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Polynomial asymptotics and approximation of Sobolev functions

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Abstract. We prove several results concerning density of C_0^∞, behaviour at infinity and integral representations for elements of the space $L^{m,p} = \{f \mid \nabla^m f \in L^p\}$.

1. Introduction. It was O. Nikodym who first introduced Sobolev type spaces. They appeared in [9] under the name of Beppo Levi spaces. Today this name is reserved for spaces of the type $L^{m,p}(\mathbb{R}^n) = \{f \in \mathcal{D}'(\mathbb{R}^n) \mid \nabla^m f \in L^p\}$, also denoted by $BL_m(L^p(\mathbb{R}^n))$. However, an interest in spaces of this type really begun with the paper of Deny and Lions [4].

The space $L^{m,p}$ is equipped with a quasinorm $\|\nabla^m f\|_{L^p}$. It is well known that elements of $L^{m,p}$ are locally integrable with exponent p. However, they need not be p-integrable in the entire space $\mathbb{R}^n$. As an example, take any polynomial of degree less than m.

In this paper we prove several results concerning behaviour at infinity, approximation by C_0^∞ and integral representations for functions from the space $L^{m,p}$. We also deal with the space $W_{r,p}^m = L^r \cap L^{m,p}$.

The general framework of the subject and the problems discussed here are certainly not new. They have been developed in many directions (cf. [1]–[3], [6], [8], [11], [13]). The most comprehensive source is [3]. However, the approach presented in these papers is very technical, based upon complicated integral representations and singular integrals. For this reason the authors deal *only* with $1 < p < \infty$.

Our approach is more elementary, because it depends only on a Poincaré type inequality. We also cover the missing case $p = 1$. The Poincaré

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inequality was first used in a similar context by Iwaniec and Martin [5, Lemma 3.4].

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2. Notation. Let $\Omega \subset \mathbb{R}^n$ be an open set, m a positive integer and $1 \le p < \infty$. We define

$$W^{m,p}(\Omega) = \{f \in \mathcal{D}'(\Omega) \mid D^\alpha f \in L^p(\Omega) \text{ for } |\alpha| \le m\},$$
$$L^{m,p}(\Omega) = \{f \in \mathcal{D}'(\Omega) \mid D^\alpha f \in L^p(\Omega) \text{ for } |\alpha| = m\}.$$

The space $W^{m,p}(\Omega)$ with the norm $\|f\|_{W^{m,p}(\Omega)} = \sum_{|\alpha|\le m} \|D^\alpha f\|_{L^p(\Omega)}$ is a Banach space. The space $L^{m,p}(\Omega)$ is equipped with a quasinorm $\|f\|_{L^{m,p}(\Omega)} = \sum_{|\alpha|=m} \|D^\alpha f\|_{L^p(\Omega)}$, vanishing on all polynomials of degree less than m. Therefore, it induces a Banach norm on the quotient space $\dot{L}^{m,p}(\Omega) = L^{m,p}(\Omega)/\mathcal{P}^{m-1}$, where $\mathcal{P}^k$ denotes the space of polynomials of degree less than or equal to k. The quasinorm $\|\ \|_{L^{m,p}(\Omega)}$ is equivalent to the following:

$$\|\nabla^m f\|_{L^p(\Omega)} = \Big(\int_\Omega \Big(\sum_{|\alpha|=m} |D^\alpha f(x)|^2 \Big)^{p/2} dx \Big)^{1/p},$$

where $\nabla^m f$ denotes the vector field with components $D^\alpha f$, $|\alpha| = m$. Replacing L^p by L^p_{loc} we obtain the definitions of $W^{m,p}_{\text{loc}}(\Omega)$ and $L^{m,p}_{\text{loc}}(\Omega)$. It is well known (see [7, Th. 1.1.2]) that $L^{m,p}(\Omega) \subset W^{m,p}_{\text{loc}}(\Omega)$.

The symbol C_0 will stand for the space of continuous functions on $\mathbb{R}^n$ vanishing at infinity, which is a Banach space equipped with supremum norm. It is clear that C_0 is the closure of C_0^∞ in L^∞ norm.

