

$$|Jf| = \sqrt{\det(Df)(Df)^T}\,. \tag{4}$$

Again it is more convenient to regard (4) as a definition and what was before as a geometric interpretation. Note that the right hand sides of (3) and (4) are slightly different.

In the same way we can define $|Jf|$ in the case in which f is a Lipschitz mapping between Riemannian manifolds $f\colon M^n \to N^m$. Indeed, the differential $Df(x)\colon T_xM^n \to T_{f(x)}N^m$ is a linear mapping between linear spaces equipped with a scalar product.

Now we can formulate both the area and co-area formulas.

THEOREM 1 (Area and Co-area formulas). *Let M^n and N^m be two Riemannian manifolds of dimensions n and m respectively. Let $f\colon M^n \to N^m$ be a Lipschitz mapping and $g\colon M^n \to \mathbb{R}$ an integrable or a nonnegative measurable function.*

(1) *If $n \leq m$, then the* area *formula holds*

$$\int_{M^n} g(x)|Jf(x)|\,d\mathcal{H}^n(x) = \int_{N^m}\left(\int_{f^{-1}(y)} g(x)\,d\mathcal{H}^0(x)\right)d\mathcal{H}^n(y)\,, \tag{5}$$

where $|Jf|$ *is defined by* (3).

(2) *If* $n \geq m$, *then the* co-area *formula holds*

$$\int_{M^n} g(x)|Jf(x)|\,d\mathcal{H}^n(x) = \int_{N^m} \left(\int_{f^{-1}(y)} g(x)\,d\mathcal{H}^{n-m}(x) \right) d\mathcal{H}^m(y)\,, \tag{6}$$

where $|Jf|$ *is defined by formula* (4).

REMARKS. 1) In most of the cases we will be interested in the situation when f is a Lipschitz mapping between Euclidean spaces $f\colon \mathbb{R}^n \to \mathbb{R}^m$. The only exceptions from this rule will appear in proofs of Theorem 7 and Theorem 26.

2) If $m = n$, then both area and co-area formulas coincide. If in addition f is a diffeomorphism, then they coincide with the change of variables formula (1).

3) The Hausdorff measure $\mathcal{H}^0$ is simply the counting measure, so we can rewrite (5) equivalently as follows

$$\int_{M^n} g(x)|Jf(x)|\,d\mathcal{H}^n(x) = \int_{N^m} \Big(\sum_{x \in f^{-1}(y)} g(x) \Big)\,d\mathcal{H}^n(y)\,.$$

Taking $g \equiv \chi_E$ we obtain the formula

$$\int_{E} |Jf(x)|\,d\mathcal{H}^n(x) = \int_{N^m} N_f(y,E)\,d\mathcal{H}^n(y)\,, \tag{7}$$

where $N_f(y,E)$ is the *Banach indicatrix* defined as a number of points in the set $f^{-1}(y) \cap E$.

4) The theorem generalizes to the case in which the mapping f is defined on an open subset of M^n as such a subset is a Riemannian manifold as well.

5) There are various generalizations of the area and co-area formulas, see e.g. [3], [34], for a far reaching generalization to the case of mappings between metric spaces.

6) Area and co-area formulas as stated in Theorem 1 are due to Federer. Area formula was proved in [15] and [16, Theorem 5.9] and the co-area in [17]. Earlier versions are due to Banach [4] when $n = m = 1$, and Kronrod [36] when $n = 2$, $m = 1$. For the proofs of Theorem 1 we refer the reader to books [11], [13], [18].

To have a better understanding of the co-area formula let us have a look at the following standard examples which show that the co-area formula is a common generalization of the change of variables formula, the Fubini theorem and the formula for the integration in the polar coordinates system!

If $n \geq m$ and $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is the orthogonal projection, then $|Jf| = 1$ and hence (6) reduces to the Fubini theorem.

If $f\colon \mathbb{R}^n \to \mathbb{R}$, $f(x) = |x|$, then $|Jf| = |\nabla f| = 1$ whenever $x \neq 0$ and hence (6) gives the integration in the polar coordinates system

$$\int_{\mathbb{R}^n} g\, dx = \int_0^\infty \left(\int_{\partial B(0,r)} g\, d\mathcal{H}^{n-1} \right) dr\,.$$

Let us also note the following important case. If $f\colon \mathbb{R}^n \to \mathbb{R}$ is an arbitrary Lipschitz function, then $|Jf| = |\nabla f|$ a.e. and hence

$$\int_{\mathbb{R}^n} |\nabla f| = \int_{-\infty}^{\infty} \mathcal{H}^{n-1}(\{f = t\})\, dt\,.$$

Countably rectifiable sets. Given integers $k \leq n$ we say that a Borel set $E \subset \mathbb{R}^n$ is *countably $\mathcal{H}^k$-rectifiable* if there exists a sequence of Lipschitz mappings $f_i\colon E_i \subset \mathbb{R}^k \to \mathbb{R}^n$ such that $\mathcal{H}^k(E \setminus \bigcup_{i=1}^\infty f_i(E_i)) = 0$.

To get another characterization of countably rectifiable sets we need the following lemma which is a special case of much more general results of Whitney [62], see also [13], [18], [58].

LEMMA 2. *If $f\colon E \subset \mathbb{R}^k \to \mathbb{R}^n$ is Lipschitz, then to every $\varepsilon > 0$ there exists a C^1 mapping $f_\varepsilon\colon \mathbb{R}^k \to \mathbb{R}^n$ such that $\mathcal{H}^k(\{x \in E :\ f(x) \neq f_\varepsilon(x)\}) < \varepsilon$.*

Now we can prove the following well known characterization of countably rectifiable sets.

THEOREM 3. *Let $E \subset \mathbb{R}^n$ be a Borel set with $\mathcal{H}^k(E) < \infty$, where $k \leq n$ is an integer. Then E is countably $\mathcal{H}^k$-rectifiable if and only if there exists a sequence of C^1 smooth, k-dimensional submanifolds M_1^k, M_2^k,... of $\mathbb{R}^n$ such that $\mathcal{H}^k(E \setminus \bigcup_{i=1}^\infty M_i^k) = 0$.*

PROOF. The condition is obviously sufficient. Now we prove necessity. Observe that if $f\colon A \subset \mathbb{R}^k \to \mathbb{R}^n$ is Lipschitz and $\mathcal{H}^k(A) = 0$, then $\mathcal{H}^k(f(A)) = 0$. Hence it follows from Lemma 2 that $f_i\colon E_i \to \mathbb{R}^n$ can be assumed to be the restriction of a C^1 mapping $f_i\colon \mathbb{R}^k \to \mathbb{R}^n$ to E_i. If $\operatorname{rank} Df_i(x) = k$, then f_i maps a neighborhood of x into a k-dimensional submanifold of $\mathbb{R}^n$. Hence the theorem will follow if we prove that $\mathcal{H}^k(f_i(\{\operatorname{rank} Df_i < k\})) = 0$. This however follows directly from the area formula applied to $f = f_i$, $g = \chi_{\{\operatorname{rank} Df_i < k\}}$. Indeed, $|Jf| = 0$ whenever $\operatorname{rank} Df_i < k$. $\square$

The following well known result can be regarded as a kind of Sard's theorem.

THEOREM 4. *If $f\colon \mathbb{R}^n \to \mathbb{R}^m$, is Lipschitz, then for almost every point $y \in \mathbb{R}^m$, $f^{-1}(y)$ is countably $\mathcal{H}^{n-m}$-rectifiable.*

REMARK. If $m > n$, then the theorem simply says that $f^{-1}(y) = \emptyset$ for a.e. $y \in \mathbb{R}^m$. Actually the Hausdorff dimension of the image $f(\mathbb{R}^n)$ does not exceed n, so in this case the claim is obvious.

PROOF. Due to the remark we can assume that $m \leq n$. Assume first that $f \in C^1$. Then $f^{-1}(y) \cap \{\operatorname{rank} Df = m\}$ is a C^1 submanifold. Moreover it follows from the co-area formula that for almost every $y \in \mathbb{R}^m$, $\mathcal{H}^{n-m}(f^{-1}(y) \cap \{\operatorname{rank} Df < m\}) = 0$. Hence $f^{-1}(y)$ is countably $(n-m)$-rectifiable for a.e. $y \in \mathbb{R}^m$. The case of a general Lipschitz mapping easily follows from the case of a C^1 mapping and from Lemma 2. □

The following lemma of McShane, [47], is frequently used in the setting of Lipschitz functions.

