Équations aux dérivées partielles/*Partial Differential Equations*
(Analyse mathématique/*Mathematical Analysis*)

Sobolev meets Poincaré

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Abstract – We prove that a very weak form of the Poincaré inequality implies a Sobolev-Poincaré inequality in the abstract setting of metric spaces.

Sobolev rencontre Poincaré

Résumé – *Nous démontrons qu'une forme très faible d'inégalité de Poincaré implique une inégalité de Sobolev-Poincaré dans la situation abstraite des espaces métriques.*

Version française abrégée – Le but de cette Note est la démonstration du théorème 1 qui affirme que, dans le cadre très général des espaces métriques, une inégalité faible de Poincaré entraîne une inégalité de Sobolev-Poincaré. Ce résultat – dont la preuve est élémentaire – a un certain nombre d'applications; parmi ses corollaires simples on trouve le théorème de Saloff-Coste [1], theorem 2.1 et une caractérisation utile des *poids p-admissibles* [2]. De plus, il y a des relations intéressantes du théorème 1 avec les résultats récents de Jerrison [3] et Franchi, Gutiérrez et Wheeden [4].

Soit X un espace métrique. Nous disons que $\Omega \subset X$ remplit la condition de chaîne $C(\lambda, M)$ avec $\lambda, M \geqq 1$ s'il existe une boule fixée $B_0 \subset \Omega$ telle que pour chaque $x \in \Omega$ on peut trouver une suite de boules $B_0, B_1, B_2, \ldots$ avec les trois propriétés suivantes.

1. $\lambda B_i \subset \Omega$ pour $I = 0, 1, 2, \ldots$ et B_i est centrée en x pour tout i suffisamment grand.
2. Pour $i \geqq 0$, B_i est de rayon r_i, $M^{-1}(\operatorname{diam}\Omega)\, 2^{-i} \leqq r_i \leqq M(\operatorname{diam}\Omega)\, 2^{-i}$.
3. Pour tout $i \geqq 0$, il y a une boule $R_i \subset B_i \cap B_{i+1}$ telle que $B_i \cup B_{i+1} \subset MR_i$.

Théorème 1. – *Soit $\Omega \in C(\lambda, M)$ un sous-ensemble d'un espace métrique X. Admettons que la mesure μ définie sur X a la propriété de doublement, $\mu(2B) \leqq C_d\, \mu(B)$; $B = B(x, r)$, $x \in \Omega$, et $r < 5 \operatorname{diam}\Omega$. Admettons encore que pour des fonctions $g > 0$, $g \in L^p$, $0 < p < \infty$, $u \in L^1_{\mathrm{loc}}(\Omega, \mu)$ la version abstraite de l'inégalité faible de Poincaré,*

$$\fint_B |u - u_B|\, d\mu \leqq C_1\, r \left(\fint_{\lambda B} g^p\, d\mu \right)^{1/p},$$

soit vérifiée pour chaque boule B telle que $\lambda B \subset \Omega$. Alors il existe $k > 1$ qui ne dépend que de p et C_d, et une constante $C_2 = C_2(C_1, C_d, p, k, \lambda, M)$ telle qu'on ait l'inégalité globale de Sobolev-Poincaré suivante

$$\left(\fint_\Omega |u - u_\Omega|^{kp}\, d\mu \right)^{1/kp} \leqq C_2 (\operatorname{diam}\Omega) \left(\fint_\Omega g^p\, d\mu \right)^{1/p}.$$

Si $kp < 1$, on remplace u_Ω par u_{B_0}.

Main result. – The purpose of this Note is the proof of theorem 1 which, roughly speaking, states that, in the very general setting of metric spaces, a weak Poincaré inequality

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implies a Sobolev-Poincaré inequality. This result has a number of applications and its proof is surprisingly elementary. In particular, the theorem of Saloff-Coste ([1], theorem 2.1) is a very special case of theorem 1; also it provides an elementary characterization of p-admissible weights [2] and it simplifies and extends some of the results by Jerison [3], and Franchi, Gutiérrez and Wheeden [4].

Let *X* be a metric space. We say that $\Omega \subset X$ satisfies the chain condition $C(\lambda, M)$, where $\lambda, M \geqq 1$, if there exists a distinguished ball $B_0 \subset \Omega$ such that for every $x \in \Omega$ there exists an infinite sequence of balls $B_0, B_1, B_2, \ldots$ (called "chain") with the following properties.