We will also be concerned with two other Sobolev type spaces, namely $W^m_{r,p}(\Omega) = L^r(\Omega) \cap L^{m,p}(\Omega)$ with the norm $\|f\|_{W^m_{r,p}} = \|f\|_{L^r} + \|\nabla^m f\|_{L^p}$ (this is relevant to Nirenberg's multiplicative inequalities [10]) and $W^{m,p}_*(\Omega)$. The latter space is defined as follows: if $mp < n$ or $m = n$, $p = 1$, then the homogeneous Sobolev space is

$$W^{m,p}_*(\Omega) = \bigcap_{k=0}^{m} L^{k,p_k^*}(\Omega),$$

where $p_k^* = np/(n-(m-k)p)$, under the convention that $np/0 = \infty$. The norm in this space is given by

$$\|f\|_{W^{m,p}_*(\Omega)} = \sum_{k=0}^{m} \|\nabla^k f\|_{L^{p_k^*}(\Omega)}.$$

Obviously, $W^m_{r,p}(\Omega)$ and $W^{m,p}_*(\Omega)$ are Banach spaces. For notational simplicity we write $p_0^* = p^*$ in the case $k = 0$.

Also, if $\Omega = \mathbb{R}^n$ the domain Ω will be suppressed in our notation. We will often use the cut-off functions $\eta \in C_0^\infty(B^n(2))$, $\eta \geq 0$, $\eta|_{B^n(1)} \equiv 1$ and $\eta_R(x) = \eta(x/R)$, for a pair of concentric balls $B^n(R) \subset B^n(2R)$. Clearly, $|D^\alpha \eta_R| \leq CR^{-|\alpha|}$ and $\operatorname{supp} D^\alpha \eta_R \subset \{x \mid R \leq |x| \leq 2R\}$ for $|\alpha| > 0$.

In the sequel the letter C denotes a constant which may change from line to line.

Our basic tool is the following Poincaré type inequality (see e.g. [7, Th. 1.1.11]):

THEOREM 1. *If Ω is a bounded (connected) domain with the cone property and $\varphi \in C_0^\infty(\Omega)$ with $\int_\Omega \varphi(x)\,dx = 1$, then every function $f \in L^{m,p}(\Omega)$, $1 \leq p < \infty$, satisfies the inequality*

$$\|f - P^{m-1}f\|_{W^{m,p}(\Omega)} \leq C\|\nabla^m f\|_{L^p(\Omega)},$$

where $P^{m-1}f \in \mathcal{P}^{m-1}$ is the polynomial given by

$$P^{m-1}f(x) = \int_\Omega \sum_{|\alpha| \leq m-1} D_y^\alpha \left(\varphi(y)\frac{(y-x)^\alpha}{\alpha!}\right) f(y)\,dy.$$

The constant C does not depend on f.

Remark. Domains with Lipschitz boundary, like a ball, an annulus $\{R_1 \leq |x| \leq R_2\}$ or a cube have the cone property.

In addition we will appeal to the classical Sobolev imbedding theorem (see e.g. [7, Th. 1.4.5]).

THEOREM 2. *If $1 \leq p < \infty$ and either $mp < n$ or $m = n$, $p = 1$, and if Ω is a bounded domain with the cone property or an infinite cone, then the space $W^{m,p}(\Omega)$ is continuously imbedded in $L^{p^*}(\Omega)$.*

In particular, we have

COROLLARY 1. *Suppose m, n, p are as in Theorem 2 and $f \in W^{m,p}$. Then*

$$\|f\|_{W_*^{m,p}} \leq C\|\nabla^m f\|_{L^p},$$

where the constant C does not depend on f.

The last prerequisite is the following representation formula (cf. [7, Th. 1.1.10/2]).

THEOREM 3. *For every $\phi \in C_0^\infty$ we have*

$$\phi = \sum_{|\alpha|=m} K_\alpha * D^\alpha \phi,$$

where $K_\alpha(x) = \frac{m}{n\omega_n \alpha!}\frac{x^\alpha}{|x|^n}$, and ω_n denotes the volume of the unit ball.

3. Density results for $L^{m,p}$. Throughout this section approximation in $L^{m,p}$ is understood with respect to the quasinorm $\| \ \|_{L^{m,p}}$.

