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SOBOLEV MAPPINGS: LIPSCHITZ DENSITY IS NOT A BI-LIPSCHITZ INVARIANT OF THE TARGET

Piotr Hajłasz

Abstract. We study a question of density of Lipschitz mappings in the Sobolev class of mappings from a closed manifold into a singular space. The main result of the paper, Theorem 1.7, shows that if we change the metric in the target space to a bi-Lipschitz equivalent one, then the property of the density of Lipschitz mappings may be lost. Other main results in the paper are Theorems 1.2, 1.3, 1.6, 1.8.

1 Introduction

Sobolev spaces have been playing a major role in the study of nonlinear partial differential equations and calculus of variations for more than seven decades now. Variational approach to mappings between manifolds leads in a natural way to the notion of Sobolev mappings between manifolds $W^{1,p}(M, N)$. Here the symbol $W^{1,p}$ stands for the Sobolev space of L^p integrable mappings with L^p integrable first-order derivatives. This class of mappings was considered for the first time as early as in 1948 by Morrey [Mo] in his celebrated paper on regularity of minimizing harmonic mappings from two-dimensional surfaces into manifolds. While it is a well-known theorem of Meyers and Serrin that smooth functions are dense in the Sobolev space $W^{1,p}(M)$, where M is a Riemannian manifold, a similar question, raised by Eells and Lemaire, [EL], about density of smooth mappings in the class of Sobolev mappings between manifolds, is no longer obvious. Actually Schoen and Uhlenbeck [SU1,2] were the first to observe that in general the answer to this question is in the negative. Namely they

demonstrated that the radial projection mapping $x/|x| \in W^{1,p}(B^n, S^{n-1})$, $1 \leq p < n$, cannot be approximated by $C^\infty(B^n, S^{n-1})$ mappings when $n-1 \leq p < n$. In the same paper they proved that $C^\infty(M, N)$ is a dense subset of $W^{1,p}(M, N)$ when $p \geq \dim M$ and M, N are compact smooth manifolds, $\partial N = \emptyset$. Later, mainly due to the work of Bethuel, [B2], it turned out that the condition for the density of smooth mappings can be formulated in terms of algebraic topology describing the topological structure of the manifolds M and N. Finally a necessary and sufficient condition for the density of smooth mappings in the space of Sobolev mappings $W^{1,p}(M, N)$ was discovered by Hang and Lin [HaL1,2]. This result corrects an earlier statement of Bethuel [B2] about a necessary and sufficient condition for the density. Other papers on this topic include [B1], [BCDH], [BZ], [BrL], [BrLMN], [BrN1,2] [CorG], [D], [Es], [GISS], [H1,2], [HIMO], [HaL1,2], [Hel], [I], [PR], [W1,2].

The theory of Sobolev mappings between manifolds has been extended to the case of Sobolev mappings with values into metric spaces. The first papers on this subject include the work of Ambrosio, [Am], on limits of classical variational problems and the work of Gromov and Schoen, [GrS], on Sobolev mappings into the Bruhat–Tits buildings, with applications to rigidity questions for discrete groups. Later the theory of Sobolev mappings with values into metric spaces was developed in a more elaborated form by Korevaar and Schoen [KS] in their approach to the theory of harmonic mappings into Alexandrov spaces of non-positive curvature. Other papers on Sobolev mappings from a manifold into a metric space include [CL], [EF], [J1,2,3], [JZ], [R], [Se]. Finally, analysis on metric spaces, the theory of Carnot–Carathéodory spaces and the theory of quasiconformal mappings between metric spaces lead to the theory of Sobolev mappings between metric spaces, [HeK], [HeKST], [Tr].

In the case of Sobolev mappings into metric spaces Heinonen, Koskela, Shanmugalingam and Tyson, [HeKST, Rem. 6.9], ask for the conditions that would guarantee the density of Lipschitz mappings. Our paper is devoted to study this problem. From the point of view of geometric analysis one is willing to identify spaces that are bi-Lipschitz equivalent. The main result of the paper, Theorem 1.7, shows that if we change the metric in the target space to a bi-Lipschitz equivalent metric, than the property of the density of Lipschitz mappings may be lost. That shows a major difficulty in attempts to develop the theory in the case of metric space valued maps. Other main results in the paper are Theorems 1.2, 1.3, 1.6, 1.8. All the main results but Theorem 1.3 are counterexamples.

Let M and N be two smooth compact manifolds, N without boundary. Assume that N is smoothly embedded into the Euclidean space $\mathbb{R}^\nu$. Then for $1 \leq p < \infty$ we define the class of Sobolev mappings between manifolds M and N as

$$W^{1,p}(M, N) = \left\{ u \in W^{1,p}(M, \mathbb{R}^\nu) : u(x) \in N \text{ a.e.} \right\}. \qquad (1.1)$$

The space $W^{1,p}(M, N)$ is equipped with the metric of the Sobolev norm $W^{1,p}$, i.e. $\rho(u, v) = \|u - v\|_{1,p}$ for $u, v \in W^{1,p}(M, N)$.

By the theorem of Meyers and Serrin every Sobolev mapping $u \in W^{1,p}(M, N)$ can be approximated by smooth mappings $C^\infty(M, \mathbb{R}^\nu)$, but it is not always possible to choose the approximating sequence to have values in N.

Clearly the space $W^{1,p}(M, N)$ depends on the embedding of N into the Euclidean space. However, if $i_1 : N \to \mathbb{R}^{\nu_1}$ and $i_2 : N \to \mathbb{R}^{\nu_2}$ are two different smooth embeddings then the mapping

$$\Phi : W^{1,p}\big(M, i_1(N)\big) \to W^{1,p}\big(M, i_2(N)\big), \quad \Phi(u) = i_2 \circ i_1^{-1} \circ u,$$

is a homeomorphism of the spaces $W^{1,p}(M, i_1(N))$ and $W^{1,p}(M, i_2(N))$. In particular the answer to the question about density of smooth mappings in $W^{1,p}(M, N)$ does not depend on the smooth embedding of N.

There are various possibilities to define metric space valued Sobolev mappings, see e.g. [HeKST], [KS], [R]. As was proved by Heinonen, Koskela, Shanmugalingam and Tyson [HeKST], in a reasonable generality different approaches are equivalent. The results in [HeKST] concern Sobolev mappings between metric spaces, but in our paper we will consider the case of mappings from a closed manifold to metric spaces $W^{1,p}(M, X)$ only. One of the definitions of such mappings mimics the definition (1.1). First one defines Sobolev mappings from M into Banach spaces. Then one considers an isometric embedding of X into a Banach space V (any metric space X admits an isometric embedding into $\ell^\infty(X)$) and defines $W^{1,p}(M, X)$ as the set of all mappings in $W^{1,p}(M, V)$ such that values of the mapping belong to $X \subset V$. For a precise definition and proof of the equivalence with other approaches, see [HeKST]. The space $W^{1,p}(M, X)$ inherits the metric from the Banach structure of the space $W^{1,p}(M, V)$. Then under reasonable assumptions every Sobolev mapping $u \in W^{1,p}(M, X)$ can be approximated by Lipschitz mappings $\mathrm{Lip}(M, V)$, [HeKST, Th. 6.7 and Rem. 6.6]. The authors ask: *It is an interesting problem to determine when one can choose the Lipschitz approximation to have values in the target X. [...] For instance, one can ask to what extent Bethuel's results have analogs for general spaces.*

In the case of Sobolev mappings between Riemannian manifolds $W^{1,p}(M,N)$ it is often very important to have a Riemannian metric on N fixed as changing the metric could result in turning a given variational problem into a different one. In the case of mappings into metric spaces $W^{1,p}(M,X)$ it is often that there is no canonical way to define the metric in X and there are several possibilities which are in some sense equally good. One of the requirements is, however, that the metrics be bi-Lipschitz equivalent. Actually, from the point of view of analysis on metric spaces, spaces which are bi-Lipschitz homeomorphic are equivalent. A fundamental example in this context is the Heisenberg group H^m. There are two canonical metrics on the Heisenberg group. The Carnot–Carathéodory metric d_C and the gauge metric d. They are bi-Lipschitz equivalent, i.e. there is a constant $C > 0$ such that for all $p, q \in H^m$

$$C^{-1}d_C(p,q) \leq d(p,q) \leq Cd_C(p,q)\,. \qquad (1.2)$$

Capogna and Lin, [CL] study Sobolev mappings from a domain Ω in the Euclidean space into the Heisenberg group. They write: *because of* (1.2) *both the definitions of the Lebesgue space* $L^p(\Omega, H^m)$, *and the Sobolev space* $W^{1,p}(\Omega, H^m)$ *do not depend on the choice between the Carnot–Carathéodory distance and the one coming from the gauge norm. To simplify the computations we will deal with the metric* $d(p,q)$. They consider the space of Sobolev mappings with values into the Heisenberg group as a set and they *do not* equip this space with any metric. They are perfectly right when they say that the set does not depend on the choice of the metric in the target as long as the metrics that are considered are bi-Lipschitz equivalent, like those in (1.2). However, if one equips the space of the Sobolev mappings $W^{1,p}(M,X)$ from a manifold into a metric space with a metric that comes from the isometric embedding of X into a Banach space (as was done in [HeKST]) it is not clear how changing the metric in X to a bi-Lipschitz equivalent one would affect the metric structure of the space $W^{1,p}(M,X)$. Our paper is devoted to study this problem.

Easy examples show that different isometric embeddings of X into Banach spaces may give different metrics in $W^{1,p}(M,X)$ ($\mathbb{R}^2$ equipped with the norm $\|(x_1,x_2)\|_1 = |x_1| + |x_2|$ is a Banach space and $i : \mathbb{R} \to (\mathbb{R}^2, \|\cdot\|_1)$, $i(x) = (x,0)$ if $x \geq 0$, $i(x) = (0,x)$ if $x < 0$ is an isometric embedding. It is easy to see that this embedding equips the space $W^{1,p}([0,1], \mathbb{R})$ with a metric which is different than the standard one), and it is not even clear what the answer to the following question is.

QUESTION 1. *Does the answer to the question about the density of Lipschitz mappings in $W^{1,p}(M, X)$ depend on the isometric embedding of X into a Banach space?*

It is easy to see that in the case of isometric embeddings of X into Hilbert spaces the answer to Question 1 is no. Since we will be concerned with metric spaces X that admit isometric embeddings to Euclidean spaces, we will be concerned with Question 2 (formulated below) rather than with Question 1 which is obvious in this case. As we assume X to have an isometric embedding to the Euclidean space, we can actually identify it with a subset of $\mathbb{R}^\nu$ and define

$$W^{1,p}(M, X) = \left\{ u \in W^{1,p}(M, \mathbb{R}^\nu) : u(x) \in X \text{ a.e.} \right\}. \qquad (1.3)$$

The space $W^{1,p}(M, X)$ is equipped with the metric of the Sobolev norm $W^{1,p}$.

Two closed manifolds can be identified if they are diffeomorphic. In the case of metric spaces a natural counterpart of a diffeomorphism is a bi-Lipschitz homeomorphism. This is the class of the *most smooth* homeomorphisms that can be considered for arbitrary metric spaces.

Many of the metric spaces X admit a bi-Lipschitz embedding into the Euclidean space $\mathbb{R}^\nu$, rather than an isometric one. In such a case one could try to use this embedding to define the space $W^{1,p}(M, X)$ by (1.3). How does the space $W^{1,p}(M, X)$ depend on the bi-Lipschitz embedding of X?