1. $\lambda B_i \subset \Omega$ for $i = 0, 1, 2, \ldots$ and B_i is centered at x for all sufficiently large i.
2. For $i \geqq 0$ the radius r_i of B_i satisfies $M^{-1}(\operatorname{diam}\Omega)\,2^{-i} \leqq r_i \leqq M(\operatorname{diam}\Omega)\,2^{-i}$.
3. For every $i \geqq 0$ there is a ball $R_i \subset B_i \cap B_{i+1}$ such that $B_i \cup B_{i+1} \subset MR_i$.

Here and in what follows, by *B* we always denote a ball and by *tB*, where $t > 0$, a ball concentric with *B* and with radius t times that of *B*. By *C* we denote a general constant which can change its value even in a single line. The above chain condition is different from the commonly used Boman's chain condition (*cf.* [4]).

If $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary then it satisfies the $C(\lambda, M)$ condition for all $\lambda \geqq 1$. The following lemma and its corollary provide us with more sophisticated examples.

LEMMA 1. – *Let (X, d) be a metric space such that bounded and closed sets are compact. Assume that the metric d has the property that for every two points $a, b \in X$ the distance $d(a, b)$ is equal to the infimum of the lengths of continuous curves that join a and b (in particular we assume that such a curve always exists). Then there exists a shortest path γ from a to b. This curve has the following segment property. For every $z \in \gamma$, $d(a, b) = d(a, z) + d(z, b)$.*

This lemma is due to Busemann [5], p. 25 (*cf.* [4], p. 592).

COROLLARY 1. – *Fix $\lambda \geqq 1$. Let the metric space (X, d) fulfill the hypothesis of the lemma above. Then every ball $B \subset X$ satisfies the $C(\lambda, M)$ condition with a certain M which depends on the choice of λ.*

The main result of this Note reads as follows.

THEOREM 1. – *Let $\Omega \subset X, \Omega \in C(\lambda, M)$. Assume that μ is a doubling measure: $\mu(2B) \leqq C_d\,\mu(B)$ whenever $B = B(x, r)$, $x \in \Omega$, $r \leqq 5\operatorname{diam}\Omega$. Assume that $g > 0$, $g \in L^p(\Omega, \mu)$, $0 < p < \infty$, $u \in L^1_{\text{loc}}(\Omega, \mu)$ are such that the following abstract version of the local weak Poincaré inequality holds*:

$$(1) \qquad \fint_B |u - u_B|\,d\mu \leqq C_1\, r \left(\fint_{\lambda B} g^p\, d\mu\right)^{1/p}$$

whenever $\lambda B \subset \Omega$ and r is the radius of the ball B. Then there exists $k > 1$ which depends on p and the doubling constant C_d only, and $C_2 = C_2(C_1, C_d, p, k, \lambda, M)$ such that the following global Sobolev-Poincaré inequality holds

$$(2) \qquad \left(\fint_\Omega |u - u_\Omega|^{kp}\, d\mu\right)^{1/kp} \leqq C_2\,(\operatorname{diam}\Omega)\left(\fint_\Omega g^p\, d\mu\right)^{1/p}.$$

If $kp < 1$, we replace u_Ω by u_{B_0}.

Here and in what follows $u_K = \fint_K u\, d\mu = \mu(K)^{-1}\int_K u\, d\mu$.

Remark. – If we know in addition that $\delta_0 = \log_2 C_d \leqq p$, then we can prove more. Namely, for $\delta_0 = p$, we get exponential integrability, and, for $\delta_0 < p$, Hölder continuity of u. Moreover, there is a two-weighted version of the above theorem where integration in the different sides in (1) and (2) is with respect to different measures, *see* [6].

Proof. – It suffices to assume $u_{B_0} = 0$ and estimate $\left(\fint_\Omega |u|^{kp}\right)^{1/kp}$. Let $x \in A_t = \{|u| > t\}$ be a Lebesgue point of u. Let $B_0, B_1, B_2, \ldots$ be a chain assigned to x. We have $|u_{B_j}| \to |u(x)| > t$, $u_{B_0} = 0$. Using the doubling property and Poincaré inequality (1) we compute

$$t \leqq |u(x) - u_{B_0}| \leqq \sum_{i=0}^{\infty} (|u_{B_i} - u_{R_i}| + |u_{B_{i+1}} - u_{R_i}|)$$
$$\leqq C \sum_{i=0}^{\infty} \fint_{B_i} |u - u_{B_i}|\, d\mu \leqq C \sum_{i=0}^{\infty} r_i \left(\fint_{\lambda B_i} g^p\, d\mu\right)^{1/p}.$$

Let $\varepsilon > 0$. Then

$$\sum_{i=0}^{\infty} r_i \left(\fint_{\lambda B_i} g^p\, d\mu\right)^{1/p} \geqq C\, t = C\, t \sum_{i=0}^{\infty} 2^{-i\varepsilon} \geqq C\, t\, (\operatorname{diam}\Omega)^{-\varepsilon} \sum_{i=0}^{\infty} r_i^{\varepsilon}.$$