THEOREM 4. *Let $1 \le p < \infty$ and $m = 1, 2, \ldots$ The subspace C_0^∞ is dense in $L^{m,p}$ if and only if either $n > 1$ or $p > 1$.*

Remark. The case $p > 1$ has been previously solved by Sobolev [13], [14] (see also [3]).

Proof of Theorem 4. First we will construct a function $f \in L^{m,1}(\mathbb{R})$ which cannot be approximated by smooth, compactly supported functions. Let f be such that $f^{(m)} = \phi$ (mth derivative), where $\phi \in C_0^\infty(\mathbb{R})$, $\int_{\mathbb{R}} \phi \neq 0$. Now assuming that $\psi_k \in C_0^\infty$, $\psi_k^{(m)} \to f^{(m)} = \phi$ in L^1 leads to a contradiction, since $0 = \int_{\mathbb{R}} \psi_k^{(m)} \to \int_{\mathbb{R}} \phi \neq 0$.

Next, we prove that if $1 < p < \infty$, then C_0^∞ is dense in $L^{m,p}(\mathbb{R})$.

LEMMA 1. *If $p > 1$, $f_0 \in L^p(\mathbb{R})$, and $f_{k+1}(x) = \int_0^x f_k(t)\,dt$, then $f_k(x)|x|^{-k} \in L^p(\mathbb{R})$ for $k = 0, 1, 2, \ldots$*

Proof. The assertion follows by induction and the Hardy inequality (see e.g. [15]).

Let $f \in L^{m,p}(\mathbb{R})$. Approximating f by convolution with standard mollifiers we can assume that $f \in C^\infty \cap L^{m,p}$. Set $F_0 = f^{(m)}$ and $F_{k+1} = \int_0^x F_k(t)\,dt$. Our goal is to show that $F_m \eta_R \to f$ in $L^{m,p}$ as $R \to \infty$.

Applying Leibniz's formula to $(F_m \eta_R)^{(m)}$ it suffices to prove that $F_m^{(m)} \eta_R \to f^{(m)}$ in L^p and $\eta_R^{(k)} F_k \to 0$ in L^p for $k = 1, \ldots, m$. The first convergence is clear. The second one follows from the estimate

$$\begin{aligned} \|\eta_R^{(k)} F_k\|_{L^p(\mathbb{R})} &\le C R^{-k} \|F_k\|_{L^p(R \le |x| \le 2R)} \\ &\le 2^k C \|F_k(x)|x|^{-k}\|_{L^p(R \le |x| \le 2R)} \to 0 \quad \text{as } R \to \infty. \end{aligned}$$

It remains to show that if $n \ge 2$ and $1 \le p < \infty$, then every $f \in L^{m,p}(\mathbb{R}^n)$ can be approximated by functions from C_0^∞. As before, we can assume that $f \in C^\infty \cap L^{m,p}$. By Theorem 1 applied to the annulus $\{x \mid 1 \le |x| \le 2\}$ there exists a polynomial $P_1 f$ such that

$$\|D^\alpha(f - P_1 f)\|_{L^p(1 \le |x| \le 2)} \le C \|\nabla^m f\|_{L^p(1 \le |x| \le 2)}$$

for $f \in L^{m,p}(\{x \mid 1 \le |x| \le 2\})$ and $|\alpha| \le m$ (the construction fails when $n = 1$, because $\{x \mid 1 \le |x| \le 2\}$ is not connected). By a simple rescaling argument we obtain the analogous inequality in the annulus $\{x \mid R \le |x| \le 2R\}$:

$$\|D^\alpha(f - P_R f)\|_{L^p(R \le |x| \le 2R)} \le C R^{m-|\alpha|} \|\nabla^m f\|_{L^p(R \le |x| \le 2R)}.$$

We will prove that $(f - P_R f)\eta_R \to f$ in $L^{m,p}$ as $R \to \infty$. According to Leibniz's formula it is enough to show that

$$D^\beta (f - P_R f) D^\gamma \eta_R \to 0 \quad \text{in } L^p \text{ as } R \to \infty,$$

for $|\beta + \gamma| = m$, $|\gamma| \geq 1$. We have

$$\begin{aligned} \|D^\beta (f - P_R f) D^\gamma \eta_R\|_{L^p} &\leq C R^{-|\gamma|} \|D^\beta (f - P_R f)\|_{L^p(R \leq |x| \leq 2R)} \\ &\leq C R^{-|\gamma|} R^{m-|\beta|} \|\nabla^m f\|_{L^p(R \leq |x| \leq 2R)} \\ &= C \|\nabla^m f\|_{L^p(R \leq |x| \leq 2R)} \to 0 \quad \text{as } R \to \infty. \end{aligned}$$