QUESTION 2. *Assume that X and Y are compact subsets of $\mathbb{R}^\nu$ that are bi-Lipschitz homeomorphic. Assume that M is a smooth closed n-dimensional manifold and that Lipschitz mappings $\mathrm{Lip}(M, X)$ are dense in $W^{1,p}(M, X)$ for some $1 \leq p < \infty$. Are the Lipschitz mappings $\mathrm{Lip}(M, Y)$ dense in $W^{1,p}(M, Y)$?*

We will examine the case of Sobolev mappings $W^{1,n}(M, X)$ from an n-dimensional closed manifold to a compact set $X \subset \mathbb{R}^\nu$. We will prove that if X is a Lipschitz neighborhood retract then Lipschitz mappings $\mathrm{Lip}(M, X)$ are dense in $W^{1,n}(M, X)$, see Theorem 1.3. Now if Y is bi-Lipschitz homeomorphic to X, then $\mathrm{Lip}(M, Y)$ is dense in $W^{1,n}(M, Y)$ because the class of Lipschitz neighborhood retracts is closed under bi-Lipschitz homeomorphisms (Lemma 1.4) and hence Y is a Lipschitz neighborhood retract as well. On the other hand we will show in Theorem 1.7 that there are two bi-Lipschitz homeomorphic compact subsets X and Y of $\mathbb{R}^{n+2}$ such that for any n-dimensional closed manifold M Lipschitz mappings $\mathrm{Lip}(M, X)$ are dense in $W^{1,n}(M, X)$, but Lipschitz mappings $\mathrm{Lip}(M, Y)$ are not dense in $W^{1,n}(M, Y)$. Obviously X cannot be a Lipschitz neighborhood retract.

Other new results are Theorems 1.2, 1.6 and 1.8. In all theorems but Theorem 1.2 we deal with the Sobolev space $W^{1,n}(M, X)$, where $n = \dim M$. One can ask similar questions for spaces $W^{1,p}(M, X)$ with $p \neq n$. I believe that our counterexamples have counterparts for other exponents as well. It would be, however, more interesting if one could prove positive results for some $p \neq n$.

Statement of main results. Consider the class of Sobolev mappings $W^{1,p}(\Omega, \mathbb{R}^\nu)$ and a bounded Lipschitz function $\varphi \in \operatorname{Lip}(\mathbb{R}^\nu) \cap L^\infty(\mathbb{R}^\nu)$. Here $\Omega \subset \mathbb{R}^n$ is an open set with finite measure, but we could also consider a closed manifold M instead of Ω. It is well known and easy to prove that the nonlinear operator

$$\Phi : W^{1,p}(\Omega, \mathbb{R}^\nu) \to W^{1,p}(\Omega) \,, \quad \Phi(u) = \varphi \circ u \,, \tag{1.4}$$

is bounded, i.e. it maps bounded sets to bounded sets. More precisely the inequality $\|\Phi(u)\|_{1,p} \leq C(1 + \|u\|_{1,p})$ is satisfied. The same holds in the case in which Ω is replaced by M.

QUESTION 3. *Is the operator Φ continuous?*

This is a crucial problem for us. Indeed, continuity would immediately imply that $\Phi : W^{1,p}(M, X) \to W^{1,p}(M, Y)$ is a homeomorphism whenever $\varphi : X \to Y$ is a bi-Lipschitz homeomorphism and $\Phi(u) = \varphi \circ u$. Marcus and Mizel [MarM] proved continuity of Φ given by (1.4) in the case $\nu = 1$. We are, however, interested mainly in the case in which $\nu > 1$. Let us start with a positive result that will be needed later.

LEMMA 1.1.   *Let M be a smooth closed manifold and φ a bounded Lipschitz function on $\mathbb{R}^\nu$ that is of the class C^1 away from one point x_0. Then the operator $\Phi : W^{1,p}(M, \mathbb{R}^\nu) \to W^{1,p}(M)$ is continuous for all $1 \leq p < \infty$.*

Unfortunately, in general, the answer to Question 3 is in the negative when $\nu > 1$. This is shown by the following result. I believe the result is known, but I was unable to locate it in the literature. In any case the proof is very easy and the phenomenon is so important for our paper that we provide a short proof.

Theorem 1.2.   *There is a Lipschitz function $\varphi \in \operatorname{Lip}(\mathbb{R}^2)$ with compact support such that the bounded operator $\Phi : W^{1,p}([0,1], \mathbb{R}^2) \to W^{1,p}([0,1])$ defined as composition $\Phi(u) = \varphi \circ u$ is not continuous for any $1 \leq p < \infty$.*

Similar claim follows from Theorem 1.7 below. Indeed, the mapping $\Phi : W^{1,n}(M, X) \to W^{1,n}(M, Y)$ induced by the bi-Lipschitz homeomorphism

$\varphi : X \to Y$ from Theorem 1.7 cannot be continuous. However, the proof of Theorem 1.2 is much simpler than that of Theorem 1.7.

Even if we know that $\Phi : W^{1,p}(M,X) \to W^{1,p}(M,Y)$ is not continuous for a bi-Lipschitz homeomorphism $\varphi : X \to Y$ it does not immediately imply that the spaces $W^{1,p}(M,X)$ and $W^{1,p}(M,Y)$ must have different properties. In fact it easily follows that if $\varphi : X \to Y$ is a bi-Lipschitz homeomorphism then the corresponding mapping Φ gives a one-to-one correspondence between the spaces $W^{1,p}(M,X)$ and $W^{1,p}(M,Y)$ and one-to-one correspondence between the spaces $\mathrm{Lip}(M,X)$ and $\mathrm{Lip}(M,Y)$. Hence one could expect a positive answer to Question 2.

The following result answers Question 2 in the positive in the case in which $p = \dim M$ and X is a Lipschitz neighborhood retract.

Theorem 1.3. *Let M be a smooth closed n-dimensional manifold and $X \subset \mathbb{R}^\nu$ a compact Lipschitz neighborhood retract. Then Lipschitz mappings $\mathrm{Lip}(M,X)$ are dense in $W^{1,n}(M,X)$.*

Recall that $X \subset \mathbb{R}^\nu$ is a *Lipschitz neighborhood retract* if there is an open set $X \subset U \subset \mathbb{R}^\nu$ and a Lipschitz mapping (called *Lipschitz retraction*) $p : U \to X$ such that $p(x) = x$ for all $x \in X$. The following result is well known.

LEMMA 1.4. *If $X \subset \mathbb{R}^\nu$ is a Lipschitz neighborhood retract and $Y \subset \mathbb{R}^\nu$ is bi-Lipschiz homeomorphic to X, then Y is a Lipschitz neighborhood retract as well.*

Proof. If $p : U \to X$ is a Lipschitz retraction, $f : Y \to X$ a bi-Lipschitz homeomorphism and $F : \mathbb{R}^\nu \to \mathbb{R}^\nu$ is a Lipschitz extension (McShane) of f, then $f^{-1} \circ p \circ F : F^{-1}(U) \to Y$ is a Lipschitz retraction. $\square$

We say that $X \subset \mathbb{R}^\nu$ is an m-dimensional *Lipschitz submanifold* if every point $x \in X$ has a neighborhood in X which is bi-Lipschitz homeomorphic to $B^m(0,1)$.

Theorem 1.3 applies to Lipschitz submanifolds because of the following result.

LEMMA 1.5 [LV, Th. 5.13]. *Lipschitz submanifolds of $\mathbb{R}^\nu$ are Lipschitz neighborhood retracts.*

Proof. We sketch the proof which is based on a localization of the argument of Almgren [Al] (cf. [He, Prop. 2.13]). Let $X \subset \mathbb{R}^\nu$ be a Lipschitz submanifold. We decompose $\mathbb{R}^\nu \setminus X$ into Whitney cubes and define the Lipschitz retraction first on the vertices of the cubes as the projection to a closest point in X. We take only those Whitney cubes which are in a

close neighborhood of X, as we require that all the vertices of each cube are mapped into a domain in X which is bi-Lipschitz homeomorphic to a ball. Next we extend the Lipschitz projection to higher-dimensional edges of the cubes by induction. $\square$

Schoen and Uhlenbeck [SU1,2] proved Theorem 1.3 in the case in which X is a compact smooth manifold. They proved that the convolution approximation of $u \in W^{1,n}(M, X)$ takes values into a tubular neighborhood of X in $\mathbb{R}^\nu$ and hence one can obtain the desired approximation by composing the convolution approximation with the smooth nearest point projection. This argument cannot be directly applied in the case of Theorem 1.3, even when X is a Lipschitz manifold, because now the retraction of a neighborhood of X onto X is Lipschitz only and according to Theorem 1.2 the composition with a Lipschitz function need not be continuous in the Sobolev norm. Proof of Theorem 1.3 will require a much more delicate approximation method than the one coming from convolution.

A nice and nontrivial class of examples of Lipschitz manifolds was provided by Toro, [T] (see also [Mü]). She proved that surfaces in $\mathbb{R}^3$ that are locally graphs of $W^{2,2}$ functions are Lipschitz submanifolds. If we replace the class of $W^{2,2}$ surfaces by continuous $W^{1,2}$ surfaces, then we can construct a counterexample to the density. Such a counterexample is provided by the case $n = 2$ of the following result.

Theorem 1.6.   *There is a subset $N \subset \mathbb{R}^{n+1}$ homeomorphic to S^n and $x_0 \in N$ such that $N \setminus \{x_0\}$ is a smooth submanifold of $\mathbb{R}^{n+1}$ and N is a graph of a continuous $W^{1,n}$ function near x_0. That means N is a regular n-dimensional submanifold of $\mathbb{R}^{n+1}$ with one singular point. This construction can be done in such a way that, for every closed n-dimensional manifold M, Lipschitz mappings $\mathrm{Lip}(M, N)$ are not dense in $W^{1,n}(M, N)$.*

One should compare Theorem 1.6 with Theorem 1.3. As we will see Theorem 1.6 is a consequence of the proof of Theorem 1.8. Namely we will see that if $M = S^n \subset \mathbb{R}^{n+1}$, then the singular manifold $N = \widetilde{M}$ constructed in the proof of Theorem 1.8 has properties required in Theorem 1.6.

If $X \subset \mathbb{R}^\nu$ is a subset, then by smooth mappings $C^\infty(M, X)$ we will mean the class of all mappings $u \in C^\infty(M, \mathbb{R}^\nu)$ such that $u(M) \subset X$.

Theorem 1.7.   *Fix an integer $n \geq 2$. There is a compact and connected set $X \subset \mathbb{R}^{n+2}$ and a global bi-Lipschitz homeomorphism $\Phi : \mathbb{R}^{n+2} \to \mathbb{R}^{n+2}$ with the property that for any closed n-dimensional manifold M smooth mappings $C^\infty(M, X)$ are dense in $W^{1,n}(M, X)$, but Lipschitz mappings $\mathrm{Lip}(M, Y)$ are not dense in $W^{1,n}(M, Y)$, where $Y = \Phi(X)$.*

According to Theorem 1.3 and Lemma 1.4, X cannot be a Lipschitz neighborhood retract. In particular there is no such example as in Theorem 1.7 in the case in which X is a closed manifold. However, if we allow a global homeomorphism Φ of the Euclidean space to be locally bi-Lipschitz everywhere but at one point, then one can find counterexamples even in the class of closed manifolds.

Theorem 1.8. *Let $M \subset \mathbb{R}^\nu$ be a closed n-dimensional manifold. Then there is a homeomorphism $\Phi \in C^\infty(\mathbb{R}^\nu, \mathbb{R}^\nu)$ which is a diffeomorphism in $\mathbb{R}^\nu \setminus \{0\}$, which is identity outside a sufficiently large ball and has the property that Lipschitz mappings $\mathrm{Lip}(M, \widetilde{M})$ are not dense in $W^{1,n}(M, \widetilde{M})$, where $\widetilde{M} = \Phi^{-1}(M)$.*

The example constructed in the proof of Theorem 1.8 will be employed in the proof of Theorem 1.7.

NOTATION.   Notation employed in the paper is rather standard. The k-dimensional Hausdorff measure will be denoted by $\mathcal{H}^k$ and the average value by

$$u_E = \fint_E u \, d\mu = \mu(E)^{-1} \int_E u \, d\mu \, .$$

Symbol B will be used to denote a ball. By σB, where $\sigma \geq 1$, we will denote a ball concentric with B and with the radius σ times that of B. We say that M is a closed manifold if it is smooth, compact and without boundary. The L^p norm of a function u will be denoted by $\|u\|_p$. The Sobolev space $W^{1,p}(M)$ consists of all $u \in L^p(M)$ such that $\nabla u \in L^p(M)$. The space is equipped with the norm $\|u\|_{1,p} = \|u\|_p + \|\nabla u\|_p$. By C we will denote a general constant which can change its value even in the same string of estimates. We will write $f : X \twoheadrightarrow Y$ to designate any function $f : X \to Y$ such that $f(X) = Y$. Coordinates in $\mathbb{R}^n$ will often be denoted by (x', x^n), where $x' \in \mathbb{R}^{n-1}$.