LEMMA 5. *A Lipschitz function defined on a subset of* $\mathbb{R}^n$ *can be extended to a Lipschitz function defined on entire* $\mathbb{R}^n$ *without increasing the Lipschitz constant.*

PROOF. If $f\colon E \subset \mathbb{R}^n \to \mathbb{R}$ is Lipschitz, then the function $\widetilde{f}\colon \mathbb{R}^n \to \mathbb{R}$ defined by

$$\widetilde{f}(x) = \inf_{y \in E} \{f(y) + \operatorname{Lip}(f|_E)|x-y|\}$$

has desired properties i.e., $\widetilde{f}|_E = f$, and $\operatorname{Lip}(\widetilde{f}) = \operatorname{Lip}(f)$. □

COROLLARY 6. *If* $f\colon E \subset \mathbb{R}^n \to \mathbb{R}^m$ *is Lipschitz, then* $f^{-1}(y) \cap E$ *is countably* $\mathcal{H}^{n-m}$*-rectifiable for a.e.* $y \in \mathbb{R}^m$.

A general countably $\mathcal{H}^k$-rectifiable set consists of a nice part that can be covered by countably many k-dimensional manifolds and a small "irregular" part of vanishing $\mathcal{H}^k$-measure. Irregular means that it cannot be covered by countably many k-dimensional submanifolds. In particular this description applies to preimages of almost all points of a Lipschitz mapping. In general one cannot avoid the irregular part in the description of $f^{-1}(y)$, see [35].

Integral-geometric measure. In this section we recall definition and basic properties of the *integral-geometric measure* $\mathcal{I}^m$ (called sometimes Favard's measure). This is one of the m-dimensional measures in $\mathbb{R}^n$. On regular sets it coincides with Hausdorff's measure $\mathcal{H}^m$, but it has different properties on some "fractal" sets.

Let $E \subset \mathbb{R}^n$ be a Borel set and $1 \leq m < n$ be an integer. Set

$$\mathcal{I}^m(E) = \frac{1}{\beta(n,m)} \int_{p \in O^*(n,m)} \int_{y \in \operatorname{Im} p} N_p(y,E)\, d\mathcal{H}^m(y)\, d\vartheta^*_{n,m}(p), \tag{8}$$

where $O^*(n,m)$ denotes the space of orthogonal projections p from $\mathbb{R}^n$ onto m-dimensional linear subspaces, $\operatorname{Im} p$ is the image of the projection p and $\vartheta^*_{n,m}$ is the Haar measure on $O^*(n,m)$ invariant under the action of $O(n)$, normalized to have total mass 1. By $N_p(y,E)$ we denote as before the Banach indicatrix.

For $m = n$, we define $\mathcal{I}^m \equiv \mathcal{H}^m$.

Thus roughly speaking $\mathcal{I}^m(E)$ is defined as follows. Fix an m-dimensional linear subspace of $\mathbb{R}^n$ and denote by p the orthogonal projection from $\mathbb{R}^n$

onto that subspace. Next we compute the measure of the image of the projection of the set E taking into account the multiplicity function N_p, and then we average the resulting integral over all m-dimensional linear subspaces of $\mathbb{R}^n$.

We still have to define the coefficient $\beta(n,m)$. If $E = Q^m$ is the m-dimensional cube in $\mathbb{R}^n$, then the double integral, which stands on the right hand side of (8) is finite and positive. The coefficient $\beta(n,m)$ is defined in such a way that $\mathcal{I}^m(Q^m) = \mathcal{H}^m(Q^m)$.

We extend the definition of $\mathcal{I}^m$ to all sets $A \subset \mathbb{R}^n$ by the formula

$$\mathcal{I}^m(A) = \inf\{\, \mathcal{I}^m(E) \,|\, A \subset E, E - \text{Borel set}\}\,.$$

and then one can prove that $\mathcal{I}^m$ is a Borel regular measure, see [18].

It follows easily from the definition that $\mathcal{I}^m$ coincides with $\mathcal{H}^m$ on polyhedral sets. Then it is not surprising that both measures coincide on countably $\mathcal{H}^m$-rectifiable sets. This is a theorem of Federer [16, Theorem 5.14] known sometimes as Crofton's theorem.

THEOREM 7. *If $A \subset \mathbb{R}^n$ is countably $\mathcal{H}^m$-rectifiable, $m \leq n$, then $\mathcal{I}^m(A) = \mathcal{H}^m(A)$.*

PROOF. Assume first that A is a subset of an m-dimensional C^1 submanifold $M^m \subset \mathbb{R}^n$. Denoting by $\widetilde{p}$ the restriction of $p \in O^*(n,m)$ to M^m area formula (7) yields

$$\int_A |J\widetilde{p}(x)|\, d\mathcal{H}^m(x) = \int_{\operatorname{Im} p} N_p(y,A)\, d\mathcal{H}^m(y)\,.$$

In this case theorem follows from the observation that

$$\int_{p\in O^*(n,m)} |J\widetilde{p}(x)|\, d\vartheta^*_{n,m}(p) = \beta(n,m)\,,$$

after averaging over $p \in O^*(n,m)$. The case of general countably $\mathcal{H}^m$-rectifiable set follows from Theorem 3 and the elementary observation that $\mathcal{H}^m(E) = 0$ implies $\mathcal{I}^m(E) = 0$. □

Although we will not need it in the sequel let us close the section by recalling some known results which say how to relate $\mathcal{I}^m$ to $\mathcal{H}^m$ on general subsets of $\mathbb{R}^n$.

If $A \subset \mathbb{R}^n$, $\mathcal{H}^m(A) < \infty$, then there is a unique decomposition up to sets of $\mathcal{H}^m$-measure zero

$$A = B \cup C\,, \tag{9}$$

where B is countably $\mathcal{H}^m$-rectifiable and C is *purely $\mathcal{H}^m$-nonrectifiable* in a sense that

$$\mathcal{H}^m(C \cap W) = 0$$

for every countably $\mathcal{H}^m$-rectifiable set W. To prove the existence of the decomposition (9) we take as B a countably $\mathcal{H}^m$-rectifiable subset of A of maximal measure. It is easy to see that such a set exists. The uniqueness is also easy.

The following theorem is a celebrated structure theorem of Besicovitch and Federer.

THEOREM 8. *If* $C \subset \mathbb{R}^n$, $\mathcal{H}^m(C) < \infty$, $m < n$, *is purely* $\mathcal{H}^m$*-nonrectifiable, then* $\mathcal{I}^m(C) = 0$.

REMARKS. 1) Besicovitch, [5], proved the theorem for case $n = 2$ and $m = 1$ and Federer, [16], proved it in the general case. Proof can be also found in books [18, Theorem 3.3.13], [46]. Recently a simplified proof of a general case has been obtained in [61] and of the case $m = n - 1$ in [32].
2) It is easily seen that if B is countably $\mathcal{H}^m$-rectifiable and $\mathcal{H}^m(B)>0$, then almost all projections of B onto m-dimensional linear subspaces of $\mathbb{R}^n$ have positive measure. Hence the structure theorem together with the decomposition (9) results in a fact that every subset of $\mathbb{R}^n$ of finite $\mathcal{H}^m$-measure, can be decomposed into two parts. The first part has the property that projections onto almost all m-dimensional linear subspaces of $\mathbb{R}^n$ have positive measure and for the second part almost all projections have measure zero.

It follows from above results that for an arbitrary set $A \subset \mathbb{R}^n$

$$\mathcal{H}^m(A) \geq \mathcal{I}^m(A).$$

There exists plenty of (even compact) sets with $\mathcal{H}^m(A) > 0$ and $\mathcal{I}^m(A) = 0$. However then the Hausdorff dimension of the set A cannot exceed m. This is a result of Mattila [44], [46].

THEOREM 9. *If* $\dim_{\mathcal{H}} A > m$, *then* $\mathcal{I}^m(A) = \infty$. *Hence* $\mathcal{I}^m(A) < \infty$ *implies that* $\dim_{\mathcal{H}} A \leq m$.

3. Sobolev mappings

In this section we will generalize both area and co-area formulas to the case in which the transformation $f\colon \mathbb{R}^n \to \mathbb{R}^m$ belongs to the Sobolev space $W^{1,p}$.

The Sobolev space $W^{1,p}(\mathbb{R}^n)$, $1 \leq p \leq \infty$ is defined as a set of all functions $f \in L^p(\mathbb{R}^n)$ with the distributional gradient in $L^p(\mathbb{R}^n)$. The space is equipped with the norm $\|f\|_{1,p} = \|f\|_p + \|\nabla f\|_p$. The notation $W^{1,p}_{\mathrm{loc}}$ is self-explanatory.

By Sobolev mappings $W^{1,p}(\mathbb{R}^n, \mathbb{R}^m)$ we will mean mappings from $\mathbb{R}^n$ to $\mathbb{R}^m$ with the property that coordinate functions belong to $W^{1,p}(\mathbb{R}^n)$.

There are many advanced monographs on Sobolev spaces, but for an introduction to a basic stuff needed in our paper we especially recommend [13], [21], [22], [43], [64].