Remarks. 1) The above theorem might be useful in the L^p theory of Hodge decomposition. For example, Lemma 3.4 of [5] follows directly from Theorem 4. In fact, our approach via the Poincaré inequality is similar to that of [5, Lemma 3.4].

2) The same arguments work if Ω is an infinite cone but instead of C_0^∞ we must take smooth functions in Ω with bounded support.

4. Imbedding theorems

4.1. *The case $mp < n$*

THEOREM 5. *Let $mp < n$ and $1 \leq p < \infty$. Then for every $f \in L^{m,p}$ there exists exactly one polynomial $P^{m-1} f \in \mathcal{P}^{m-1}$ such that $f - P^{m-1} f \in W_*^{m,p}$ and*

$$\|f - P^{m-1} f\|_{W_*^{m,p}} \leq C \|\nabla^m f\|_{L^p}. \tag{1}$$

Moreover,

$$P^{m-1} f = f - \sum_{|\alpha|=m} K_\alpha * D^\alpha f$$

with K_α as in Theorem 3.

Remark. In the case $p > 1$ the inequality (1) has already been obtained by Sedov [11] (see also [3, Th. 14.4]).

Proof of Theorem 5. The uniqueness part is evident. Let $\phi_n \in C_0^\infty$, $\phi_n \to f$ in $L^{m,p}$ (see Theorem 4). By Corollary 1 applied to $\phi_n - \phi_m$, we see that ϕ_n converges in $W_*^{m,p}$ to a function u. Clearly, $D^\alpha u = D^\alpha f$ for $|\alpha| = m$. Thus $u = f - P^{m-1} f$ for some polynomial $P^{m-1} f \in \mathcal{P}^{m-1}$. Applying again Corollary 1 to $\{\phi_n\}_n$ and letting n go to infinity we obtain the desired inequality

$$\|f - P^{m-1} f\|_{W_*^{m,p}} \leq C \|\nabla^m f\|_{L^p}.$$

It remains to show that $u = \sum_{|\alpha|=m} K_\alpha * D^\alpha f$. By Theorem 3 we have

$$\phi_k = \sum_{|\alpha|=m} K_\alpha * D^\alpha \phi_k.$$

Let $\psi \in C_0^\infty$. Since $|K_\alpha(x)| \le C|x|^{m-n}$, it follows that $\overline{K}_\alpha * \psi \in L^{p'}$, where $1/p + 1/p' = 1$, $\overline{K}_\alpha(x) = K_\alpha(-x)$. Thus, by the Fubini Theorem,

$$(\phi_k, \psi) = \sum_{|\alpha|=m} \int_{\mathbb{R}^n} D^\alpha \phi_k(y)(\overline{K}_\alpha * \psi)(y)\, dy.$$

Passing to the limit as $k \to \infty$ we arrive at the formula

$$(u, \psi) = \sum_{|\alpha|=m} \int_{\mathbb{R}^n} D^\alpha f(y)(\overline{K}_\alpha * \psi)(y)\, dy = \Big(\sum_{|\alpha|=m} K_\alpha * D^\alpha f, \psi \Big),$$

which completes the proof, since ψ was taken arbitrarily.

Remark. An analogous statement holds if Ω is an infinite cone. In this case, instead of Theorem 3, one uses the representation formula from [12, Th. 5.3] for $C^\infty(\Omega)$-functions with bounded support. The formula applied to the family of operators $P_\alpha f = D^\alpha f$.

COROLLARY 2. *If $mp < n$ and $p > 1$, then $W_*^{m,p}$ coincides with the space of Riesz potentials*

$$I_m f(x) = \int_{\mathbb{R}^n} f(y)|x-y|^{m-n}\, dy$$

for all $f \in L^p(\mathbb{R}^n)$.