2   Proof of Lemma 1.1

For $f_n \to f$ in $W^{1,p}(M, \mathbb{R}^\nu)$ and a.e. we will show that $\Phi \circ f_n \to \Phi \circ f$ in $W^{1,p}$. We split M into three pairwise disjoint sets $A = f^{-1}(x_0)$,

$$B_k = \left(\bigcup_{n=k}^\infty f_n^{-1}(x_0) \right) \setminus f^{-1}(x_0) \quad C_k = M \setminus \left(f^{-1}(x_0) \cup \bigcup_{n=k}^\infty f_n^{-1}(x_0) \right).$$

Note that the measure of B_k goes to 0 as $k \to \infty$ and hence for every $\varepsilon > 0$ there is k such that

$$\int_{B_k} \left|D(\Phi \circ f)\right|^p + \sup_{n \geq k} \int_{B_k} \left|D(\Phi \circ f_n)\right|^p < \varepsilon \,,$$

because the family $|D(\Phi \circ f_n)|^p \leq C|Df_n|^p$ is equiintegrable as $Df_n \to Df$ in L^p. In particular

$$\int_{B_k} \left|D(\Phi \circ f_n) - D(\Phi \circ f)\right|^p < C\varepsilon$$

for all $n \geq k$. The chain rule applies to $D(\Phi \circ f_n)$ and to $D(\Phi \circ f)$ on C_k when $n \geq k$ and hence $D(\Phi \circ f_n) \to D(\Phi \circ f)$ in L^p on C_k. Since f is constant on A, $Df = 0$ a.e. on A and $D(\Phi \circ f) = 0$ a.e. on A. Hence $Df_n \to 0$ in L^p on A. Thus $|D(\Phi \circ f_n)| \leq C|Df_n| \to 0$ in L^p on A. This, however, implies that $D(\Phi \circ f_n) \to 0 = D(\Phi \circ f)$ in L^p on A. All the facts put together easily imply that $D(\Phi \circ f_n) \to D(\Phi \circ f)$ in $L^p(M)$. The proof is complete.   □

3 Proof of Theorem 1.2

We will construct a bounded Lipschitz function $\varphi : \mathbb{R}^2 \to \mathbb{R}$ with compact support $\operatorname{supp} \varphi = [0,2] \times [0,1]$ such that the bounded operator

$$\Phi : W^{1,p}\big([0,1], \mathbb{R}^2\big) \to W^{1,p}\big([0,1], \mathbb{R}\big) \,, \quad \Phi(u) = \varphi \circ u \,,$$

is not continuous for any choice of p in the interval $1 \leq p < \infty$.

To this end we first define a compact set $K \subset [0,2] \times [0,1]$ as union of an infinite family of segments defined below.

Let $a_i = 2 - 2^{-i+1}$ for $i = 0, 1, 2, \ldots$ We define

$$I_i = \{a_i\} \times [0,1] \quad \text{and} \quad I_\infty = \{2\} \times [0,1] \,.$$

This is a family of vertical segments. Next for each $i = 0, 1, 2, \ldots$ we define a finite family of horizontal segments $\{J_{i,k}\}_{k=0}^{2^i}$ that join I_i with I_{i+1} Namely we set

$$J_{i,k} = [a_i, a_{i+1}] \times \{k/2^i\} \,.$$

Now we define the set K as follows

$$K = I_\infty \cup \left(\bigcup_{i=0}^{\infty} I_i \right) \cup \left(\bigcup_{i=0}^{\infty} \bigcup_{k=0}^{2^i} J_{i,k} \right).$$

The set K is the union of boundaries an infinite family of squares. Now we define the function φ by the formula

$$\varphi(x) = \begin{cases} \operatorname{dist}(x, K) & \text{if } x \in [0,2] \times [0,1] \,, \\ 0 & \text{if } x \in \mathbb{R}^2 \setminus [0,2] \times [0,1] \,. \end{cases}$$

The graph of φ over each of the squares has the shape of a pyramid, so the graph of φ is an infinite collection of pyramids. The Lipschitz constant of φ is 1. Fix $1 \leq p < \infty$. We will show that the operator Φ is not continuous in $W^{1,p}$. Let $u_i : [0,1] \to \mathbb{R}^2$, $u_i(x) = ((a_i + a_{i+1})/2, x)$. The image of u_i is the vertical segment that is in the middle between I_i and I_{i+1}. Obviously $u_i \to u$ in $W^{1,p}$ as $i \to \infty$, where $u : [0,1] \to \mathbb{R}^2$, $u(x) = (2,x)$. However, $\varphi \circ u_i$ does not converge to $\varphi \circ u \equiv 0$ in $W^{1,p}$ because $(\varphi \circ u_i)' = \pm 1$ a.e. The proof is complete. $\qquad\square$

4   Proof of Theorem 1.3.

The approximation technique employed here is similar to that used in [BoHS], [HK]. Fix a Riemannian tensor on the manifold M. Denote the induced distance and the gradient by $d(x,y)$ and ∇u respectively. $\mathcal{M}g(x) = \sup_{r>0} \fint_{B(x,r)} |g(z)|\, dz$ will be used to denote the Hardy–Littlewood maximal function.

There exists r_0 depending on the Riemannian structure of M such that the following inequalities:

$$\fint_B |u - u_B| \leq Cr \fint_B |\nabla u|\,,$$

and

$$\left| u(x) - u_B \right| \leq Cr\mathcal{M}|\nabla u|(x) \quad \text{for a.e. } x \in B\,, \tag{4.1}$$

hold true whenever the radius r of B is less than r_0 and $u \in W^{1,p}(B)$, $1 \leq p < \infty$. The first inequality is known as the Poincaré inequality. The second inequality follows, e.g. from Lemma 1.50 and Theorem 1.32(i) with $\alpha = 1$ in [MaZ].

As a corollary one can easily prove the following version of a well-known pointwise inequality, see e.g. [AF], [H3], [HKo, Ths. 3.2, 3.3] and references therein.

LEMMA 4.1.   *Let M be a closed manifold and $u \in W^{1,p}(M)$, where $1 \leq p < \infty$. Then*

$$\left| u(x) - u(y) \right| \leq Cd(x,y)\big(\mathcal{M}|\nabla u|(x) + \mathcal{M}|\nabla u|(y)\big) \quad \text{for a.e. } x, y \in M\,.$$

Let $u \in W^{1,n}(M, X)$. Our aim is to construct a family of Lipschitz mappings $u_t \in \mathrm{Lip}(M, \mathbb{R}^\nu)$ such that

(A) The Lipschitz constant of u_t is bounded by Ct;
(B) $t^n \mathcal{H}^n(\{u \neq u_t\}) \to 0$ as $t \to \infty$;
(C) $\sup_{x \in M} \mathrm{dist}\,(u_t(x), X) \to 0$ as $t \to \infty$.

Before we construct a family $\{u_t\}$ we will show how to use the properties (A), (B) and (C) to complete the proof of the theorem.

Let $p : U \to X$ be a Lipschitz neighborhood retraction. By (C), $u_t(M) \subset U$ for all sufficiently large t and hence $p \circ u_t : M \to X$ is a family of Lipschitz mappings well defined for large t. We will show now that $p \circ u_t \to u$ in $W^{1,n}$ when $t \to \infty$. Indeed, $p \circ u_t \to u$ a.e. by (B) and hence $p \circ u_t \to u$ in L^n (mappings are bounded). Finally (A) and (B) imply

$$\int_M \left| D(p \circ u_t) - Du \right|^n = \int_{\{p \circ u_t \neq u\}} \left| D(p \circ u_t) - Du \right|^n$$

$$\leq Ct^n \mathcal{H}^n\left(\{p \circ u_t \neq u\}\right) + C \int_{\{p \circ u_t \neq u\}} |Du|^n \to 0, \quad \text{as } t \to \infty.$$

Now we proceed to the construction of the family $\{u_t\}$. Let $E_t = \{x \in M : \mathcal{M}|\nabla u|(x) \leq t\}$. The set $M \setminus E_t$ is open. There is a Whitney decomposition of $M \setminus E_t$ into balls and subordinated Lipschitz partition of unity. This is to say there is a constant $C \geq 1$ depending on the Riemannian structure of M only and a sequence $\{x_i\}_{i \in I}$ of points in $M \setminus E_t$ such that with $r_i = \text{dist}\,(x_i, E_t)/10$ we have

(a) $\bigcup_{i \in I} B(x_i, r_i) = M \setminus E_t$;
(b) $B(x_i, 5r_i) \subset M \setminus E_t$ for all $i \in I$;
(c) For every $i \in I$ and all $x \in B(x_i, 5r_i)$ we have $5r_i \leq \text{dist}\,(x, E_t) \leq 15r_i$;
(d) No point of $M \setminus E_t$ belongs to more than C balls $\{B(x_i, 5r_i)\}_{i \in I}$;
(e) There is a family of Lipschitz continuous functions $\{\varphi_i\}_{i \in I}$ such that $\text{supp}\,\varphi_i \subset B(x_i, 2r_i)$, $0 \leq \varphi_i \leq 1$, $\sum_{i \in I} \varphi_i = 1$ and the Lipschitz constant of φ_i is bounded by Cr_i^{-1}.

The proof of the existence of a Whitney decomposition of a domain in $\mathbb{R}^n$ into cubes with pairwise disjoint interiors can be found in [St]. Standard modification of the method gives the above decomposition into balls. Actually the construction is so general that it can be carried out on arbitrary doubling metric spaces, see [CoW], [MS, Lem. 2.9]. In what follows we will use the notation $B_i = B(x_i, r_i)$ and $2B_i = B(x_i, 2r_i)$. We define

$$u_t = \begin{cases} u(x) & \text{for } x \in E_t, \\ \sum_{i \in I} \varphi_i(x) u_{B_i} & \text{for } x \in M \setminus E_t. \end{cases}$$

Note that by Lemma 4.1 the function $u_t|_{E_t} = u|_{E_t}$ is Lipschitz continuous with the Lipschitz constant Ct.

By the Hardy–Littlewood theorem, [St], $\mathcal{M}|\nabla u| \in L^n$, and hence $t^n \mathcal{H}^n(M \setminus E_t) \to 0$ as $t \to \infty$. This immediately implies property (B) of u_t.

Now we prove property (A). Given $x \in M \setminus E_t$, let $\overline{x} \in E_t$ be such that $d(x, \overline{x}) = \operatorname{dist}(x, E_t)$. Note that whenever B_i is such that $x \in 2B_i$, then $B_i \subset B(\overline{x}, 2d(x, \overline{x})) := \widetilde{B}(\overline{x})$. Observe that radius of $\widetilde{B}(\overline{x})$ is comparable to r_i whenever $x \in 2B_i$. Choose t large enough to make the measure $\mathcal{H}^n(M \setminus E_t)$ so small that radii of all of the balls $\widetilde{B}(\overline{x})$ are less than r_0. For $x \in M \setminus E_t$ we have

$$\left| u_t(x) - u_t(\overline{x}) \right| = \left| \sum_{i \in I} \varphi_i(x) \big(u(\overline{x}) - u_{B_i} \big) \right| \leq \sum_{i \in I_x} \left| u(\overline{x}) - u_{B_i} \right|, \qquad (4.2)$$

where $I_x = \{ i \in I : x \in \operatorname{supp} \varphi_i \}$ (observe that $x \in 2B_i$ for $i \in I_x$). We will prove now that

$$\left| u(\overline{x}) - u_{B_i} \right| \leq C t r_i \quad \text{for all } i \in I_x. \qquad (4.3)$$

This, together with inequality (4.2), the observation that $r_i \leq d(x, \overline{x})$ for $i \in I_x$ and the fact that the number of elements in I_x is bounded by a constant that does not depend on x, will give

$$\left| u_t(x) - u_t(\overline{x}) \right| \leq C t d(x, \overline{x}). \qquad (4.4)$$

To prove (4.3) note that $|u(\overline{x}) - u_{B_i}| \leq |u(\overline{x}) - u_{\widetilde{B}(\overline{x})}| + |u_{\widetilde{B}(\overline{x})} - u_{B_i}|$. Applying inequality (4.1) yields

$$\left| u(\overline{x}) - u_{\widetilde{B}(\overline{x})} \right| \leq C r_i \mathcal{M} |\nabla u|(\overline{x}) \leq C t r_i$$

because $\overline{x} \in E_t = \{ \mathcal{M} |\nabla u| \leq t \}$. To estimate the second term we apply the Poincaré inequality

$$\left| u_{\widetilde{B}(\overline{x})} - u_{B_i} \right| \leq \fint_{B_i} |u - u_{\widetilde{B}(\overline{x})}| \leq C \fint_{\widetilde{B}(\overline{x})} |u - u_{\widetilde{B}(\overline{x})}|$$

$$\leq C r_i \fint_{\widetilde{B}(\overline{x})} |\nabla u| \leq C r_i \mathcal{M} |\nabla u|(\overline{x}) \leq C t r_i.$$

The above inequalities complete the proof of (4.3) and hence that for (4.4). We proved inequality (4.4) for $x \in M \setminus E_t$. The inequality holds, however, for all $x \in M$ because $x = \overline{x}$ whenever $x \in E_t$.