One can prove that $W^{1,\infty}(\mathbb{R}^n)$ coincides with the space of bounded Lipschitz functions. Since we have treated the Lipschitz case separately we will assume in what follows that $1 \leq p < \infty$.

All the theory discussed below can be easily extended to the case of Sobolev functions defined on domains $\Omega\subset\mathbb{R}^n$ or even on Riemannian manifolds. However we decided to deal with the case $\Omega=\mathbb{R}^n$ just for the sake of simplicity.

In order to generalize area and co-area formulas to the case in which $f\in W^{1,p}(\mathbb{R}^n,\mathbb{R}^m)$ we need the following well known lemma, [1], [7] [13], [26], [38], [43], [48], [64].

LEMMA 10. *If $f\in W^{1,p}(\mathbb{R}^n)$, $1\leq p<\infty$, then to every $\varepsilon>0$ there exists a Lipschitz function $g\in \mathrm{Lip}\,(\mathbb{R}^n)$ such that*

(1) $|\{f\neq g\}|<\varepsilon$;
(2) $\|f-g\|_{1,p}<\varepsilon$.

PROOF. Given $f\in W^{1,p}(\mathbb{R}^n)$ the following pointwise inequality holds a.e.

$$|f(x)-f(y)|\leq C|x-y|(\mathcal{M}|\nabla f|(x)+\mathcal{M}|\nabla f|(y))\,,$$

where $\mathcal{M}g(x)=\sup_{r>0}|B(x,r)|^{-1}\int_{B(x,r)}|g|$ is the Hardy–Littlewood maximal function, see [6], [7], [26]. Let $E_t=\{x:\,|f(x)|\leq t \text{ and } \mathcal{M}|\nabla f|(x)\leq t\}$. Then $f|_{E_t}$ is a bounded Lipschitz function with the Lipschitz constant bounded by $2Ct$. Let $\widetilde{f}_t$ be the extension of $f|_{E_t}$ given by Lemma 5. Next we define f_t as a truncation of $\widetilde{f}_t$

$$f_t(x)=\begin{cases} t & \text{if} \quad \widetilde{f}_t(x)\geq t,\\ \widetilde{f}_t(x) & \text{if} \quad |\widetilde{f}_t(x)|\leq t,\\ -t & \text{if} \quad \widetilde{f}_t(x)\leq -t\,.\end{cases}$$

Observe that f_t has the same Lipschitz constant as $\widetilde{f}_t$ (which is bounded by $2Ct$). Moreover f_t is a bounded function with $\|f_t\|_\infty\leq t$ and $f_t|_{E_t}=f|_{E_t}$. We will prove that f_t gives the desired Lipschitz approximation as t goes to infinity. First note that $\{f\neq f_t\}\subset\{|f|>t\}\cup\{\mathcal{M}|\nabla f|>t\}:=F_t$. Thus

$$\begin{aligned}\int_{\mathbb{R}^n}|f-f_t|^p+|\nabla f-\nabla f_t|^p\,dx &= \int_{F_t}|f-f_t|^p+|\nabla f-\nabla f_t|^p\,dx\\ &\leq C\int_{F_t}(|f|^p+|\nabla f|^p)\,dx+Ct^p|F_t|\,.\end{aligned}$$

Now we will conclude that both terms on the right hand side converge to zero as $t\to\infty$ as soon as we show that $t^p|F_t|\to 0$ as $t\to\infty$. This will be the consequence of

$$t^p|\{|f|>t\}|+t^p|\{\mathcal{M}|\nabla f|>t\}|\to 0 \quad \text{as } t\to\infty. \tag{10}$$

The convergence of the first term in (10) follows from the fact that $f\in L^p$ (Chebyschev's inequality), while the convergence of the second term from the fact that $\mathcal{M}|\nabla f| \in L^p$, when $p > 1$, [59, Theorem 1, page 5], and from the following weak type estimate, when $p = 1$, [59, (5) on page 7],

$$|\{\mathcal{M}|\nabla f| > t\}| \leq \frac{C}{t}\int_{\{|\nabla f|>t/2\}} |\nabla f|\, dx\,.$$

The proof is complete. □

Now we can formulate and prove a version of Theorem 1 valid for Sobolev mappings. This seems to be a folklore result.

THEOREM 11. *Let $f \in W^{1,p}_{\rm loc}(\mathbb{R}^n, \mathbb{R}^m)$, $1 \leq p < \infty$, and let $g\colon \mathbb{R}^n \to \mathbb{R}$ be either nonnegative measurable or measurable and such that $g(x)|Jf(x)| \in L^1(\mathbb{R}^n)$. Then there exists a representative of f such that both area* (5) *and co-area* (6) *formulas hold.*

REMARK. The theorem can be easily generalized to the case of Sobolev mappings between manifolds.

The theorem is a consequence of the following more detailed result.

THEOREM 12. *Let $f \in W^{1,p}_{\rm loc}(\mathbb{R}^n, \mathbb{R}^m)$, $1 \leq p < \infty$, be an arbitrary representative. Then there exists a sequence of closed sets $F_1 \subset F_2 \subset \ldots \subset \mathbb{R}^n$ such that $A = \mathbb{R}^n \setminus \bigcup_i F_i$ has the Lebesgue measure zero and $f|_{F_i}$ is Lipschitz continuous for every $i = 1, 2, \ldots$ Hence $f^{-1}(y) \cap (\mathbb{R}^n \setminus A)$ is countably $\mathcal{H}^{n-m}$-rectifiable for a.e. $y \in \mathbb{R}^m$. Moreover if $g\colon \mathbb{R}^n \to \mathbb{R}$ is either nonnegative measurable or measurable and such that $g(x)|Jf(x)| \in L^1(\mathbb{R}^n)$, then the following area when $n \leq m$*

$$\int_{\mathbb{R}^n} g(x)|Jf(x)|\, d\mathcal{H}^n(x) = \int_{\mathbb{R}^m} \Big(\sum_{x \in f^{-1}(y)\cap(\mathbb{R}^n\setminus A)} g(x)\Big)\, d\mathcal{H}^n(y)\,,$$

and co-area when $n \geq m$

$$\int_{\mathbb{R}^n} g(x)|Jf(x)|\, d\mathcal{H}^n(x) = \int_{\mathbb{R}^m} \left(\int_{f^{-1}(y)\cap(\mathbb{R}^n\setminus A)} g(x)\, d\mathcal{H}^{n-m}(x)\right) d\mathcal{H}^m(y) \tag{11}$$

formulas hold.

PROOF. We will prove the co-area formula as the proof for the area formula follows exactly the same argument.

Let $f_k\colon \mathbb{R}^n \to \mathbb{R}^m$ be a sequence of Lipschitz mappings and $F_1 \subset F_2 \subset \ldots \subset \mathbb{R}^n$ be a sequence of closed sets such that $f|_{F_k} = f_k|_{F_k}$ and $\mathcal{H}^n(\mathbb{R}^n \setminus \bigcup_k F_k) = 0$. The existence of such a sequence follows from Lemma 10. Denote $A = \mathbb{R}^n \setminus \bigcup_k F_k$.

The co-area formula applied to each of the mappings f_k and $g_k \equiv g\chi_{F_k}$ yields

$$(12)\quad \int_{F_k} g(x)|Jf_k(x)|\,d\mathcal{H}^n(x) = \int_{\mathbb{R}^m}\left(\int_{f_k^{-1}(y)\cap F_k} g(x)\,d\mathcal{H}^{n-m}(x)\right)d\mathcal{H}^m(y)\,.$$

The equality $f_k \equiv f$ in F_k implies $Jf_k = Jf$ a.e. in F_k. Hence passing to the limit in (12) as $k\to\infty$ yields

$$\int_{\bigcup_k F_k} g(x)|Jf(x)|\,d\mathcal{H}^n(x) = \int_{\mathbb{R}^m}\left(\int_{f^{-1}(y)\cap\bigcup_k F_k} g(x)\,d\mathcal{H}^{n-m}(x)\right)d\mathcal{H}^m(y)\,,$$

which readily implies (11).

The fact that the set $f^{-1}(y)\cap(\mathbb{R}^n\setminus A) = f^{-1}(y)\cap\bigcup_k F_k$ is countably $\mathcal{H}^{n-m}$-rectifiable for a.e. $y\in\mathbb{R}^m$ follows from Corollary 6. □

Proof of Theorem 11. Given an arbitrary representative f it suffices to choose another representative which sends the set A from Theorem 12 to a single point and remains unchanged in $\mathbb{R}^n\setminus A$. □

Observe that we did not use the whole strength of Lemma 10 as the only property of the Sobolev function that we employed was the fact that the Sobolev function coincides with a Lipschitz function outside a set of an arbitrary small measure. This property is true for a much larger class of functions which are approximately differentiable a.e. [62], and so one can generalize both the area and the co-area formulas to that class of mappings. We will not go however that far with our generalizations, as we are interested mainly in the case of Sobolev mappings; see [15], [18], [20], [23] for some related results in the case of the a.e. approximately differentiable mappings.