Remark. This theorem has been established by Lizorkin [6].

Proof of Corollary 2. The standard application of Marcinkiewicz's Multiplier Theorem implies that the space of Riesz potentials is equal to the closure of C_0^∞ in the norm $\|g\|_{L^{p^*}} + \|\nabla^m g\|_{L^p}$. It follows from Theorems 4 and 5 that C_0^∞ is dense in $W^m_{p^*,p}$. This completes the proof.

4.2. *The case $m = n$, $p = 1$.* As we will see this case is more subtle than that of $mp < n$. Note that $W_*^{n,1} \cap C_0$ is a closed subspace of $W_*^{n,1}$, because $W_*^{n,1} \subset L^\infty$.

THEOREM 6. *Let $f \in L^{n,1}$.*

(i) *If $n > 1$, then there exists a unique polynomial $P^{n-1} f \in \mathcal{P}^{n-1}$ such that $f - P^{n-1} f \in W_*^{n,1} \cap C_0$ and*

$$\|f - P^{n-1} f\|_{W_*^{n,1}} \le C \|\nabla^n f\|_{L^1}.$$

Moreover,

$$P^{m-1} f = f - \sum_{|\alpha|=m} K_\alpha * D^\alpha f.$$

(ii) *If* $n = 1$, *then*

$$\|f - f(y)\|_{W_*^{1,1}} \leq 2\|f'\|_{L^1},$$

for any fixed $y \in \mathbb{R}$.

Remarks. 1) Since $W^{1,1}(\mathbb{R})$ consists of continuous functions, it follows that the value of f at any point is well defined.

2) Note that in the case $n = 1$ we do not get an imbedding into $W_*^{1,1} \cap C_0$. A smooth function f such that $f(x) = 1$ for $x > 1$ and $f(x) = 0$ for $x < 0$ belongs to $L^{1,1}(\mathbb{R})$, while $f - C$ does not belong to C_0 for any constant C.

Proof of Theorem 6. The result for $n > 1$ is obtained in much the same way as in the case $mp < n$. The case $n = 1$ follows from the simple estimate

$$|f(x) - f(y)| = \Big| \int_{\min\{x,y\}}^{\max\{x,y\}} f'(t)\, dt \Big| \leq \int_{\mathbb{R}} |f'(t)|\, dt.$$

4.3. *Polynomial asymptotics at infinity.* Theorems 5 and 6 state that if either $mp < n$, or $m = n > 1$ and $p = 1$, then every function f from $L^{m,p}$ has a polynomial behaviour at infinity in the sense that there exists a polynomial $P \in \mathcal{P}^{m-1}$ such that $f - P$ belongs to a certain L^r space or to C_0.

In the case $m = n = p = 1$ we know that f is bounded (Theorem 6), but we have no imbedding in C_0, as follows from the example given in the remark after Theorem 6.

The following examples show that in all other cases there exist functions in $L^{m,p}$ without polynomial behaviour at infinity in any reasonable sense.

EXAMPLE 1 (The case $mp > n$ and $1 \leq p < \infty$). Any smooth function f such that $f(x) = |x|^\varepsilon$ for $|x| > 1$ (where $1 > \varepsilon > 0$ satisfies $(m - \varepsilon)p > n$) belongs to $L^{m,p}$. In this case $\lim_{x\to\infty} |f(x) - P(x)| = \infty$ for any polynomial P.

EXAMPLE 2 (The case $mp = n$ and $p > 1$). Any smooth function such that $f(x) = \log\log|x|$ for $|x| > e$ is a member of $L^{m,p}$. In this case $\lim_{x\to\infty} |f(x) - P(x)| = \infty$ for any polynomial P.

5. Density results for $W^m_{r,p}$

THEOREM 7. *If* $1 \leq p, r < \infty$, *then* C_0^∞ *is dense in* $W^m_{r,p}$.

Remark. For $1 < r, p < \infty$ this result was already known in [3, Th. 14.14].

Proof of Theorem 7. Let $f \in W^m_{r,p}$. As before, it can be assumed that $f \in C^\infty \cap W^m_{r,p}$. Clearly, $f\eta_R \to f$ in L^r as $R \to \infty$. We will prove that $f\eta_R \to f$ in $L^{m,p}$ as $R \to \infty$.