Now we are ready to estimate the Lipschitz constant of u_t. If $d(x, y) \geq \min\{\operatorname{dist}(x, E_t), \operatorname{dist}(y, E_t)\}$, then

$$\left| u_t(x) - u_t(y) \right| \leq \left| u_t(x) - u_t(\overline{x}) \right| + \left| u_t(\overline{x}) - u_t(\overline{y}) \right| + \left| u_t(\overline{y}) - u_t(y) \right|$$

$$\leq C t \big(d(x, \overline{x}) + d(\overline{x}, \overline{y}) + d(\overline{y}, y) \big) \leq C t d(x, y).$$

We employed here the fact that $u_t|_{E_t} = u|_{E_t}$ is Lipschitz continuous with the Lipschitz constant Ct. Thus we can assume that $d(x, y) \leq \min\{\operatorname{dist}(x, E_t), \operatorname{dist}(y, E_t)\}$. Since $\sum_{i \in I}(\varphi_i(x) - \varphi_i(y)) = 0$ we have

$$\left| u_t(x) - u_t(y) \right| = \left| \sum_{i \in I} \varphi_i(x) u_{B_i} - \sum_{i \in I} \varphi_i(y) u_{B_i} \right|$$

$$= \Big| \sum_{i \in I} \big(\varphi_i(x) - \varphi_i(y) \big) \big(u(\overline{x}) - u_{B_i} \big) \Big| \leq C d(x,y) \sum_{i \in I_x \cup I_y} r_i^{-1} \big| u(\overline{x}) - u_{B_i} \big|$$

$$\leq C d(x,y) \sum_{i \in I_x \cup I_y} r_i^{-1} C t r_i \leq C t d(x,y) \,.$$

In the inequality second to last we employed the fact that $|u(\overline{x}) - u_{B_i}| \leq C t r_i$ for all $i \in I_x \cup I_y$. The proof of this fact is very similar to the proof of inequality (4.3). The proof of the second claim is complete.

Now we will prove property (C) of u_t. For $x \in E_t$ we have $u_t(x) = u(x) \in X$ and hence we can assume that $x \in M \setminus E_t$. Given $\varepsilon > 0$ there is r_0 such that for every ball B in M with radius $r \leq r_0$ we have

$$\left(\int_B |\nabla u|^n \right)^{1/n} \leq \varepsilon \,. \tag{4.5}$$

Hence if we choose t large enough we can guarantee that (4.5) holds for each of the balls $B = \widetilde{B}(\overline{x})$. Since $\operatorname{dist}(u_{\widetilde{B}(\overline{x})}, X) \leq |u_{\widetilde{B}(\overline{x})} - u(z)|$ for $z \in \widetilde{B}(\overline{x})$, the Poincaré inequality yields

$$\operatorname{dist}(u_{\widetilde{B}(\overline{x})}, X) \leq \fint_{\widetilde{B}(\overline{x})} |u - u_{\widetilde{B}(\overline{x})}|$$

$$\leq C r_i \fint_{\widetilde{B}(\overline{x})} |\nabla u| \leq C \left(\int_{\widetilde{B}(\overline{x})} |\nabla u|^n \right)^{1/n} \leq C\varepsilon \,. \tag{4.6}$$

This means for sufficiently large t all averages $\widetilde{B}(\overline{x})$ are very close to X. Since

$$\operatorname{dist}\big(u_t(x), X \big) \leq \big| u_t(x) - u_{\widetilde{B}(\overline{x})} \big| + C\varepsilon$$

it suffices to prove that $|u_t(x) - u_{\widetilde{B}(\overline{x})}| < C\varepsilon$. We have

$$\big| u_t(x) - u_{\widetilde{B}(\overline{x})} \big| = \Big| \sum_{i \in I} \varphi_i(x) \big(u_{B_i} - u_{\widetilde{B}(\overline{x})} \big) \Big|$$

$$\leq \sum_{i \in I_x} \fint_{B_i} |u - u_{\widetilde{B}(\overline{x})}| \leq C \sum_{i \in I_x} \fint_{\widetilde{B}(\overline{x})} |u - u_{\widetilde{B}(\overline{x})}| \leq C\varepsilon \,.$$

The last inequality follows from the Poincaré inequality applied as in (4.6) and the fact that the number of indices in I_x is bounded by a constant independent of x. This completes the proof of the last claim and hence that of the theorem. $\qquad\square$

5   Proof of Theorem 1.8.

Let us start with a brief description of the main steps in the proof. Let $M \subset \mathbb{R}^\nu$ be a closed n-dimensional manifold. In the first step we define

a continuous $W^{1,n}$ function γ on $\mathbb{R}^n$ with compact support contained in the unit ball $B^n(0,1)$. In the second step we use the fact that M contains a subset diffeomorphic to $\overline{B^n}(0,1)$ to replace a part of M by the graph of γ. The resulting space is denoted by $\widetilde{M}$. It is still a subset of $\mathbb{R}^\nu$. The function γ is constructed in such a way that it is smooth everywhere but at one point and hence $\widetilde{M}$ is a smooth manifold with one nonsmooth point on it. Obviously $\widetilde{M}$ is homeomorphic to M, actually a careful construction of $\widetilde{M}$ can make it C^∞ homeomorphic in the sense that there is a homeomorphism $\Phi \in C^\infty(\mathbb{R}^\nu, \mathbb{R}^\nu)$ such that $\Phi(\widetilde{M}) = M$. Moreover Φ can be chosen to be a diffeomorphism everywhere but at one point and identity outside a sufficiently large ball. While $\Phi|_{\widetilde{M}} : \widetilde{M} \to M$ is a smooth homeomorphism, it is shown that there is no smooth homeomorphism from M onto $\widetilde{M}$. Actually due to high oscillations of γ near the nonsmooth point, it is proved that there is no Lipschitz mapping $\psi : M \twoheadrightarrow \widetilde{M}$.

In the last step we show that Lipschitz mappings $\mathrm{Lip}(M, \widetilde{M})$ are not dense in $W^{1,n}(M, \widetilde{M})$. The reason being the following: $\gamma \in W^{1,n}$ and hence there is a homeomorphism $\varphi : M \twoheadrightarrow \widetilde{M}$ of the class $W^{1,n}$. It follows from the homotopy properties of $W^{1,n}$ mappings that every Lipschitz mapping $\psi : M \to \widetilde{M}$ sufficiently close to φ in the $W^{1,n}$ norm is homotopic to φ and hence $\psi(M) = \widetilde{M}$. This contradicts, however, the lack of Lipschitz mappings $\psi : M \twoheadrightarrow \widetilde{M}$.

Actually the steps of the proof will be organized in a slightly different order due to the idea of collecting technical constructions at the end of the proof.

Construction of γ. The construction of γ presented below is closely related to the fact that the n-capacity of a point is zero. Those who know what it means can avoid the part of the construction involving truncation and convolution, but just invoke basic definitions.

It is well known and easy to prove that $\eta(x) = \log|\log|x|| \in W^{1,n}(B^n(0, e^{-1}))$. Define the truncation of η between levels s and t, $0 < s < t < \infty$, by

$$\eta_s^t(x) = \begin{cases} t - s & \text{if } \eta(x) \geq t, \\ \eta(x) - s & \text{if } s \leq \eta(x) \leq t, \\ 0 & \text{if } \eta(x) \leq s. \end{cases}$$

Fix an arbitrary $\tau > 0$. For every $\varepsilon > 0$ there is a sufficiently large s such that $\widetilde{\gamma}_{\varepsilon,\tau} := \eta_s^{s+\tau}$ is a Lipschitz function on $\mathbb{R}^n$ with the properties:

$$\mathrm{supp}\,\widetilde{\gamma}_{\varepsilon,\tau} \subset B^n(0, \varepsilon/2)\,,$$

$$0 \leq \widetilde{\gamma}_{\varepsilon,\tau} \leq \tau \text{ and } \widetilde{\gamma}_{\varepsilon,\tau} = \tau \text{ in a neighborhood of } 0\,,$$

$$\int_{\mathbb{R}^n} |\nabla \widetilde{\gamma}_{\varepsilon,\tau}|^n < \varepsilon^n.$$

The function $\widetilde{\gamma}_{\varepsilon,\tau}$ is not smooth because it is defined as a truncation; however, mollifying $\widetilde{\gamma}_{\varepsilon,\tau}$ gives a smooth function, denoted by $\gamma_{\varepsilon,\tau}$, with the same properties as those of $\widetilde{\gamma}_{\varepsilon,\tau}$ listed above.

Let (x', x^n) denote the coordinates in $\mathbb{R}^n$, where $x' = (x^1, \ldots, x^{n-1})$. Let $x_k = (x'_k, x^n_k) = (0, 2^{-k})$, $k = 6, 7, 8 \ldots$, and let $B_k = B^n(x_k, 2^{-(k+2)})$. The balls B_k are pairwise disjoint and all contained in $B^n(0, 2^{-5})$.

Observe that we can find $N_k = c(n)2^{(k^2-k)n}$ pairwise disjoint balls of radius 2^{-k^2} inside B_k. Select such a family for each B_k and denote the balls in the family by $B_{k,i}$, $i = 1, 2, \ldots, N_k$. Denote the centers of $B_{k,i}$ by $x_{k,i}$ and define

$$\gamma(x) = \sum_{k=6}^{\infty} \sum_{i=1}^{N_k} \eta_{k,i}(x), \tag{5.1}$$

where $\eta_{k,i}(x) = \gamma_{2^{-k^2}, 2^{-k}}(x - x_{k,i})$. Each of the functions $\eta_{k,i}$ is supported in $\frac{1}{2}B_{k,i}$ and hence the supports of the functions in the double sum (5.1) are pairwise disjoint. This immediately implies that $\gamma \in C^{\infty}(\mathbb{R}^n \setminus \{0\})$, γ is continuous on $\mathbb{R}^n$, $\gamma(0) = 0$ and $\operatorname{supp} \gamma \subset B^n(0, 2^{-5})$. Moreover

$$\int_{\mathbb{R}^n} |\nabla \gamma|^n = \sum_{k=6}^{\infty} \sum_{i=1}^{N_k} \int_{\mathbb{R}^n} |\nabla \eta_{k,i}|^n < \sum_{k=6}^{\infty} \sum_{i=1}^{N_k} 2^{-k^2 n} = c(n) \sum_{k=6}^{\infty} 2^{-kn} < \infty,$$

and hence $\gamma \in W^{1,n}(\mathbb{R}^n)$. Denote the graph of γ over $\overline{B^n}(0,1)$ by Γ i.e.

$$\Gamma = \left\{ (x, \gamma(x)) \in \mathbb{R}^{n+1} : x \in \overline{B^n}(0,1) \right\}.$$

We equip Γ with the metric induced from $\mathbb{R}^{n+1}$. The following lemma is a consequence of high oscillations of γ near the origin.