There is something very delicate in Theorem 11. We will explain it on an example.

If $f\in W^{1,p}(\mathbb{R}^n,\mathbb{R}^n)$ is a homeomorphism and $g\equiv\chi_E$ is a characteristic function of a measurable set, then both the area and co-area formulas seem to yield the identity

$$(13)\qquad \int_E |Jf|\,d\mathcal{H}^n = \mathcal{H}^n(f(E))\,.$$

However Ponomarev, [51], [52], provided an example of a homeomorphism f which belongs to all the Sobolev spaces $W^{1,p}(\mathbb{R}^n,\mathbb{R}^n)$, for $1\le p<n$, and has the property that for some set $E\subset\mathbb{R}^n$ of the measure zero, $\mathcal{H}^n(f(E))>0$. Then applying (13) to that particular set E yields a contradiction. This means some of our arguments were not correct. Indeed, we were not very careful when applying Theorem 11. The theorem says that there exists a representative of f for which (13) is true and it does not say that the representative has to be the homeomorphism. Actually the above example shows that the representative has to be different from the homeomorphic one and thus discontinuous.

Examples of Malý and Martio, [42], Reshetnyak, [56], and Väisälä, [60], show that the situation is not much better even when f is a continuous mapping in $W^{1,n}(\mathbb{R}^n, \mathbb{R}^n)$.

Thus how can one characterize those representatives for which the area and the co-area formulas hold?

We will discuss the case of the area formula. We say that the mapping $f\colon \mathbb{R}^n \to \mathbb{R}^m$, $m \geq n$, has the *Lusin property* if the following implication holds

$$\mathcal{H}^n(E) = 0 \quad \Longrightarrow \quad \mathcal{H}^n(f(E)) = 0\,. \tag{14}$$

It follows easily from the proof of Theorem 11 that given representative of $f \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^m)$, $m \geq n$, verifies the area formula (5) if and only if it has the Lusin property. It follows also that each Sobolev mapping for $m \geq n$ admits a representative with the Lusin property.

We assumed that $m \geq n$ as in the case $m < n$ condition (14) is always satisfied and thus trivial.

In the examples discussed above the assumption was $p \leq n$ and this is the limiting case. Indeed, it is well known, [8], [63], that if $p > n$, and $m \geq n$ then the continuous representative of a Sobolev map $f \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^m)$ has the Lusin property, see Lemma 21.

Moreover if $f \in W^{1,n}(\mathbb{R}^n, \mathbb{R}^n)$ is a homeomorphism, a theorem of Reshetnyak, [53], [56] says that f has the Lusin property.

For more details concerning the Lusin condition and its applications to the area formula we recommend the paper of Jan Malý, [41], that appears in the same volume.

We close this section with an application of Lemma 10 and the area formula to a problem of a different nature. Namely following [27] we will prove a generalization of the classical Brouwer fixed point theorem. We will need this result in the last section.

THEOREM 13. *If $B^n \subset \mathbb{R}^n$ is a ball, then for $f \in W^{1,n}(B^n, B^n)$, with $f|_{\partial B^n} = \mathrm{id}$, we have $|B^n \setminus f(B^n)| = 0$.*

Of course $f \in W^{1,n}(B^n, B^n)$ means $f = (f_1, \ldots, f_n)$, $f_i \in W^{1,n}(B^n)$ for $i = 1, 2, \ldots, n$ and $\sum_i f_i^2 \leq 1$ a.e. The restriction of f to the boundary is understood as a trace of a Sobolev function.

PROOF. By Lemma 10, there exists a sequence $f_k \in \mathrm{Lip}\,(B^n, B^n)$, such that $f_k \to f$ in $W^{1,n}$ and $|\{f_k \neq f\}| \to 0$ as $k \to \infty$. It is not difficult to see that we can assume in addition that $f_k|_{\partial B^n} = \mathrm{id}$ (restriction in the classical sense). According to the classical Brouwer theorem $f_k(B^n) = B^n$, and hence there exists $E_k \subset B^n$ such that $f_k(E_k) = B^n \setminus f(B^n)$. Obviously $E_k \subset \{f_k \neq f\}$, and hence $|E_k| \to 0$, as $k \to \infty$. Now the area formula implies

$$|B^n \setminus f(B^n)| = |f_k(E_k)| \leq \int_{E_k} |Jf_k| \to 0,$$

since $Jf_k \to Jf$ in L^1 (by the Hölder inequality), and $|E_k| \to 0$. $\square$

4. Quasicontinuous representatives

A generic measurable function $f\colon \mathbb{R}^n \to \mathbb{R}$ has some continuity properties that are described by the Lusin theorem. Namely to every $\varepsilon > 0$ there exists an open set $U \subset \mathbb{R}^n$ of the Lebesgue measure less than ε and such that f restricted to $\mathbb{R}^n \setminus U$ is continuous.

If $p \leq n$, then a generic function from $W^{1,p}(\mathbb{R}^n)$ is essentially discontinuous everywhere. However if $p > n$, then all the functions in $W^{1,p}(\mathbb{R}^n)$ are Hölder continuous.

Thus we may expect that for $1 \leq p \leq n$ there is a result which would be a kind of an interpolation between the Lusin theorem and the everywhere continuity. This leads to notions of capacity and quasicontinuity that we next describe.

Let $1 \leq p < n$. If $K \subset \mathbb{R}^n$ is a compact set, then we define its *p-capacity* as

$$\mathrm{Cap}_p(K) = \inf\Big\{ \int_{\mathbb{R}^n} |\nabla u|^p : \ u \in C_0^\infty(\mathbb{R}^n), \ u|_K \geq 1 \Big\}.$$

For an open set $U \subset \mathbb{R}^n$ we put

$$\mathrm{Cap}_p(U) = \sup_{\substack{K \subset U \\ K-\mathrm{compact}}} \mathrm{Cap}_p(K)\,,$$

and finally for an arbitrary set $E \subset \mathbb{R}^n$ we define

$$\mathrm{Cap}_p(E) = \inf_{\substack{U \supset E \\ U-\mathrm{open}}} \mathrm{Cap}_p(U)\,.$$

The reader can find more information about the p-capacity in [13], [30], [43], [64].

It is also possible to define the p-capacity for $p \geq n$, but it requires some changes in the definitions and we will not go into details here.

We say that the function $f\colon \mathbb{R}^n \to \mathbb{R}$ is *p-quasicontinuous* if it is Borel measurable and to every $\varepsilon > 0$ there exists an open set $U \subset \mathbb{R}^n$ with $\mathrm{Cap}_p(U) < \varepsilon$ and such that $f|_{\mathbb{R}^n \setminus U}$ is continuous. The following result was proved by Reshetnyak, [54].

THEOREM 14. *If $1 \leq p < n$, then any function $f \in W^{1,p}(\mathbb{R}^n)$ has a p-quasicontinuous representative.*

The theorem generalizes to the case $p \geq n$ as the capacity can be defined there. If $p > n$, then the p-capacity of a point is positive and hence an open set U of a sufficiently small capacity is empty. Thus the continuity of f

in the set $\mathbb{R}^n \setminus U$ means simply continuity in the entire $\mathbb{R}^n$, which is the Sobolev embedding.

Theorem 14 is a desired interpolation between everywhere continuity and the Lusin theorem. Sets of small p-capacity are "smaller" than generic sets of small Lebesgue measure as it is well known that the Hausdorff dimension of sets of vanishing p-capacity is less than or equal to $n-p$. Roughly speaking this implies that if $p>k$, where k is an integer, then a p-quasicontinuous function is continuous when restricted to almost every k-dimensional affine subspace of $\mathbb{R}^n$. This is because almost all such planes does not meet the set of the Hausdorff dimension $n-p$.

For those who do not know the capacity theory we will provide a direct and elementary proof of the last statement i.e. we will prove that $f \in W^{1,p}(\mathbb{R}^n)$ has a representative which is continuous on almost all k-dimensional affine subspaces of $\mathbb{R}^n$, for $k<p$. This is the only consequence of Theorem 14 that will be needed in the sequel. Since we will show an elementary proof of this fact, from now on, the reader may forget what we said about the capacity and p-quasicontinuous representatives.