Evidently, there exists a term in the first sum which is greater than or equal to the corresponding term in the latter sum. Combining this with the fact that for certain *C*, which does not depend on i, $\lambda B_i \subset B(x, Cr_i)$, and using the doubling property we conclude that there exists $r_x > 0$ with

$$(\operatorname{diam}\Omega)^{p\varepsilon} \int_{B(x, r_x) \cap \Omega} g^p\, d\mu \geqq Ct^p\, r_x^{p(\varepsilon - 1)}\, \mu(B(x, r_x)).$$

It is well known that the iteration of the doubling condition implies that $\mu(B(x, r)) \geqq 2^{-\delta}\, \mu(\Omega)\, (r/\operatorname{diam}\Omega)^{\delta}$ for all $\delta \geqq \delta_0 = \log_2 C_d$, $x \in \Omega$ and $r < \operatorname{diam}\Omega$. We can assume that $0 < \varepsilon < 1$, so $p(\varepsilon - 1) < 0$. Then

$$\mu(\Omega)^{p(\varepsilon-1)/\delta} (\operatorname{diam}\Omega)^p \int_{B(x, r_x) \cap \Omega} g^p\, d\mu \geqq Ct^p\, \mu(B(x, r_x))^{1 + (p(\varepsilon - 1)/\delta)}. \tag{3}$$

Applying a Vitali type lemma we obtain a collection of pairwise disjoint balls $B_{x_i} = B(x_i, r_{x_i})$, $i = 1, 2, 3, \ldots$, with $A_t \subset \bigcup_{i=1}^{\infty} 5\, B_{x_i}$. We can assume that $\delta > p$ and hence $0 < (\delta + p(\varepsilon - 1))/\delta < 1$. Now the doubling property and (3) give

$$\mu(A_t)^{1 + (p(\varepsilon-1)/\delta)} \leqq C \sum_{i=1}^{\infty} \mu(B_{x_i})^{1 + (p(\varepsilon - 1)/\delta)}$$
$$\leqq Ct^{-p}\, L \sum_{i=1}^{\infty} \int_{B_{x_i} \cap \Omega} g^p\, d\mu \leqq Ct^{-p}\, L \int_\Omega g^p\, d\mu,$$

where $L = (\operatorname{diam}\Omega)^p\, \mu(\Omega)^{p(\varepsilon - 1)/\delta}$. This is a weak type estimate. Now the theorem follows by a standard argument (which goes back to Marcinkiewicz) involving integration with respect to t.

The method used in the above proof can be easily modified to obtain an imbedding theorem in domains with very irregular boundaries [7].

Applications. – Consider the following degenerate equation in $\Omega \subset \mathbb{R}^n$

$$\operatorname{div} A(x, \nabla u) = 0, \tag{4}$$

where $A(x, \xi) . \xi \geqq C_1 \omega(x) |\xi|^p$, $|A(x, \xi)| \leqq C_2 \omega(x) |\xi|^{p-1}$ and $\omega > 0$, $\omega \in L^1_{\text{loc}}(\Omega)$. Weak solutions are defined in the weighted Sobolev space $W^{1,p}_{\text{loc}}(\Omega, \omega)$. In order to extend Moser's technique to the case of equation (4) one needs to put some conditions on ω. These conditions are listed in the following definition, *see* [2].

Definition. – We say that $\omega \in L^1_{\text{loc}}(\mathbb{R}^n)$, $\omega > 0$ a.e. is p-admissible, $1 < p < \infty$, if the measure defined by $d\mu = \omega(x)\, dx$ satisfies the following four conditions:

1. (Doubling condition) $\mu(2B) \leqq C_1 \mu(B)$ for all balls $B \subset \mathbb{R}^n$.
2. (Uniqueness condition) If Ω is an open subset of $\mathbb{R}^n$ and $\varphi_i \in C^\infty(\Omega)$ is a sequence such that $\int_\Omega |\varphi_i|^p\, d\mu \to 0$ and $\int_\Omega |\nabla \varphi_i - v|^p\, d\mu \to 0$, where $v \in L^p(\mu)$, then $v \equiv 0$.
3. (Sobolev inequality) There exists a constant $k > 1$ such that for all balls $B \subset \mathbb{R}^n$ and all $\varphi \in C_0^\infty(B)$
$$\left(\fint_B |\varphi|^{kp}\, d\mu \right)^{1/kp} \leqq C_2\, r \left(\fint_B |\nabla \varphi|^p\, d\mu \right)^{1/p}.$$
4. (Poincaré inequality) If $B \subset \mathbb{R}^n$ is a ball and $\varphi \in C^\infty(B)$, then
$$\int_B |\varphi - \varphi_B|^p\, d\mu \leqq C_3\, r^p \int_B |\nabla \varphi|^p\, d\mu.$$

Moser's technique extends to the case when ω is p-admissible weight, and one also obtains a rich potential theory, *see* [2].