LEMMA 5.1.   *Let N be a closed n-dimensional Riemannian manifold. Then there is no Lipschitz mapping $\psi : N \twoheadrightarrow \Gamma$.*

REMARK.   The claim of the lemma remains the same (with the same proof) if we replace N by any metric space X equipped with a finite Borel measure μ satisfying the lower bound for the growth $\mu(B(x,r)) \geq Cr^n$ for all $x \in X$ and all $0 < r \leq r_0$. Clearly N equipped with the Hausdorff measure $\mathcal{H}^n$ is an example.

Proof of Lemma 5.1.   We argue by contradiction. Suppose $\psi : N \twoheadrightarrow \Gamma$ is an L-Lipschitz mapping. Fix a positive integer k. The graph of $\eta_{k,i}$ over $\frac{1}{2}B_{k,i}$ is of height 2^{-k} and hence we can find 2^{k^2-k} points on that graph whose mutual distances are at least 2^{-k^2}. Choose such points and denote them

by $y_{k,i,j}$, $j = 1, 2, \ldots, 2^{k^2-k}$. Since distances between the balls $\left\{\frac{1}{2} B_{k,i}\right\}_i$ are no less than 2^{-k^2} we conclude that mutual distances of the points in the family $\{y_{k,i,j}\}_{i,j}$ are at least 2^{-k^2}. The family $\{y_{k,i,j}\}_{i,j}$ consists of

$$c(n) 2^{(k^2-k)n} 2^{k^2-k} = c(n) 2^{(k^2-k)(n+1)}$$

points. Choose $z_{i,j} \in \psi^{-1}(y_{k,i,j})$ arbitrarily. Since ψ is L-Lipschitz the distances between points $\{z_{i,j}\}_{i,j}$ are at least $L^{-1} 2^{-k^2}$. Hence the balls $B(z_{i,j}, L^{-1} 2^{-k^2}/2)$ are pairwise disjoint. This in turn gives the lower estimate for the volume of N.

$$\mathcal{H}^n(N) \geq \sum_{i,j} \mathcal{H}^n\big(B(z_{i,j}, L^{-1} 2^{-k^2}/2)\big) \geq C(L^{-1} 2^{-k^2}/2)^n 2^{(k^2-k)(n+1)} \to \infty$$

as $k \to \infty$. This clearly contradicts the finiteness of the volume of N. The proof of the lemma is complete. $\qquad\square$

The construction of $\widetilde{M}$. Take a subset of M that is diffeomorphic to $\overline{B^n}(0,1)$ and replace it by the graph Γ. Denote the resulting space by $\widetilde{M} \subset \mathbb{R}^\nu$. The construction is quite obvious and does not require further explanation. In any case, we will present all details in the next subsection because it will be needed for the construction of Φ.

The following properties of $\widetilde{M}$ are obvious. The property (c) below is a consequence of the fact that the mapping

$$\Gamma \to \overline{B^n}(0,1), \quad \big(x, \gamma(x)\big) \mapsto x,$$

is Lipschitz continuous, and property (d) is a consequence of Lemma 5.1.

  (a) There is $\widetilde{x}_0 \in \widetilde{M}$ such that $\widetilde{M} \setminus \{\widetilde{x}_0\}$ is a smooth submanifold of $\mathbb{R}^\nu$.
  (b) There is a homeomorphism $\varphi : M \twoheadrightarrow \widetilde{M}$ in the class $\varphi \in W^{1,n}(M, \widetilde{M})$ with the property that $\varphi : M \setminus \{x_0\} \twoheadrightarrow \widetilde{M} \setminus \{\widetilde{x}_0\}$ is a C^∞ diffeomorphism of manifolds, where $x_0 = \varphi^{-1}(\widetilde{x}_0)$.
  (c) φ can be chosen in such a way that $\varphi^{-1} : \widetilde{M} \twoheadrightarrow M$ is Lipschitz continuous.
  (d) There is no Lipschitz mapping $\psi : M \twoheadrightarrow \widetilde{M}$.

The above properties do not immediately guarantee the existence of a homeomorphism $\Phi \in C^\infty(\mathbb{R}^\nu, \mathbb{R}^\nu)$ being a diffeomorphism everywhere but at one point and such that $\Phi(\widetilde{M}) = M$. One could try to construct Φ as an extension of $\varphi^{-1} : \widetilde{M} \twoheadrightarrow M$, however, high oscillations of $\widetilde{M}$ near $\widetilde{x}_0$ seem to produce a singularity of $D\Phi$ at $\widetilde{x}_0$, and this is just one of the technical problems that we will have to face.

We postpone the technically complicated construction of Φ for a while and show that Lipschitz mappings $\mathrm{Lip}(M, \widetilde{M})$ are not dense in $W^{1,n}(M, \widetilde{M})$.

We will actually show that the homeomorphism $\varphi \in W^{1,n}(M, \widetilde{M})$ cannot be approximated by Lipschitz mappings $\text{Lip}(M, \widetilde{M})$ in the $W^{1,n}$ norm. Suppose to the contrary that $\varphi_k : M \to \widetilde{M}$ is a sequence of Lipschitz mappings such that $\varphi_k \to \varphi$ in $W^{1,n}$. Then $\varphi^{-1} \circ \varphi_k \to \varphi^{-1} \circ \varphi = \text{id} : M \to M$ in $W^{1,n}(M, M)$. Indeed, by Lemma 1.1 the composition with φ^{-1} is continuous in the $W^{1,n}$ norm. (Note that in the construction that will be carried out later φ^{-1} admits an extension to $\Phi \in C^\infty(\mathbb{R}^\nu, \mathbb{R}^\nu)$. Hence φ^{-1} is smooth everywhere and there is no problem with the continuity of the operator of composition.) We will also need the following result of White [W1, Th. 2].

LEMMA 5.2. *Let M and N be closed manifolds and let $n = \dim M$. Then, for every $f \in W^{1,n}(M, N)$, there is $\varepsilon > 0$ such that any two Lipschitz mappings $g_1, g_2 : M \to N$ satisfying $\|f - g_i\|_{1,n} < \varepsilon$, for $i = 1, 2$, are homotopic.*

The degree modulo 2 of the identity mapping $\text{id} : M \to M$ equals 1 (we do not assume M to be orientable, so the Brouwer degree needs to be replaced by degree modulo 2, see [Mi]). Lemma 5.2 implies that, for sufficiently large k, $\varphi^{-1} \circ \varphi_k : M \to M$ is homotopic to the identity map. Hence by the homotopy lemma [Mi], the degree modulo 2 of $\varphi^{-1} \circ \varphi_k$ equals 1 as well, and the definition of the degree modulo 2 yields $(\varphi^{-1} \circ \varphi_k)(M) = M$. This, in turn, implies $\varphi_k(M) = \varphi(M) = \widetilde{M}$ which contradicts property (d).

The construction of Φ. To complete the proof of the theorem it remains to show that there is a C^∞ homeomorphism $\Phi : \mathbb{R}^\nu \to \mathbb{R}^\nu$ which is a diffeomorphism everywhere but at one point, identity outside a sufficiently large ball, and satisfies $\Phi(\widetilde{M}) = M$. To this end we will repeat the construction of $\widetilde{M}$ providing more details. In what follows $B^k(0, 1)$, $1 \leq k < \nu$, will be identified with a subset of $\mathbb{R}^\nu$ being the ball in the subspace generated by the first k coordinates.

We use (x', x^{n+1}, x'') to designate coordinates in $\mathbb{R}^\nu = \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^{\nu-n-1}$. With no loss of generality we may assume that

$$M \cap \overline{B^\nu}(0, 1) = \overline{B^n}(0, 1) \times \{0\} \times \{0\} \subset \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^{\nu-n-1} = \mathbb{R}^\nu.$$

Let γ be as before. Define $\widetilde{M}$ by replacing $\overline{B^n}(0, 1) \times \{0\} \times \{0\} \subset M$ with the graph of γ

$$\Gamma = \left\{ (x', x^{n+1}, x'') : x' \in \overline{B^n}(0, 1), \ x^{n+1} = \gamma(x'), \ x'' = 0 \right\}.$$

Now we will extend this graph to a graph of a function defined in $\mathbb{R}^{\nu-1} = \mathbb{R}^n \times \{0\} \times \mathbb{R}^{\nu-n-1}$. Let $\eta \in C^\infty(\mathbb{R})$ be such that $0 \leq \eta \leq 1$ and

$$\eta(t) = \begin{cases} 1 & \text{if } |t| \geq 2, \\ 0 & \text{if } |t| \leq 1. \end{cases}$$

Define $\lambda : \mathbb{R}^n \times \mathbb{R}^{\nu-n-1} \to \mathbb{R}$ by

$$\lambda(x', x'') = \begin{cases} \gamma(x')\eta\left(\frac{x'}{|x''|}\right) & \text{if } x'' \neq 0, \\ \gamma(x') & \text{if } x'' = 0. \end{cases} \tag{5.2}$$

It is easily seen that $\lambda \in C^{\infty}(\mathbb{R}^n \times \mathbb{R}^{\nu-n-1} \setminus \{(0,0)\})$, λ is continuous on $\mathbb{R}^{\nu-1}$ and equals zero outside the ball $B^{\nu-1}(0, 2^{-4})$. Now define the graph of λ over $\overline{B^{\nu-1}}(0,1)$ by

$$\widetilde{\Gamma} = \left\{ (x', x^{n+1}, x'') \in \mathbb{R}^{\nu} : x^{n+1} = \lambda(x', x''), \ (x', x'') \in \overline{B^{\nu-1}}(0,1) \right\}.$$

Since $|\lambda| \leq 2^{-6}$ it follows that $\widetilde{\Gamma} \subset B^{\nu}(0, 2^{-3})$. Clearly $\widetilde{\Gamma}$ is an extension of Γ in the sense that the intersection of $\widetilde{\Gamma}$ with the linear subspace $x'' = 0$ coincides with Γ.

Observe that

$$\widetilde{\Phi}(x', x^{n+1}, x'') = \left(x', x^{n+1} - \lambda(x', x''), x'' \right)$$

is a homeomorphism $\widetilde{\Phi} : \mathbb{R}^{\nu} \twoheadrightarrow \mathbb{R}^{\nu}$, $\widetilde{\Phi}(\widetilde{\Gamma}) = \overline{B^{\nu-1}}(0,1)$, $\widetilde{\Phi}(\Gamma) = \overline{B^n}(0,1)$ and $\widetilde{\Phi}$ is a smooth diffeomorphism off the line $\{x' = 0, x'' = 0\}$. However, $\widetilde{\Phi}$ is *not* differentiable in points of the line $\{x' = 0, x'' = 0\}$. Moreover $\widetilde{\Phi}$ is not identity outside any finite ball. Hence we have to modify $\widetilde{\Phi}$ in order to improve its properties. Actually we will prove

LEMMA 5.3.   *There is a C^{∞} homeomorphism $\Phi : \mathbb{R}^{\nu} \twoheadrightarrow \mathbb{R}^{\nu}$ which is diffeomorphism in $\mathbb{R}^{\nu} \setminus \{0\}$, $\Phi(\widetilde{\Gamma}) = \overline{B^{\nu-1}}(0,1)$, $\Phi(\Gamma) = \overline{B^n}(0,1)$ and Φ is identity in $\mathbb{R}^{\nu} \setminus B^{\nu}(0,1)$.*

It is obvious now that Lemma 5.3 implies Theorem 1.8, so we are left with the proof of the lemma. Since a version of Lemma 5.3 holds for a much larger class of graphs, and not just for the graph of the function λ constructed above, it makes sense to formulate the lemma in a more general form and deduce Lemma 5.3 as a special case. Lemma 5.3 is a direct consequence of Lemma 5.4 and Lemma 5.5 below.

Notation (x', x^{n+1}, x'') for the coordinates in $\mathbb{R}^{\nu}$ will no longer be convenient for us, so in the remaining part of section 5 we will denote the coordinates in $\mathbb{R}^{\nu}$ by (x', x^{ν}), where $x' = (x^1, \dots, x^{\nu-1})$.