When saying that some property holds on almost all k-dimensional affine subspaces we mean that for almost every k-dimensional linear subspace of $\mathbb{R}^n$ the property holds for almost all affine k-dimensional subspaces parallel to the given linear subspace. More precisely the space of all k-dimensional affine subspaces form a vector bundle which is equipped with a measure that provides the notion of a.e. Given integer $1 \leq k \leq n$, by $G(n,k)$ we will denote the Grassmannian of all k-dimensional linear subspaces of $\mathbb{R}^n$. This is a manifold on which the orthogonal group $O(n)$ acts transitively. Let $\gamma_{n,k}$ be the unique Haar measure on $G(n,k)$ normalized to have total mass 1, invariant under the action of $O(n)$. The space of all k-dimensional affine subspaces in $\mathbb{R}^n$ can be identified with the $(n-k)$-dimensional vector bundle over $G(n,k)$ with orthogonal complements of elements of $G(n,k)$ as fibers. Denote this vector bundle by $E(n,k)$. The identification of $E(n,k)$ with all affine k-dimensional subspaces is the natural one. Elements of $G(n,k)$ indicate directions of subspaces and each fiber parameterizes all k-dimensional affine subspaces parallel to the given element of $G(n,k)$. The vector bundle is equipped with a measure which is $\gamma_{n,k}$ on the base-manifold and $\mathcal{H}^{n-k}$ in each fiber. We will denote such a measure by $\mathcal{H}^{n-k} \mathbin{\widehat{\times}} \gamma_{n,k}$. "Hat" means that this is not the Cartesian product, but a "twisted" product of measures since this is not a trivial vector bundle.

The following lemma is a version of the Fubini theorem adopted to the setting of Sobolev spaces.

Lemma 15. *Let $f \in W^{1,p}(\mathbb{R}^n)$. If $1 \leq k \leq n$ is an integer, then $f|_{P^k} \in W^{1,p}(P^k)$ for almost all $P^k \in E(n,k)$. Moreover if $f_i, f \in W^{1,p}(\mathbb{R}^n)$, $f_i \to f$ in $W^{1,p}(\mathbb{R}^n)$, then there exists a subsequence f_{i_j} such that $f_{i_j}|_{P^k} \to f|_{P^k}$ in $W^{1,p}(P^k)$ for almost all $P^k \in E(n,k)$.*

PROOF. Fix $\Gamma \in G(n,k)$. The fact that $f|_{P^k} \in W^{1,p}(P^k)$ for almost all k-dimensional affine subspaces P^k parallel to Γ is an easy consequence of the Fubini theorem applied to a sequence of smooth functions approximating f in $W^{1,p}(\mathbb{R}^n)$. We leave details to the reader.

Now let $f_i \to f$ in $W^{1,p}(\mathbb{R}^n)$. Fix $\Gamma \in G(n,k)$ and denote by $E_\Gamma = \Gamma^\perp$ the fiber of $E(n,k)$ over Γ. Then given $y \in E_\Gamma$ denote by Γ_y the k-dimensional affine subspace parallel to Γ and intersecting E_Γ at a point y. The Fubini theorem yields

$$\|f - f_i\|_{1,p}^p = \int_{E_\Gamma} \int_{\Gamma_y} |f - f_i|^p + |\nabla f - \nabla f_i|^p \, d\mathcal{H}^k(x) \, d\mathcal{H}^{n-k}(y) \to 0 \, .$$

Now averaging over $G(n,k)$ gives

$$\begin{aligned} &\|f - f_i\|_{1,p}^p \\ &= \int_{G(n,k)} \int_{E_\Gamma} \int_{\Gamma_y} |f - f_i|^p + |\nabla f - \nabla f_i|^p \, d\mathcal{H}^k(x) \, d\mathcal{H}^{n-k}(y) \, d\gamma_{n,k}(\Gamma) \to 0 \, . \end{aligned}$$

Let

$$F_i = \int_{\Gamma_y} |f - f_i|^p + |\nabla f - \nabla f_i|^p \, d\mathcal{H}^k(x) \, .$$

Then $F_i \to 0$ in $L^1(E(n,k))$ and hence $F_{i_j} \to 0$ a.e. on $E(n,k)$ for a suitable subsequence. The proof is complete. □

THEOREM 16. *Let $f \in W^{1,p}(\mathbb{R}^n)$, $1 \le p < \infty$, and let $1 \le k \le n$ be an integer such that $k < p$ or $k = p = 1$. Then there exists a representative of f such that $f|_{P^k} \in W^{1,p}(P^k) \cap C^0(P^k)$ for almost all $P^k \in E(n,k)$.*

PROOF. It follows directly from the previous lemma applied to a sequence of smooth functions approximating f and from the Sobolev embedding theorem which implies that a sequence of smooth functions which converges in the $W^{1,p}$ norm on a k-dimensional space converges uniformly on that space. □

The above theorem provides us with the desired representative of a Sobolev function. As we have already mentioned a p-quasicontinuous representative also verifies the property from the above theorem. In what follows we will still use the name p-quasicontinuous representative, but the only property we will actually need is the property from Theorem 16.

Let us also mention that Theorem 16 directly follows from results of Reshetnyak, [54], [55].

5. Studying preimage

A Sard type theorem for Sobolev mappings. The following theorem will play a crucial role in the description of the set $f^{-1}(y)$ where f is a quasicontinuous representative of a Sobolev mapping. For related results see [24], [25, Lemma 3].

THEOREM 17. *Let $A \subset \mathbb{R}^n$ be a Borel measurable set and $f\colon A \to \mathbb{R}^m$ a Borel measurable mapping. Let $1 \le k \le n$ be an integer. Then the following two conditions are equivalent.*

(1) *For almost all k-dimensional affine subspaces P^k of $\mathbb{R}^n$*

$$\mathcal{H}^m(f(P^k \cap A)) = 0.$$

(2) *For $\mathcal{H}^m$ almost all $y \in \mathbb{R}^m$*

$$\mathcal{I}^{n-k}(f^{-1}(y) \cap A) = 0.$$

Very roughly speaking the theorem says that images of small sets are small if and only if preimages of points are small.

To explain the idea of the proof let us consider a simple example. Given a measurable set $\mathfrak{S} \subset \mathbb{R}^2$, the following two conditions are equivalent: (a) for almost all lines ℓ parallel to the x-axis, $\mathcal{H}^1(\mathfrak{S} \cap \ell) = 0$; (b) for almost all lines $\widetilde{\ell}$ parallel to the y-axis $\mathcal{H}^1(\mathfrak{S} \cap \widetilde{\ell}) = 0$. Indeed, by the Fubini theorem both conditions are equivalent to $\mathcal{H}^2(\mathfrak{S}) = 0$. Now the idea of the proof of the theorem is the following: find a set $\mathfrak{S} \subset X \times Y$ and measures μ_X and μ_Y on X and Y such that both conditions from the theorem can be interpreted as vanishing of the measure of X-slices and Y-slices of $\mathfrak{S}$ respectively. Then it will follow directly from the Fubini theorem that both of the conditions from the theorem are equivalent to $(\mu_X \times \mu_Y)(\mathfrak{S}) = 0$.

The set $\mathfrak{S}$ will be defined as a kind of projection of a set constructed from the graph of f. To ensure measurability of the set we have to invoke the following well known results.

LEMMA 18. *Let $f\colon X \to Y$ be a mapping between separable metric spaces X and Y. If f is Borel measurable, then the graph $G_f = \{(x, f(x)) \mid x \in X\} \subset X \times Y$ is Borel measurable as well.*

PROOF. Let $\{A_i^{(n)}\}_{i=1}^\infty$ be a disjointed family of Borel subsets of Y such that $\bigcup_{i=1}^\infty A_i^{(n)} = Y$ and $\operatorname{diam} A_i^{(n)} < 1/n$. Then sets

$$B_n = \bigcup_{i=1}^\infty (f^{-1}(A_i^{(n)}) \times A_i^{(n)}) \subset X \times Y$$

are Borel and hence $G_f = \bigcap_{n=1}^\infty B_n$ is also a Borel set. $\square$

We also need the following celebrated and deep result of Lusin and Sierpiński [39], [40], [18, Theorem 2.2.13].

LEMMA 19. *If $h\colon M^n \to N^m$ is a continuous mapping between Riemannian manifolds and $A \subset M^n$ is a Borel measurable set, then $h(A)$ is $\mathcal{H}^m$-measurable.*

REMARKS. 1) Observe that this lemma is no longer true if we assume that A is $\mathcal{H}^n$-measurable instead of being Borel. Indeed, taking a projection from $\mathbb{R}^2$ to $\mathbb{R}$ as h and a 1-dimensional non-measurable set A as a subset of $\mathbb{R}^2$ we see that $\mathcal{H}^2(A) = 0$ and thus A is $\mathcal{H}^2$-measurable, but $h(A) = A$ is not $\mathcal{H}^1$-measurable.
2) For a general Borel measurable set A, the set $h(A)$ need not be Borel.
3) Lemma 19 is a particular case of a much more general result about continuous mappings between metric spaces equipped with measures, see [18, Theorem 2.2.13].

Theorem 17 is a direct consequence of a more general result that we will prove now.