First assume that $mp < n$. It follows from Theorem 5 that $\|f\|_{W^{m,p}_*} \leq C\|\nabla^m f\|_{L^p}$. Let α and β be multiindices such that $|\alpha| = k \geq 1$ and $|\beta| = m - k$. Since $D^\beta f \in L^{p^*_{m-k}}$, by Hölder's inequality, we obtain

$$\begin{aligned}\|D^\alpha \eta_R D^\beta f\|_{L^p} &\leq \frac{C}{R^k}\|\chi_{\{R<|x|<2R\}} D^\beta f\|_{L^p} \\ &\leq \|D^\beta f\|_{L^{p^*_{m-k}}(R<|x|<2R)} \to 0 \quad \text{as } R \to \infty.\end{aligned}$$

This implies the desired convergence.

Assume now that $mp \geq n$. We distinguish between two cases: $n = 1$ and $n \geq 2$.

Case $n \geq 2$. It follows from the proof of Theorem 4 that

$$(f - P_R f)\eta_R \to f \quad \text{in } L^{m,p} \text{ as } R \to \infty,$$

where $P_R f$ are the polynomials from the proof of Theorem 4. Therefore, it remains to prove that $(P_R f)\eta_R \to 0$ in $L^{m,p}$.

Recall that $P_R f$ was obtained from $P_1 f$ by a rescaling argument, where $P_1 f$ is defined in Theorem 1 and depends on the choice of a function φ supported in $\{x \mid 1 \leq |x| \leq 2\}$. Hence, we have the explicit formula,

$$P_R f(x) = \sum_{|\alpha| \leq m-1} \left(\frac{x}{R}\right)^\alpha \int_{\mathbb{R}^n} \psi_\alpha(y) f(Ry)\,dy,$$

where $\psi_\alpha \in C_0^\infty(\{1 \leq |x| \leq 2\})$ depends on φ only.

Let $|\beta| = m$. We have to prove that $D^\beta((P_R f)\eta_R) \to 0$ in L^p. It suffices to show that $D^\gamma(P_R f) D^\delta \eta_R \to 0$, whenever $\gamma + \delta = \beta$. If $\gamma = \beta$, then $D^\gamma(P_R f) = 0$, so we can assume that $|\delta| \geq 1$. We have

$$\|D^\gamma(P_R f) D^\delta \eta_R\|_{L^p} \leq C R^{-|\delta|} \|D^\gamma(P_R f)\|_{L^p(R \leq |x| \leq 2R)}.$$

We need only estimate each of the monomials of $P_R f$. The problem reduces to showing that the quantity

$$I_R = R^{-(|\delta|+|\alpha|)} \|x^{\alpha-\gamma}\|_{L^p(R \leq |x| \leq 2R)} \left| \int \psi_\alpha(y) f(Ry)\,dy \right|$$

tends to zero as $R \to \infty$. We can assume that $\alpha \geq \gamma$. Note that

$$\|x^{\alpha-\gamma}\|_{L^p(R \leq |x| \leq 2R)} \leq C R^{|\alpha|-|\gamma|} R^{n/p}.$$

Hence, denoting $\{x \mid R \leq |x| \leq 2R\}$ by Ω_R, we have

$$I_R \le CR^{n/p-m} \int_{\Omega_1} |f(Ry)|\,dy = CR^{n/p-m-n} \int_{\Omega_R} |f(y)|\,dy$$
$$\le CR^{n/p-m-n} R^{n(1-1/r)} \|f\|_{L^r(\Omega_R)} \to 0 \quad \text{as } R \to \infty,$$

because the exponent of R is negative.

In the case $n = 1$ the proof is similar, with a slight difference: there is no Poincaré inequality (Theorem 1) for the one-dimensional annulus $\{x \mid 1 \le |x| \le 2\}$, but we can use the Poincaré inequality twice, applied to the intervals $[-2, -1]$ and $[1, 2]$.

Remarks. 1) It is easy to see that if $r = \infty$ or $p = \infty$, then C_0^∞ is not dense in $W^m_{r,p}$.

2) It follows from the above arguments that C_0^∞ is dense in $W^m_{\infty,p} \cap C_0$.

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