LEMMA 5.4.   *Assume that a nonnegative function λ defined on $\mathbb{R}^{\nu-1}$ is continuous, $\operatorname{supp}\lambda \subset B^{\nu-1}(0, 2^{-3})$, $\lambda(0) = 0$, and $\lambda \in C^{\infty}(\mathbb{R}^{\nu-1} \setminus \{0\})$. Denote the graph of λ over $\overline{B^{\nu-1}}(0,1)$ by $\widetilde{\Gamma}$, i.e.*

$$\widetilde{\Gamma} = \left\{ (x', x^{\nu}) \in \mathbb{R}^{\nu} : x' \in \overline{B^{\nu-1}}(0,1), x^{\nu} = \lambda(x') \right\}.$$

Assume that the graph of λ over $B^{\nu-1}(0, 2^{-3})$ is contained in the ball $B^{\nu}(0, 2^{-3})$. Then there is a homeomorphism $\Phi^* : \mathbb{R}^{\nu} \twoheadrightarrow \mathbb{R}^{\nu}$ of the class $C^{\infty}(\mathbb{R}^{\nu} \setminus \{0\})$ with the following properties:

1. Φ^* is of the form $\Phi^*(x', x^\nu) = (x', \xi(x', x^\nu))$;
2. $\Phi^*(0) = 0$;
3. Φ^* is a *diffeomorphism* in $\mathbb{R}^\nu \setminus \{0\}$;
4. $\Phi^*(\widetilde{\Gamma}) = \overline{B^{\nu-1}}(0, 1)$;
5. Φ^* is identity in $\mathbb{R}^\nu \setminus B^\nu(0, 1)$.

The mapping Φ^* constructed in the above lemma has all the properties needed in Lemma 5.3 except for the smoothness at the origin. The following lemma shows how to modify the mapping Φ^* in a way that it will have all the properties needed in Lemma 5.3.

LEMMA 5.5. *Let* $\Lambda : \mathbb{R}^\nu \to \mathbb{R}^\nu$ *be a continuous mapping such that* $\Lambda(0) = 0$ *and* $\Lambda \in C^\infty(\mathbb{R}^\nu \setminus \{0\})$. *Then there is a homeomorphism* $\Theta : \mathbb{R}^\nu \twoheadrightarrow \mathbb{R}^\nu$ *such that*

(a) $\Theta \in C^\infty(\mathbb{R}^\nu)$;
(b) Θ *is a diffeomorphism in* $\mathbb{R}^\nu \setminus \{0\}$;
(c) $\Theta(0) = 0$ *and* $\Theta(x) = x$ *for all* $|x| \geq 1$;
(d) $\Theta(x)$ *is of the form*

$$\Theta(x) = \begin{cases} \frac{x}{|x|}\eta(|x|), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases}$$

for some homeomorphism $\eta : [0, \infty) \twoheadrightarrow [0, \infty)$ *and hence* $\Theta(\overline{B^k}(0, 1)) = \overline{B^k}(0, 1)$ *for* $k = 1, 2, \ldots, \nu$.

(e) $\Theta \circ \Lambda \in C^\infty(\mathbb{R}^\nu)$.

If we set $\Lambda = \Phi^*$, where Φ^* is defined for λ given by (5.2), then $\Phi = \Theta \circ \Phi^*$ has the properties required in Lemma 5.3. Indeed, it follows from the property (1) of Φ^* that $\Phi^*(\Gamma) = \overline{B^n}(0, 1)$ and hence $(\Theta \circ \Phi^*)(\Gamma) = \Theta(\overline{B^n}(0, 1)) = \overline{B^n}(0, 1)$ by the property (d) of Θ.

Proof of Lemma 5.4. Let $f(x') = 4\lambda(x') + |x'|^2$. Then $f \in C^0(\mathbb{R}^{\nu-1}) \cap C^\infty(\mathbb{R}^{\nu-1} \setminus \{0\})$ is strictly positive in $\mathbb{R}^{\nu-1} \setminus \{0\}$, $f(0) = 0$ and $f \geq 4\lambda$ in $\mathbb{R}^{\nu-1}$.

Fix a function $\varphi \in C_0^\infty(\mathbb{R})$ such that $0 < \varphi(\tau) < 2$ for $0 < \tau < 1$, $\operatorname{supp}\varphi = [0, 1]$ and $\int \varphi(\tau)\, d\tau = 1$. For $x' \neq 0$ we define

$$g(x', x^\nu) = \frac{-\lambda(x')}{f(x')}\varphi\left(\frac{x^\nu + f(x')}{f(x')}\right).$$

Then $\int g(x', x^\nu)\, dx^\nu = -\lambda(x')$, the support of the function $x^\nu \mapsto g(x', x^\nu)$ is contained in the interval $[-f(x'), 0]$, the function is nonpositive in the interval, but $g(x', x^\nu) \geq -1/2$. For $x' \neq 0$ we also define

$$h(x', x^\nu) = \frac{\lambda(x')}{f(x') - \lambda(x')}\varphi\left(\frac{x^\nu - \lambda(x')}{f(x') - \lambda(x')}\right).$$

Then the support of the function $x^\nu \mapsto h(x', x^\nu)$ is contained in the interval $[\lambda(x'), f(x')]$, the function is nonnegative in the interval and $\int h(x', x^\nu) dx^\nu = \lambda(x')$.

Now we define

$$\xi(x', x^\nu) = \begin{cases} \int_{\lambda(x')}^{x^\nu} \big(1 + g(x', \tau) + h(x', \tau)\big) d\tau & \text{if } x' \neq 0 \,, \\ x^\nu & \text{if } x' = 0 \,. \end{cases}$$

The integrand is greater than or equal to $1/2$ and hence the function ξ is strictly increasing in x^ν with $\partial \xi / \partial x^\nu \geq 1/2$. Moreover $\xi(x', x^\nu) = x^\nu$ for $|x^\nu| \geq f(x')$ and $\xi(x', x^\nu) = x^\nu - \lambda(x')$ for $0 \leq x^\nu \leq \lambda(x')$.

Clearly the mapping $\Phi^*(x', x^\nu) = (x', \xi(x', x^\nu))$ is a homeomorphism of $\mathbb{R}^\nu$ onto $\mathbb{R}^\nu$ satisfying claims (1), (2) and (4). Property (5) follows from the formula for ξ because $\xi(x', x^\nu) = x^\nu$ whenever $\lambda(x') = 0$, and the fact that support of λ is concentrated near 0.

We still have to prove that Φ^* is smooth in $\mathbb{R}^\nu \backslash \{0\}$ and that the Jacobian of Φ^* is positive in $\mathbb{R}^\nu \setminus \{0\}$.

Smoothness of Φ^* at (x_0', x_0^ν), $x_0' \neq 0$ follows from the definition of ξ and smoothness of Φ^* at $(0, x_0^\nu)$, $x_0^\nu \neq 0$ follows from the observation that Φ^* is the identity in a neighborhood of $(0, x_0^\nu)$. The mapping Φ^* is a diffeomorphism in $\mathbb{R}^\nu \setminus \{0\}$ because direct computation shows that $\det D\Phi^*(x', x^\nu) = \partial \xi / \partial x^\nu > 0$. This completes the proof of all the properties of Φ^*.   □

Proof of Lemma 5.5.   In the proof we will need the following result.

LEMMA 5.6.   *Let $\{N_n\}_{n=1}^\infty$ be an arbitrary increasing sequence of positive numbers. Then there is a homeomorphism*

$$\eta : [0, \infty) \twoheadrightarrow [0, \infty)$$

such that $\eta \in C^\infty((0, \infty))$, $\eta'(t) > 0$ for $t > 0$, $\eta(t) = t$ for $t \geq 1$ and

$$N_n \sup_{0 < t \leq 2^{-n}} \sup_{0 \leq \ell \leq n} \big|\eta^{(\ell)}(t)\big| \to 0 \qquad (5.3)$$

as $n \to \infty$.

Proof. There is a nonnegative function $\varphi \in C_0^\infty(\mathbb{R})$ with $\operatorname{supp} \varphi = [1/2, 2]$ and such that

$$\sum_{n=-\infty}^{\infty} \varphi(2^n t) = 1 \qquad (5.4)$$

for all $t > 0$. Indeed, take an arbitrary function $\widetilde{\varphi} \in C_0^\infty(\mathbb{R})$ such that $\operatorname{supp} \widetilde{\varphi} = [1/2, 2]$, and $\widetilde{\varphi} > 0$ in $(1/2, 2)$. Then $\varphi(t) = \widetilde{\varphi}(t)/\alpha(t)$, where

$$\alpha(t) = \sum_{n=-\infty}^{\infty} \widetilde{\varphi}(2^n t)$$

satisfies (5.4). This follows from the observation that α is a strictly positive C^∞ function for all $t > 0$ and $\alpha(2^n t) = \alpha(t)$ for all integers n.

Let $\{a_n\}_{n=2}^\infty$ be a sequence of positive numbers less than 1 such that $a_n \to 0$ as $n \to \infty$. Let also $a_n = 1$ for all integers $n \leq 1$. The function

$$\psi(t) = \sum_{n=-\infty}^\infty a_n \varphi(2^n t)$$

is smooth and strictly positive. Moreover $\psi(t) \to 0$ as $t \to 0+$. Define

$$\widetilde{\eta}(t) = \int_0^t \psi(s) ds \,.$$

Obviously $\widetilde{\eta} : [0, \infty) \twoheadrightarrow [0, \infty)$ is a homeomorphism, $\widetilde{\eta} \in C^\infty((0, \infty))$ and $\eta'(t) > 0$ for $t > 0$. Moreover $\widetilde{\eta}'(t) = 1$ for $t \geq 1$. However, $\widetilde{\eta}(t) < t$ for $t \geq 1$ and hence $\widetilde{\eta}(t) = t - a$ for all $t \geq 1$ and some constant a. Let $\delta \in C^\infty(\mathbb{R})$ be an arbitrary nondecreasing function such that $\delta(t) = 0$ for $t \leq 1/2$ and $\delta(t) = 1$ for $t \geq 3/4$. Now the function $\eta(t) = \widetilde{\eta}(t) + a\delta(t)$ has all the properties required in the lemma except perhaps for (5.3). We will show that an appropriate choice of $\{a_n\}_{n=2}^\infty$ will guarantee (5.3) as well.

Since $\delta(t) = 0$ for $t \leq 1/2$ we have that $\widetilde{\eta}(t) = \eta(t)$ for small t and hence it suffices to prove (5.3) for $\widetilde{\eta}$. Let

$$M_n = \sup_{\ell \leq n-1} \|\varphi^{(\ell)}\|_\infty \,.$$

We will show that the function $\widetilde{\eta}$ constructed for the sequence

$$a_n = 2^{-2n^2} N_n^{-1} M_n^{-1}, \qquad n = 2, 3, 4, \dots$$

satisfies (5.3).

Since for every $t > 0$ no more than two of the functions $\{\varphi(2^n t)\}_{n=-\infty}^\infty$ are different from zero we conclude that for any $t \leq 2^{-n}$ there is $i \geq n$ such that for any $1 \leq \ell \leq n$

$$\eta^{(\ell)}(t) = a_i 2^{i(\ell-1)} \varphi^{(\ell-1)}(2^i t) + a_{i+1} 2^{(i+1)(\ell-1)} \varphi^{(\ell-1)}(2^{i+1} t) \,.$$

Hence

$$\left| \eta^{(\ell)}(t) \right| \leq M_n \big(2^{-2i^2} 2^{i(\ell-1)} N_n^{-1} M_n^{-1} + 2^{-2(i+1)^2} 2^{(i+1)(\ell-1)} N_n^{-1} M_n^{-1} \big)$$

$$\leq 2 \cdot 2^{-n^2} N_n^{-1} \,,$$

and thus

$$N_n \sup_{0 < t \leq 2^{-n}} \sup_{1 \leq \ell \leq n} \left| \eta^{(\ell)}(t) \right| \leq 2 \cdot 2^{-n^2} \to 0$$

as $n \to \infty$. It remains to show that

$$N_n \sup_{0 < t \leq 2^{-n}} |\eta(t)| \to 0$$

as $n \to \infty$. This is easy and we leave it to the reader. The proof of the lemma is complete.   □

Now we can complete the proof of Lemma 5.5.