THEOREM 20. *Let $A \subset \mathbb{R}^n$ be a Borel measurable set and $f\colon A \to \mathbb{R}^m$ a Borel measurable mapping. Let $1 \le k \le n$ and $1 \le \ell \le m$ be integers. Then the following two conditions are equivalent*

(1) *For almost all k-dimensional affine subspaces P^k of $\mathbb{R}^n$*

$$\mathcal{I}^\ell(f(P^k \cap A)) = 0.$$

(2) *For almost all $(m-\ell)$-dimensional affine subspaces $P^{m-\ell}$ of $\mathbb{R}^m$*

$$\mathcal{I}^{n-k}(f^{-1}(P^{m-\ell}) \cap A) = 0.$$

PROOF. Denote by $V^*(m,\ell)$ the ℓ-dimensional vector bundle over the manifold $O^*(m,\ell)$ (considered in (8)) with images of projections $p \in O^*(m,\ell)$ as fibers. This vector bundle is equipped with the measure which is $\mathcal{H}^\ell$ in each fiber and $\vartheta^*_{m,\ell}$ on the base-manifold. Denote this measure by $\mathcal{H}^\ell \mathbin{\widehat{\times}} \vartheta^*_{m,\ell}$.

Observe that $(O^*(m,\ell), \vartheta^*_{m,\ell})$ can be identified with the Grassmannian $(G(m, m-\ell), \gamma_{m,m-\ell})$, as the orthogonal projection onto an ℓ-dimensional linear subspace of $\mathbb{R}^m$ can be identified with its $(m-\ell)$-dimensional kernel. Denote this identification by $\ker\colon O^*(m,\ell) \to G(m, m-\ell)$. This induce the identification of vector bundles $V^*(m,\ell)$ and $E(m, m-\ell)$ (considered in Section 4), together with the identification of measures.

Consider two mappings

$$\pi_{m,\ell}\colon \mathbb{R}^m \times O^*(m,\ell) \to V^*(m,\ell), \qquad \pi_{m,\ell}(y,q) = (q(y), q)\,,$$

and

$$\widetilde{\pi}_{n,n-k}\colon \mathbb{R}^n \times O^*(n, n-k) \to E(n,k), \qquad \widetilde{\pi}_{n,n-k}(x,p) = (p(x), \ker p)\,,$$

and their product

$$\pi = \widetilde{\pi}_{n,n-k} \times \pi_{m,\ell} : \mathbb{R}^n \times O^*(n,n-k) \times \mathbb{R}^m \times O^*(m,\ell) \to E(n,k) \times V^*(m,\ell)\,.$$

The space $E(n,k)\times V^*(m,\ell)$ is equipped with the measure

$$\mathcal{H}^{n-k}\,\widehat{\times}\,\gamma_{n,k}\times\mathcal{H}^{\ell}\,\widehat{\times}\,\vartheta^*_{m,\ell}\,.$$

Let $G_f\subset A\times\mathbb{R}^m\subset\mathbb{R}^n\times\mathbb{R}^m$ be the graph of the mapping f. We have a natural inclusion

$$i:G_f\times O^*(n,n-k)\times O^*(m,\ell)\hookrightarrow\mathbb{R}^n\times O^*(n,n-k)\times\mathbb{R}^m\times O^*(m,\ell)\,.$$

By Lemma 18, G_f is a Borel subset of $\mathbb{R}^n\times\mathbb{R}^m$. Hence Lemma 19 implies that

$$\mathfrak{S}=(\pi\circ i)(G_f\times O^*(n,n-k)\times O^*(m,\ell))$$

is a measurable subset of $E(n,k)\times V^*(m,\ell)$. Now it follows from the Fubini theorem that both conditions of the theorem are equivalent to

$$\left(\mathcal{H}^{n-k}\,\widehat{\times}\,\gamma_{n,k}\times\mathcal{H}^{\ell}\,\widehat{\times}\,\vartheta^*_{m,\ell}\right)(\mathfrak{S})=0\,.$$

Indeed, since

$$\mathfrak{S}=\left\{(p(x),\ker p,q(f(x)),q):\,x\in A,\ p\in O^*(n,n-k),\ q\in O^*(m,\ell)\right\}$$

it easily follows that the section of $\mathfrak{S}$ corresponding to $P^k\in E(n,k)$ is given by

$$\mathfrak{S}_{P^k}=\left\{(q(f(x)),q):\,x\in P^k\cap A,\ q\in O^*(m,\ell)\right\}.$$

Hence

$$\left(\mathcal{H}^{\ell}\,\widehat{\times}\,\vartheta^*_{m,\ell}\right)(\mathfrak{S}_{P^k})=0$$

if and only if $\mathcal{I}^{\ell}(f(P^k\cap A))=0$.

Now fix an element of $V^*(m,\ell)$. As we already noticed this element can be identified with an affine subspace $P^{m-\ell}\in E(m,m-\ell)$. Denote corresponding section of the set $\mathfrak{S}$ by $\mathfrak{S}^{P^{m-\ell}}$. It is easy to see that

$$\mathfrak{S}^{P^{m-\ell}}=\left\{(p(x),\ker p):\,x\in f^{-1}(P^{m-\ell})\cap A,\ p\in O^*(n,n-k)\right\}.$$

Hence

$$\left(\mathcal{H}^{n-k}\,\widehat{\times}\,\gamma_{n,k}\right)(\mathfrak{S}^{P^{m-\ell}})=0$$

if and only if $\mathcal{I}^{n-k}(f^{-1}(P^{m-\ell})\cap A)=0$. The proof is complete. □

In order to apply above results to Sobolev mappings we need the following folklore result.

Lemma 21. *Let $f\in W^{1,p}(\mathbb{R}^n,\mathbb{R}^m)$, where $m\geq n$ and $p>n$ or $p=n=1$, be a continuous representative. If $A\subset\mathbb{R}^n$, $\mathcal{H}^n(A)=0$, then $\mathcal{H}^n(f(A))=0$.*

See [8], [63] for the case $p>n$. In the case $p=n=1$ this is a well known property of absolutely continuous functions, [57, Theorem 7.18], since $W^{1,1}(\mathbb{R})$ functions are absolutely continuous [13, 4.9.1]. The quoted proofs are in the case $n=m$ however the same arguments work for when $m>n$.

Now we can prove a theorem about the structure of a generic preimage for a p-quasicontinuous representative of a Sobolev mapping.

THEOREM 22. *If $f \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^m)$ is p-quasicontinuous and $p > k$ or $p = k = 1$, where $1 \le k \le \min\{n, m\}$ is an integer, then for almost all $(m-k)$-dimensional affine subspaces P^{m-k} of $\mathbb{R}^m$, $f^{-1}(P^{m-k}) = W \cup V$, where W is countably $\mathcal{H}^{n-k}$-rectifiable and $\mathcal{I}^{n-k}(V) = 0$.*

REMARKS. 1) If we replace the condition $1 \le k \le \min\{n, m\}$ by $n < k \le m$, then $f^{-1}(P^{m-k}) = \emptyset$ for a.e. $P^{m-k} \in E(m, m-k)$.
2) The only property of the p-quasicontinuity representative that will be employed in the proof is the property from Theorem 16. Thus the theorem holds for any representative that verifies the claim of Theorem 16.

PROOF. By Lemma 10 there is a sequence of closed sets $F_1 \subset F_2 \subset \ldots \subset \mathbb{R}^n$ such that $|\mathbb{R}^n \setminus \bigcup_i F_i| = 0$ and $f|_{F_i}$ are Lipschitz continuous. Denote $A = \mathbb{R}^n \setminus \bigcup_i F_i$ and define

$$W = f^{-1}(P^{m-k}) \cap \bigcup_i F_i, \qquad V = f^{-1}(P^{m-k}) \cap A\,.$$

Now it remains to prove that W is countably $\mathcal{H}^{n-k}$-rectifiable and $\mathcal{I}^{n-k}(V)$=0 for a.e. P^{m-k}. Rectifiability of the set W is an easy consequence of Corollary 6. Indeed, if we compose f with a projection $p \in O^*(m, k)$ onto a k-dimensional subspace we obtain the Lipschitz mappings $p \circ f \colon F_i \to \operatorname{Im} p \approx \mathbb{R}^k$. And hence it follows from Corollary 6 that $(p \circ f)^{-1}(y) \cap \bigcup_i F_i = f^{-1}(p^{-1}(y)) \cap \bigcup_i F_i$ is countably $\mathcal{H}^{n-k}$-rectifiable for all p and a.e. $y \in \operatorname{Im} p$. Observe however that $p^{-1}(y)$ is a generic $(m-k)$-dimensional affine subspace of $\mathbb{R}^m$. The equality $\mathcal{I}^{n-k}(V) = 0$ follows directly from Theorem 20 and from Lemma 21. $\square$

As we have seen in Section 3, co-area formula, Theorem 11, holds for a carefully chosen representative of a Sobolev mapping. Examples mentioned in the section show that it may happen that the Sobolev mapping is continuous, but the representative we choose has to be discontinuous. This means, Theorem 11 does not hold, in general, for quasicontinuous representatives. It would be, however, convenient to know whether there is a counterpart of the co-area formula valid for quasicontinuous representatives. The next theorem provides such a result. This is a generalization of a theorem of Ziemer [63]. Ziemer proved the theorem, by a different method, in the case $m = n - 1$. Then he employed it in a very elegant way to study the inverse of a homeomorphism from the Sobolev space.