We will construct a homeomorphism Θ of the form

$$\Theta(x) = \begin{cases} \frac{x}{|x|}\eta(|x|), & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

where η is as in Lemma 5.6 for an appropriate sequence $\{N_n\}_{n=1}^\infty$. The sequence will be chosen later. Obviously $\Theta : \mathbb{R}^\nu \twoheadrightarrow \mathbb{R}^\nu$ is a homeomorphism such that $\Theta(0) = 0$, $\Theta(x) = x$ for $|x| \geq 1$ and Θ is a diffeomorphism of $\mathbb{R}^\nu \setminus \{0\}$ of the class C^∞. Observe that

$$\left|\nabla^m \Theta(x)\right| \leq C(m) \sum_{i=0}^m |x|^{-i}\left|\eta^{(m-i)}(|x|)\right|,$$

where $\nabla^m \Theta$ denotes the vector whose components are all partial derivatives of order m. Hence if $\{N_n\}_{n=1}^\infty$ is a sequence such that $N_n \geq 2^{n^2}$, then

$$\left|\nabla^m \Theta(x)\right|/|x| \to 0 \quad \text{as } x \to 0$$

for each positive integer m. This, in turn, implies that Θ has partial derivatives at 0 of an arbitrary order and all the derivatives are equal to 0. Hence $\Theta \in C^\infty(\mathbb{R}^\nu)$.

Obviously $\Theta \circ \Lambda \in C^\infty(\mathbb{R}^\nu \setminus \{0\})$. Now it remains to prove that $\Theta \circ \Lambda \in C^\infty(\mathbb{R}^\nu)$ for some increasing sequence $\{N_n\}_{n=1}^\infty$, such that $N_n \geq 2^{n^2}$. To this end it suffices to show that for some sequence $\{N_n\}_{n=1}^\infty$ we have

$$\left|\nabla^m(\Theta \circ \Lambda)(x)\right|/|x| \to 0, \qquad \text{as } x \to 0,$$

for each nonnegative integer m. A multiple application of the Leibniz rule gives the following variant of the Faá di Bruno[1] formula

$$\left|\nabla^m(\Theta \circ \Lambda)(x)\right| \leq C\bigg(\sum_{i=1}^m \left|(\nabla^i\Theta)(\Lambda(x))\right|\bigg)$$
$$\times \bigg(\sum \left|\nabla\Lambda(x)\right|^{i_1}\left|\nabla^2\Lambda(x)\right|^{i_2} \cdot \ldots \cdot \left|\nabla^m\Lambda(x)\right|^{i_m}\bigg),$$

where the last sum is taken over all nonnegative integers $i_1, i_2, \ldots, i_m$ such that

$$i_1 + 2i_2 + 3i_3 + \ldots + mi_m = m.$$

Define

$$M_n = \sup\bigg(\sum \left|\nabla\Lambda(x)\right|^{i_1}\left|\nabla^2\Lambda(x)\right|^{i_2} \cdot \ldots \cdot \left|\nabla^m\Lambda(x)\right|^{i_m}\bigg)|x|^{-1},$$

where the supremum is taken over all x satisfying $|x| \leq 1$ and $2^{-(n+1)} \leq |\Lambda(x)| \leq 2^{-n}$. For such x we have

$$\left|\nabla^m(\Theta \circ \Lambda)(x)\right|/|x| \leq CM_n \sum_{i=1}^m \left|(\nabla^i\Theta)(\Lambda(x))\right|$$

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[1]Faá di Bruno was declared a Saint by Pope John Paul II on September 25, 1988.

$$\leq C M_n \sum_{i=1}^{m} \left(\sum_{j=0}^{i} |\Lambda(x)|^{-j} \big| \eta^{(i-j)}(|\Lambda(x)|) \big| \right)$$

$$\leq C M_n 2^{(n+1)m} m^2 \sup_{\ell \leq m} \big| \eta^{(\ell)}(|\Lambda(x)|) \big| \, .$$

Hence with a new constant C depending on m for $n \geq m$ we have

$$\big| \nabla^m (\Theta \circ \Lambda)(x) \big| / |x| \leq C M_n 2^{nm} \sup_{0 < t \leq 2^{-n}} \sup_{\ell \leq n} \big| \eta^{(\ell)}(t) \big| \, .$$

Now it suffices to take any increasing sequence $\{N_n\}_{n=1}^{\infty}$, such that $N_n \geq M_n 2^{n^2}$. This completes the proof of Lemma 5.5 and hence that for Lemma 5.3. Thus the proof of Theorem 1.8 is complete. $\square$

6   Proof of Theorem 1.6.

Let $M = S^n \subset \mathbb{R}^{n+1}$ and let $N = \widetilde{M}$ be the singular submanifold of $\mathbb{R}^{n+1}$ constructed in the proof of Theorem 1.8. Let $\varphi : S^n = M \to \widetilde{M} = N$ be the $W^{1,n}$ homeomorphism constructed in the proof of Theorem 1.8. Let W be a closed n-dimensional manifold and let $B \subset W$ be a subset diffeomorphic to $B^n(0,1)$. Then there is a smooth mapping $\psi : W \to S^n$ which is one-to-one on B and which maps all of $W \setminus B$ into one point. Clearly such a mapping has degree modulo 2 equal 1. Now the same topological argument based on Lemma 5.1, as in the proof of Theorem 1.8 shows that $\varphi \circ \psi \in W^{1,n}(W,N)$ cannot be approximated by Lipschitz mappings $\mathrm{Lip}(W,N)$. $\square$

7   Proof of Theorem 1.7.

Construction of X.   Let N be a submanifold of $\mathbb{R}^{n+1}$ diffeomorphic to S^n having $\overline{B^n}(0,1)$ as a subset. Let $\eta_{k,i}$ be defined as in the proof of Theorem 1.8 and define

$$\gamma_m(x) = \sum_{k=6}^{m} \sum_{i=1}^{N_k} \eta_{k,i}(x) \, .$$

Thus the sequence $\{\gamma_m\}$ is a smooth approximation of γ defined by formula (5.1) in the $W^{1,n}$ norm. Let Γ_m be the graph of γ_m over the unit ball $\overline{B^n}(0,1)$.

Denote by $\widetilde{N}^m$ the manifold obtained from N by substituting $\overline{B^n}(0,1)$ with the graph Γ_m. Since each of the functions γ_m is smooth, $\widetilde{N}^m$ is a smooth manifold diffeomorphic to S^n. The manifolds $\widetilde{N}^m$ converge in some

sense to N^∞, the space obtained from N by substituting $\overline{B^n}(0,1)$ with Γ, the graph of γ defined by formula (5.1).

Let B_k denote balls defined as in the definition of the function γ. Denote the coordinates in $\mathbb{R}^{n+1}$ by (x', x^{n+1}), $x' \in \mathbb{R}^n$, $x^{n+1} \in \mathbb{R}$. There is a natural projection π_m of the graph Γ onto Γ_m, namely

$$\pi_m\big((x, \gamma(x))\big) = \left\{ \begin{array}{ll} (x, 0) & \text{if } x \in \bigcup_{k=m+1}^\infty B_k\,, \\ (x, \gamma(x)) & \text{otherwise}\,. \end{array} \right.$$

The projection extends in a natural way to a homeomorphism $\pi_m : N^\infty \twoheadrightarrow \widetilde{N}^m$.

Let A_m be the graph of γ over the set $\bigcup_{k=m+1}^\infty B_k$. Hence A_m is a decreasing sequence of subsets of Γ (and hence subsets of N^∞) with empty intersection.

Denote the orthogonal projection from $\mathbb{R}^{n+1}$ onto $\mathbb{R}^n$ by π. Observe that $\pi_m|_{\overline{A}_m} = \pi|_{\overline{A}_m}$. Hence π_m restricted to $\overline{A}_m$ is smooth with the constant norm of the derivative that does not depend on m. Since π_m is identity outside A_m, π_m extends to a smooth mapping of a neighbourhood of N^∞ in $\mathbb{R}^{n+1}$ onto $\widetilde{N}^m$.

LEMMA 7.1.   *Let M be an n-dimensional closed manifold. If $f \in W^{1,n}(M, N^\infty)$, then $\pi_m \circ f \in W^{1,n}(M, \widetilde{N}^m)$ and $\pi_m \circ f \to f$ in $W^{1,n}(M, \mathbb{R}^{n+1})$ as $m \to \infty$.*

Proof. Since π_m extends to a smooth mapping we conclude that $\pi_m \circ f \in W^{1,n}(M, \widetilde{N}^m)$. Now $\pi_m \circ f \neq f$ on the set $f^{-1}(A_m)$. This is a decreasing sequence of subsets of M with empty intersection and hence $\mathcal{H}^n(f^{-1}(A_m)) \to 0$ as $m \to \infty$. This, in turn, easily implies that $\pi_m \circ f \to f$ in $W^{1,n}$ as $m \to \infty$ because

$$\|f - \pi_m \circ f\|_{W^{1,n}(M)} = \left(\int_{f^{-1}(A_m)} |f - \pi_m \circ f|^n \right)^{1/n}$$

$$+ \left(\int_{f^{-1}(A_m)} |D(f - \pi_m \circ f)|^n \right)^{1/n}$$

$$\leq \left(\mathcal{H}^n(f^{-1}(A_m)) \right)^{1/n} + \left(\int_{f^{-1}(A_m)} |D(f - \pi \circ f)|^n \right)^{1/n}$$

$$\leq \left(\mathcal{H}^n(f^{-1}(A_m)) \right)^{1/n} + C \left(\int_{f^{-1}(A_m)} |Df|^n \right)^{1/n} \to 0$$

as $m \to \infty$. We employed here the fact that $|f(x) - (\pi_m \circ f)(x)| \leq 1$ for all $x \in f^{-1}(A_m)$. The proof of the lemma is complete. $\square$

$\widetilde{N}^m$ is a subset of $\mathbb{R}^{n+1}$ and hence it is a subset of $\mathbb{R}^{n+2} = \mathbb{R}^{n+1} \times \mathbb{R}$. Let N^m be the translation of $\widetilde{N}^m$ by the vector $\langle 0, \ldots, 0, 2^{-m} \rangle$ in $\mathbb{R}^{n+2}$. Now we define

$$\widetilde{X} = N^\infty \cup \bigcup_{m=6}^{\infty} N^m.$$

We equip $\widetilde{X}$ with the metric induced from $\mathbb{R}^{n+2}$. $\widetilde{X}$ is a compact set, but it is not connected. We define X by connecting all parts of $\widetilde{X}$ by a curve each part of which is non-rectifiable.

Fix an n-dimensional closed manifold M arbitrarily. We can assume that M is connected since different components are treated separately.

It is an easy consequence of the absolute continuity of f on lines [Z, Th. 2.1.4] that images of almost all lines under f are rectifiable. Hence Sobolev mappings from M into a space that contains no rectifiable curves (except constant ones) must be constant. Accordingly, if $f \in W^{1,n}(M, X)$ is not constant, then $f \in W^{1,n}(M, \widetilde{X})$.

In the next two steps we will prove the density of smooth mappings $C^\infty(M, X)$ in $W^{1,n}(M, X)$ and we will construct a global bi-Lipschitz homeomorphism $\Phi : \mathbb{R}^{n+2} \twoheadrightarrow \mathbb{R}^{n+2}$ such that Lipschitz mappings $\mathrm{Lip}(M, \Phi(X))$ are not dense in $W^{1,n}(M, \Phi(X))$.

Density. Constant mappings are smooth, so it suffices to prove the density of $C^\infty(M, \widetilde{X})$ in $W^{1,n}(M, \widetilde{X})$. We will need the following fact.