THEOREM 23. *Let $f \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^m)$, where $n \ge m$ and $p > m$ or $p = m = 1$, be a p-quasicontinuous representative. If $g \colon \mathbb{R}^n \to \mathbb{R}$ is nonnegative measurable or measurable and such that $g(x)|Jf| \in L^1(\mathbb{R}^n)$, then the following version of the co-area formula holds*

$$\int_{\mathbb{R}^n} g(x)|Jf(x)|\, d\mathcal{H}^n(x) = \int_{\mathbb{R}^m} \left(\int_{f^{-1}(y)} g(x)\, d\mathcal{I}^{n-m}(x) \right) d\mathcal{H}^m(y)\,.$$

REMARK. The second remark to Theorem 22 applies here as well.

PROOF. Let A be a set defined as in Theorem 12. We have

$$\int_{\mathbb{R}^n} g(x)|Jf(x)|\, d\mathcal{H}^n(x) = \int_{\mathbb{R}^m} \left(\int_{f^{-1}(y)\cap(\mathbb{R}^n\setminus A)} g(x)\, d\mathcal{H}^{n-m}(x) \right) d\mathcal{H}^m(y)\,.$$

By Theorem 12 the set $f^{-1}(y) \cap (\mathbb{R}^n \setminus A)$ is countably $\mathcal{H}^{n-m}$-rectifiable for a.e. $y \in \mathbb{R}^m$. Since by Theorem 7, $\mathcal{H}^{n-m}$ coincides with $\mathcal{I}^{n-m}$ on countably $\mathcal{H}^{n-m}$-rectifiable sets we conclude that

$$\int_{\mathbb{R}^n} g(x)|Jf(x)|\, d\mathcal{H}^n(x) = \int_{\mathbb{R}^m} \left(\int_{f^{-1}(y)\cap(\mathbb{R}^n\setminus A)} g(x)\, d\mathcal{I}^{n-m}(x) \right) d\mathcal{H}^m(y)\,.$$

Now it suffices to show that $\mathcal{I}^{n-m}(f^{-1}(y) \cap A) = 0$ for a.e. $y \in \mathbb{R}^m$. This is however a direct consequence of Theorem 17 and Lemma 21. □

The classical co-area formula is formulated for mappings between manifolds. However in Theorem 23 we consider mappings between Euclidean spaces only. One could expect that it is easy to generalize the theorem to the case of mappings between manifolds. It is however not the case. The problem is that it is not possible to define the measure $\mathcal{I}^k$ on manifolds. One of the reasons is that, by results of Mattila [45], sets of integral geometric measure zero are not invariant under diffeomorphisms.

Minor modifications of the proof of Theorem 22 give

THEOREM 24. *Under the assumptions of Theorem 22 for a.e.* $y \in \mathbb{R}^m$, $f^{-1}(y) = W \cup V$, *where W is countably $\mathcal{H}^{n-m}$-rectifiable and* $\mathcal{I}^{n-k}(V) = 0$. *Hence if in addition $k < m$, then* $\mathcal{I}^{n-k}(f^{-1}(y)) = 0$ *for a.e.* $y \in \mathbb{R}^m$.

COROLLARY 25. *If the representative of* $f \in W^{1,1}(\mathbb{R}^n, \mathbb{R}^m)$, $n, m \geq 2$ *is absolutely continuous on almost all 1-dimensional lines in* $\mathbb{R}^n$, *then for a.e.* $y \in \mathbb{R}^m$, $\mathcal{I}^{n-1}(f^{-1}(y)) = 0$ *i.e., almost all lines in* $\mathbb{R}^n$ *do not intersect* $f^{-1}(y)$.

REMARKS. 1) The second remark to Theorem 22 applies to Theorem 24 and hence the only property of a 1-quasicontinuous representative needed in the proof of Corollary 25 is the absolute continuity on almost all lines.
2) Even if $k < m$, the decomposition $f^{-1}(y) = W \cup V$ to a countably $\mathcal{H}^{n-m}$-rectifiable part and a part of vanishing $\mathcal{I}^{n-k}$-measure provides more information than the condition $\mathcal{I}^{n-k}(f^{-1}(y)) = 0$. This is because we know from the proof that the part V is a subset of some fixed set $A \subset \mathbb{R}^n$ with $\mathcal{H}^n(A) = 0$.
3) Results like Corollary 25 have applications that we shortly describe next. Let $f \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^m)$ and let $g_z \in C^\infty(\mathbb{R}^m \setminus \{z\})$. Assume that g_z is discontinuous at z. According to the characterization of the Sobolev space $W^{1,p}$ by the absolute continuity on lines, [13, 4.9.2], in order to show that $g_z \circ f \in W^{1,p}$ we have to verify two conditions: a) the function $g_z \circ f$ is absolutely continuous on almost all lines parallel to coordinate axes; b) the

partial derivatives of $g_z \circ f$ that we can formally compute belong to L^p. Of course $g_z \circ f$ need not be absolutely continuous on lines. However if we can push the singularity z a little bit, then the set $f^{-1}(z)$ is disjoint with almost all lines. This means $g_z \circ f$ is absolutely continuous on almost all lines. Such an argument was employed in [25]. Actually one needs a slightly weaker version of Corollary 25 since we need to worry about the lines parallel to coordinate axes only.

The assumptions in Theorem 24 are exactly the same as assumptions in Theorem 22. However now we take the preimage of a much smaller set, namely the preimage of a point instead of the preimage of an affine subspace. Thus the preimage should be smaller. And indeed, we obtain a smaller dimension for the countably rectifiable part W. Unfortunately estimates for the "bad" part V are in both theorems the same. One could expect that this is because the argument in the proof was not sharp enough. However, surprisingly, the estimate $\mathcal{I}^{n-k}(V) = 0$ from Theorem 24 cannot be improved.

The example we provide below shows sharpness of Theorems 22–24.

Let $f \in W^{1,p}([0,1]^n, \mathbb{R}^m)$ be p-quasicontinuous. Assume that $1 \leq k \leq \min\{n, m-1\}$ and $p > k$ or $p = k = 1$.

Then it follows from Theorem 24 that $\mathcal{I}^{n-k}(f^{-1}(y)) = 0$ for a.e. $y \in \mathbb{R}^m$. The estimate is sharp since the example discussed below shows that if $p = k > 1$, then, in general, one cannot obtain the estimate $\mathcal{I}^{n-k}(f^{-1}(y)) = 0$ (actually one cannot obtain $\mathcal{I}^{n-k}(V) = 0$).

Given $k, m \geq 2$ there exists a continuous mapping $u\colon [0,1]^k \to \mathbb{R}^m$ of the class $W^{1,k}([0,1]^k, \mathbb{R}^m)$ and such that for some 1-dimensional interval $I \subset [0,1]^k$ we have $u(I) = [0,1]^m$. Moreover $Ju = 0$ a.e.

Such example in the case $k = m \geq 2$ was provided by Malý and Martio [42]. However obvious modifications of their construction lead to the general case $k, m \geq 2$. For details see [42, pp. 24–25].

Assume now that $2 \leq k \leq \min\{n, m-1\}$ and define a mapping $f : [0,1]^n \to \mathbb{R}^m$ by the formula $f(x_1, \ldots, x_n) = u(x_1, \ldots, x_k)$. Obviously $f \in W^{1,k}([0,1]^n, \mathbb{R}^m)$ is continuous and $Jf = 0$ a.e. Since images of k-dimensional slices of $[0,1]^n$ fill the cube $[0,1]^m$ we conclude that for any $y \in [0,1]^m$, $f^{-1}(y)$ contain an $(n-k)$-dimensional slice of $[0,1]^n$. Hence $\mathcal{I}^{n-k}(f^{-1}(y)) > 0$ for all $y \in [0,1]^m$.

Moreover since $Jf = 0$ a.e. the example can be used to show sharpness of Theorem 23.

Co-area and harmonic mappings. We close the paper with one application of the co-area formula to harmonic mappings. We will not prove anything new, but we will modify known arguments. We will carefully explain in which sense our arguments are different.