LEMMA 7.2. *For each $f \in W^{1,n}(M, \widetilde{X})$ we can choose a representative (in the class of functions equal a.e.) such that $f(M) \subset N^k$ for some $k = 6, 7, 8, \ldots$ or $k = \infty$.*

Proof. The lemma is a fairly easy consequence of the absolute continuity of f on almost all lines. Indeed, if we would find two subsets $A, B \subset M$ of positive measure such that $f(A) \subset N^{k_1}$ and $f(B) \subset N^{k_2}$, $k_1 \neq k_2$, then we would find a curve joining A and B such that f is continuous along that curve. Hence the image of f would contain a curve joining $f(A)$ and $f(B)$ inside $\widetilde{X}$. This is, however, impossible. □

Fix $f \in W^{1,n}(M, \widetilde{X})$. We will show that f can be approximated by mappings from M to $\widetilde{X}$ which are of the class C^∞ as mappings from M to $\mathbb{R}^{n+2}$.

We know that $f(M) \subset N^k$ for some k finite or $k = \infty$. If k is finite, then there is no problem with the approximation because N^k is a closed manifold and the smooth mappings $C^\infty(M, N^k)$ are dense in the class $W^{1,n}(M, N^k)$ according to Schoen and Uhlenbeck's theorem (cf. Theorem 1.3). Thus we

are left with the case in which $f(M) \subset N^\infty$. Let

$$f_m = (\pi_m \circ f) + \langle 0, \dots, 0, 2^{-m} \rangle, \tag{7.1}$$

i.e. first we compose f with the projection π_m from N^∞ onto $\widetilde{N}^m$ and then we translate the image by the vector $\langle 0, \dots, 0, 2^{-m} \rangle$, so that the resulting mapping f_m maps M to N^m. It follows from Lemma 7.1 that $f_m \to f$ in $W^{1,n}$ as $m \to \infty$. Now it suffices to observe that $f_m \in W^{1,n}(M, N^m)$ can be approximated by smooth mappings $C^\infty(M, N^m)$. This completes the proof of density of $C^\infty(M, \widetilde{X})$ in $W^{1,n}(M, \widetilde{X})$.

Construction of Φ. We can assume that $[1,2]^n$ is a subset of N. Since $\widetilde{N}^m$ was obtained from N by modifying it on $B^n(0,1)$ only, $[1,2]^n$ is a subset of $\widetilde{N}^m$ as well. Thus

$$I_m = [1,2]^n \times \{0\} \times \{2^{-m}\}$$

is a subset of N^m. Moreover the intersection of some $(n+2)$-dimensional box containing I_m with $\widetilde{X}$ equals I_m. More precisely we can assume that

$$\widetilde{X} \cap K_m = I_m \,,$$

where

$$K_m = [1,2]^n \times [-2^{-(m+2)}, 2^{-(m+2)}] \times [2^{-m} - 2^{-(m+2)}, 2^{-m} + 2^{-(m+2)}] \,.$$

Now for each m we will construct an L-bi-Lipschitz homeomorphism $\Phi_m : K_m \twoheadrightarrow K_m$ which is identity on the boundary ∂K_m, and with the constant L that does not depend on m. It follows that the mapping

$$\Phi(x) = \begin{cases} \Phi_m(x) & \text{if } x \in K_m, \ m = 6, 7, 8, \dots \\ x, & \text{otherwise}. \end{cases}$$

is bi-Lipschitz. Indeed, it is enough to observe that the norm of the derivative is bounded from below by a positive constant and bounded from above.

The set $\Phi_m(I_m)$ will be a subset of K_m with *wrinkles*. It will be a graph of a Lipschitz function φ_m constructed later. Thus the mapping Φ transforms each N^m into a manifold with *wrinkles* added to its surface. Note that $\Phi(N^\infty) = N^\infty$.

Let M be a closed n-dimensional manifold. According to Theorem 1.6 there is a mapping $f \in W^{1,n}(M, N^\infty)$ which cannot be approximated by Lipschitz mappings $\text{Lip}(M, N^\infty)$. Since $\Phi(N^\infty) = N^\infty$, we have $f \in W^{1,n}(M, \Phi(X))$. Our aim is to construct a bi-Lipschitz mapping $\Phi : \mathbb{R}^{n+2} \twoheadrightarrow \mathbb{R}^{n+2}$ of the form described above which has the property that there is $\varepsilon > 0$ such that every Lipschitz mapping $f_m : M \to \Phi(N^m)$ satisfies $\|f - f_m\|_{1,n} > \varepsilon$. This will readily imply that the mapping $f \in W^{1,n}(M, \Phi(X))$ cannot be approximated by Lipschitz mappings $\text{Lip}(M, \Phi(X))$. Note that f

can be approximated by Lipschitz mappings into N^m (see (7.1)), however, adding *wrinkles* to N^m by applying Φ will have the effect that mappings into $\Phi(N^m)$ stay away from f.

Now we will construct Lipschitz functions φ_m whose graphs will be the sets $\Phi(I_m)$.

Let $E_m = \{x \in K_m : x_{n+1} = 0\}$ be an $(n+1)$-dimensional box contained in K_m and let $E_m^i = \{x \in K_m : x_{n+1} = 0 \text{ and } x_n = 1 + i2^{-(m+10)}\}$ for $i = 0, 1, 2, \ldots, 2^{m+10}$. The set

$$F_m = \partial E_m \cup \bigcup_{i=0}^{2^{m+10}} E_m^i$$

is an n-dimensional subset of E_m. Let $\varphi_m : E_m \to \mathbb{R}$ be the function defined by the formula

$$\varphi_m(x) = \text{dist}(x, F_m) \qquad \text{for } x \in E_m.$$

Using notation (x', x^{n+1}, x^{n+2}), $x' \in \mathbb{R}^n$, for the coordinates in $\mathbb{R}^{n+2}$, we define the graph of φ_m by

$$G_m = \left\{ (x', \varphi_m(x', 0, x^{n+2}), x^{n+2}) : (x', 0, x^{n+2}) \in E_m \right\}.$$

G_m is a compact subset of K_m. The Lipschitz constant of φ_m is 1 as it is the distance function.

LEMMA 7.3.   *There is a constant $L > 1$ depending on n only such that for every integer $m = 6, 7, 8, \ldots$ there is an L-bi-Lipschitz homeomorphism $\Phi_m : K_m \twoheadrightarrow K_m$ which is identity on the boundary and which satisfies $\Phi_m(E_m) = G_m$.*

Proof. Let $E_m \times \mathbb{R} = \{(x', x^{n+1}, x^{n+2}) : (x', 0, x^{n+2}) \in E_m\}$. With this notation $K_m = E_m \times [-2^{-(m+2)}, 2^{-(m+2)}] \subset E_m \times \mathbb{R}$. The mapping $\widetilde{\Phi}_m : E_m \times \mathbb{R} \twoheadrightarrow E_m \times \mathbb{R}$ given by $\widetilde{\Phi}_m(x', x^{n+1}, x^{n+2}) = (x', x^{n+1} + \varphi_m(x', 0, x^{n+2}), x^{n+2})$ is L-bi-Lipschitz for some constant L that does not depend on m. Indeed, the inverse mapping is $\widetilde{\Phi}_m^{-1}(x', x^{n+1}, x^{n+2}) = (x', x^{n+1} - \varphi_m(x', 0, x^{n+2}), x^{n+2})$ and both $\widetilde{\Phi}_m$ and $\widetilde{\Phi}_m^{-1}$ have partial derivatives bounded by 1, because the Lipschitz constant of φ_m is 1. The mapping $\widetilde{\Phi}_m$ is identity on the part of the boundary of K_m (because φ_m is zero on ∂E_m), however, it is not identity on the entire boundary of K_m. Fortunately, one can easily use an argument similar to that employed in the proof of Lemma 5.4 and construct a mapping Φ_m of the form $\Phi_m(x', x^{n+1}, x^{n+2}) = (x', \xi_m(x', x^{n+1}, x^{n+2}), x^{n+2})$ being a modification of $\widetilde{\Phi}_m$ and satisfying the claim of Lemma 7.3. Indeed, one can write the formula for ξ_m explicitly making it linear as a function of x^{n+1} on each of the segments $[-2^{-(m+2)}, 0]$ and $[0, 2^{-(m+2)}]$. We leave easy details to the reader. $\square$

Lack of density. Now we can complete the proof of Theorem 1.7 by showing that $f \in W^{1,n}(M, \Phi(X))$ cannot be approximated by mappings $\text{Lip}(M, \Phi(X))$. As we have already observed it suffices to prove that there is no sequence $f_{m_k} \in \text{Lip}(M, \Phi(N^{m_k}))$, such that $\|f - f_{m_k}\|_{1,n} \to 0$ as $k \to \infty$. Let us briefly recall the construction of the mapping f (cf. the proof of Theorem 1.6; in that proof W plays the role of M). N^∞ is homeomorphic to S^n and $N^\infty \setminus \{0\}$ is diffeomorphic to S^n with one point removed. We choose a subset $\overline{B}$ of M diffeomorphic to a closed Euclidean ball. The mapping f maps $M \setminus B$ into one point $0 \neq x^* \in N^\infty$ and $f|_B : B \to N^\infty \setminus \{x^*\}$ is a homeomorphism. Moreover $f|_{B \setminus f^{-1}(0)} : B \setminus f^{-1}(0) \to N^\infty \setminus \{x^*, 0\}$ is a diffeomorphism. We can assume that $[1,2]^n \subset N^\infty \setminus \{x^*, 0\}$ as the point $0 \neq x^* \in N^\infty$ can be chosen arbitrarily. That means, applying a suitable change of variables in M, we can assume that $[1,2]^n \subset M$ and that $f : M \to N^\infty$ satisfies $f(x) = x$ for $x \in [1,2]^n$. Suppose that $f_{m_k} \in \text{Lip}(M, \Phi(N^{m_k}))$, $\|f - f_{m_k}\|_{1,n} \to 0$ as $k \to \infty$. Let us denote the coordinates in $[1,2]^n$ by

$$x = (\hat{x}, x^n) \in [1,2]^{n-1} \times [1,2] = [1,2]^n.$$

Fubini's theorem implies that for a suitable subsequence of f_{m_k} (also denoted by f_{m_k}) and for almost all $\hat{x} \in [1,2]^{n-1}$, the sequence of functions of one variable

$$[1,2] \ni x^n \mapsto f_{m_k}(\hat{x}, x^n) \tag{7.2}$$

converges to the function on one variable

$$[1,2] \ni x^n \mapsto f(\hat{x}, x^n) \tag{7.3}$$

in $W^{1,n}([1,2])$. In particular it implies that the functions converge uniformly and that the lengths of the curves (7.2) converge to the length of the curve (7.3) as $k \to \infty$. The length of the curve (7.3) equals 1, because f is identity on $[1,2]^n$. However, for k sufficiently large, the length of the curve (7.2) is at least $\sqrt{2}$, because it goes across the *wrinkles* $\Phi(I_{m_k})$. Indeed, the projection of the curve (7.2) onto the coordinate plane generated by coordinates (x^n, x^{n+1}) contains the projection of the set $\Phi(I_{m_k})$ on that plane. This projection is, however, isometric to the curve $[1,2] \ni t \mapsto (t, \text{dist}\,(t, S))$, where $S = \{1 + i2^{-(m+10)} : i = 0, 1, 2, \ldots, 2^{m+10}\}$ and the length of that curve equals $\sqrt{2}$. Since the orthogonal projection does not increase the length, we conclude that the length of the curve (7.2) is at least $\sqrt{2}$. This is a contradiction with the convergence of the length. The proof of Theorem 1.7 is complete. $\square$

REMARK. The homeomorphisms $\Phi_m : K_m \twoheadrightarrow K_m$ are not smooth. However, one can easily modify the construction in a way that the they are

diffeomorphism. Then the mapping $\Phi : \mathbb{R}^{n+2} \twoheadrightarrow \mathbb{R}^{n+2}$ is a diffeomorphism in $\mathbb{R}^{n+2} \setminus [1,2]^n$, i.e. it is a diffeomorphism outside a square of dimension n.

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Piotr Hajłasz, Department of Mathematics, University of Pittsburgh, 301 Thack-eray Hall, Pittsburgh, PA 15260, USA   `hajlasz@pitt.edu`