Let B^3 be the unit ball in $\mathbb{R}^3$ and $S^2 = \partial B^3$. Let $W^{1,2}_{\mathrm{id}}(B^3, S^2)$ denote the class of those mappings $f = (f_1, f_2, f_3)\colon B^3 \to \mathbb{R}^3$ such that $f_i \in$

$W^{1,2}(B^3)$, $f_1^2 + f_2^2 + f_3^2 = 1$ a.e. and $f|_{\partial B^3} = \mathrm{id}$ as a trace in the Sobolev space.

Observe that by the Brouwer theorem there are no continuous mappings from the closed ball onto its boundary which are identity on the boundary. However Sobolev mappings need not be continuous and indeed the class $W^{1,2}_{\mathrm{id}}(B^3, S^2)$ is not empty since it contains the radial projection $x/|x| \in W^{1,2}_{\mathrm{id}}(B^3, S^2)$ (cf. Theorem 13).

Next, consider the following energy functional

$$E_2(f) = \int_{B^3} |\nabla f|^2\, dx = \int_{B^3} \sum_{i,j} \left(\frac{\partial f_i}{\partial x_j} \right)^2 dx.$$

defined on the space $W^{1,2}_{\mathrm{id}}(B^3, S^2)$.

A mapping which minimizes the energy E_2 over the class $W^{1,2}_{\mathrm{id}}(B^3, S^2)$ is called *minimizing harmonic map*. This is a particular case of a much more general definition. However, we need this particular case only and we will not provide general definition here.

It seems that the first who considered Sobolev mappings between manifolds and minimizing harmonic mappings was Morrey, [50]. There is a vast literature on the subject. The reader may find a general introduction in books [31], [33]. For harmonic mappings with singularities see the papers [28] and [29].

The following result was proved by Brezis, Coron and Lieb [10].

THEOREM 26. *The radial projection $x/|x|$ minimizes the energy E_2 in the class $W^{1,2}_{\mathrm{id}}(B^3, S^2)$.*

There are several proofs of the theorem [10], [2], [9], [12], [37], see also references in [28].

The proofs given in [2], [9], [12] are based on the co-area formula. The rough idea of these proofs is the following. If $f \in W^{1,2}_{\mathrm{id}}(B^3, S^2)$ is smooth except for a finite number of points where f is discontinuous, then applying the co-area in a clever way one can prove that $E_2(f) \geq E_2(x/|x|)$. Now the theorem follows by showing that smooth mappings with a finite number of discontinuity points are dense in the space $W^{1,2}_{\mathrm{id}}(B^3, S^2)$.

The last approximation result is rather technical especially its generalization needed for Theorem 28.

Our aim is to show that one can apply the co-area formula for Sobolev mappings directly, thus avoiding the approximation procedure. Our proof is not simpler, but in a sense more direct and perhaps the method of our proof can be used for other related problems. Moreover it shows some deep geometric properties of Sobolev mappings which are not seen in the proof with the approximation method.

PROOF OF THEOREM 26. Let $f \in W^{1,2}_{\rm id}(B^3, S^2)$. We want to show that $E_2(f) \geq E_2(x/|x|)$.

If f is a correctly chosen representative then Theorem 11 implies that

$$\int_{B^3} |Jf|(x)\, d\mathcal{H}^3(x) = \int_{S^2} \mathcal{H}^1(f^{-1}(y))\, d\mathcal{H}^2(y).$$

Elementary inequality $2|Jf|(x) \leq |\nabla f(x)|^2$ yields

$$\begin{aligned} E_2(f) = \int_{B^3} |\nabla f(x)|^2 \geq 2 \int_{S^2} \mathcal{H}^1(f^{-1}(y))\, d\mathcal{H}^2(y) \\ = \int_{S^2} \mathcal{H}^1(f^{-1}(y) \cup f^{-1}(-y))\, d\mathcal{H}^2(y). \end{aligned}$$

Observe that in the case $f_0(x) = x/|x|$, there is $2|Jf_0|(x) = |\nabla f_0(x)|^2$ and hence we have equality in the above formula. Moreover $\mathcal{H}^1(f_0^{-1}(y) \cup f_0^{-1}(-y)) = 2$ for every $y \in S^2$ and thus it remains to prove the following lemma which is of independent interest.

LEMMA 27. *If $f \in W^{1,2}_{\rm id}(B^3, S^2)$, then $\mathcal{H}^1(f^{-1}(y) \cup f^{-1}(-y)) \geq 2$ for a.e. $y \in S^2$.*

If f is smooth in $\overline{B^3} \setminus \{a_1, \ldots, a_k\}$, for some finite set $\{a_1, \ldots, a_k\} \subset B^3$ and $f|_{\partial B^n} = {\rm id}$, then the proof of the lemma can be considerably simplified and then Theorem 26 follows from the approximation argument mentioned earlier. Our aim is to provide a proof of Lemma 27 for a general case of a Sobolev mapping thus avoiding the approximation argument in the proof of the theorem. The proof of the lemma is in the spirit of Theorem 20.

PROOF OF LEMMA 27. Let $P\colon S^2 \to \mathbb{RP}^2$, $P(y) = \{y, -y\}$. Fix an arbitrary $y \in S^2$ and $\varepsilon > 0$. It suffices to show that for a.e. w from a neighborhood of y, the length of the projection of the set $f^{-1}(w) \cup f^{-1}(-w)$ onto the segment $\overline{(-y)y}$ is at least $2 - \varepsilon$.

Given $z \in \overline{(-y)y}$ denote by D_z the disc obtained as the intersection of B^3 with the plane perpendicular to $\overline{(-y)y}$ passing through z. By the Fubini theorem (Lemma 15), $f|_{D_z} \in W^{1,2}_{\rm id}(D_z, S^2)$ for a.e. $z \in \overline{(-y)y}$.

Let $I_\varepsilon \subset \overline{(-y)y}$ be the interval of the length $2 - \varepsilon$ centered at the origin. Denote the ends of I_ε by z_1 and z_2. Discs D_{z_1} and D_{z_2} cut two small caps from the sphere S^2. Those caps induce an open set $U \subset \mathbb{RP}^2$.

Any disc D_z cuts the sphere S^2 into two parts. It follows from Theorem 13 (a tricky exercise for the reader) that for a.e. $z \in \overline{(-y)y}$, $f(D_z)$ covers almost all points of at least one of the two parts of S^2 cut by D_z. Hence for a.e. $z \in I_\varepsilon$

$$\mathcal{H}^2(U \setminus (P \circ f)(D_z)) = 0. \tag{15}$$

We can assume that f is a Borel representative and thus the graph $G_{P\circ f} \subset B^3 \times \mathbb{RP}^2$ is a Borel set by Lemma 18.

Let $\widetilde{\pi}: B^3 \to \overline{(-y)y}$ be the orthogonal projection. Define

$$\pi: B^3 \times \mathbb{RP}^2 \to \overline{(-y)y} \times \mathbb{RP}^2$$

by the formula $\pi(x,v)=(\widetilde{\pi}(x),v)$. According to Lemma 19 the set $\pi(G_{P\circ f}) \subset \overline{(-y)y} \times \mathbb{RP}^2$ is $\mathcal{H}^1 \times \mathcal{H}^2$-measurable. Hence the Fubini theorem and (15) imply

$$(\mathcal{H}^1 \times \mathcal{H}^2)(\pi(G_{P\circ f}) \cap (I_\varepsilon \times U)) = \mathcal{H}^1(I_\varepsilon) \cdot \mathcal{H}^2(U)$$

i.e. $\pi(G_{P\circ f}) \cap (I_\varepsilon \times U)$ is a subset of $I_\varepsilon \times U$ of the full measure. Applying the Fubini theorem for the second time we obtain that for a.e. $\{-w,w\} = v \in U$ corresponding slices of the set have the full length i.e. for a.e. $z \in I_\varepsilon$,

$$(P \circ f)^{-1}(v) \cap D_z \neq \emptyset.$$

This means the length of the projection of the set $f^{-1}(w) \cup f^{-1}(-w) = (P \circ f)^{-1}(v)$ on $\overline{(-y)y}$ is at least $2 - \varepsilon$. This completes the proof of the lemma and hence the proof of the theorem. □

One can generalize the above arguments to cover the general case proved in [12].

THEOREM 28. *Let $2 \le p \le n \le m-1$, be integers. Let $f_0 \in W^{1,p}(B^m, S^n)$, be defined by the formula $f_0(y,z) = y/|y|$, where $y \in \mathbb{R}^{n+1}$ and $z \in \mathbb{R}^{m-n-1}$. If $f \in W^{1,p}(B^m, S^n)$ and $f|_{\partial B^m} = f_0|_{\partial B^m}$ then*

$$\int_{B^m} |\nabla f|^p \ge \int_{B^m} |\nabla f_0|^p .$$

The proof given in [12] employs (among other things) the co-area formula for smooth mappings with singularities and then the approximation theorem. Again, it is possible to modify the argument and apply the co-area formula directly to general Sobolev mappings.

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