As an application of theorem 1 we have the following characterization.

Theorem 2. – *Let $\omega > 0$ be a locally integrable function. Then the weight ω is p-admissible if and only if the measure μ associated with ω is doubling* (i.e., $\mu(2B) \leqq C_1 \mu(B)$ *for all balls* $B \subset \mathbb{R}^n$) *and*

$$\fint_B |u - u_B|\, d\mu \leqq C_2\, r \left(\fint_{2B} |\nabla u|^p\, d\mu \right)^{1/p},$$

whenever B is a ball with radius r and $u \in C^\infty(2B)$.

Proof. – Evidently the conditions stated in the theorem are necessary. For sufficiency notice first that the Poincaré and Sobolev inequalities follow from theorem 1. We are left with the uniqueness property. This has been shown by Semmes (*cf.* [8], [6]).

Our second application concerns the results of Jerison [3], Saloff-Coste [1], and Franchi, Gutiérrez and Wheeden [4]. If L is a subelliptic operator on a manifold as in [1] or if L is a strongly degenerated Grushin type operator considered in [4], then there is a certain metric ρ and a gradient ∇_L canonically associated with the operator L. This metric induces the standard topology. The metric is defined as an infimum of the length of subunit curves and hence corollary 1 applies. Also, very often, we have a natural doubling measure μ associated with L (doubling with respect to the metric ρ) (*cf.* [3], [9], [1], [4]). Hence given such a measure μ and metric ρ, theorem 1 applies when Ω is an arbitrary ball. In such a special case, under the assumption that L is a subelliptic operator on a smooth manifold, has real C^∞ coefficients, is formally self adjoint with respect to μ, and $p = 2$, the fact that a Poincaré inequality, with $g = |\nabla_L u|$, implies a Sobolev inequality was proved by Saloff-Coste ([1], theorem 2.1). His result is an immediate consequence of theorem 1. Moreover, even in this setting, theorem 1 is more general than Saloff-Coste's result.

Theorem 1 also directly applies to Grushin type operators considered by Franchi, Gutiérrez and Wheeden in [4] and gives some extensions of their results. Moreover, theorem 1 gives a new, simpler proof of a result of Jerison [3] according to which a strong form of the Poincaré inequality follows from its weak form.

FURTHER RESULTS. – In theorem 1 we put the geometric condition $C(\lambda, M)$ on the metric. If we do not make any assumptions concerning the metric, then we arrive at a weaker version of theorem 1 which roughly speaking reads as follows. If the measure μ is doubling and (1) holds in every ball $B \subset X$, then also the following weak version of the Sobolev-Poincaré inequality holds

$$\left(\fint_B |u - u_B|^{kp}\, d\mu\right)^{1/kp} \leqq Cr \left(\fint_{2B} g^p\, d\mu\right)^{1/p}$$

in every ball $B \subset X$, where $k > 1$ depends on p and the doubling constant only. We have two independent proofs for this fact (*see* [6]). One is a modification of the method used above and the second is an application of the Sobolev imbedding on metric spaces proved in [10]. These arguments clearly also establish versions of the Sobolev inequality, *i.e.*, inequalities for compactly supported functions.

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REFERENCES

[1] L. SALOFF-COSTE, A Note on Poincaré, Sobolev, and Harnack inequalities, *Internat. Mat. Res. Notices,* No. 2, 1992, pp. 27-38.

[2] J. HEINONEN, T. KILPELÄINEN and O. MARTIO, *Nonlinear potential theory of degenerate elliptic equations,* Oxford Univ. Press, 1993.

[3] D. JERISON, The Poincaré inequality for vector fields satisfying Hörmander's condition, *Duke Math. J.,* 53, 1986, pp. 503-523.

[4] B. FRANCHI, C. E. GUTIÉRREZ and R. L. WHEEDEN, Weighted Sobolev-Poincaré inequalities for Grushin type operators, *Comm. Partial Differential Equations,* 19, 1994, pp. 523-604.

[5] H. BUSEMANN, *The geometry of geodesics,* Academic Press, New York, 1955.

[6] P. HAJŁASZ and P. KOSKELA, Sobolev met Poincaré (in preparation).

[7] P. HAJŁASZ and P. KOSKELA, Isoperimetric inequalities in irregular domains and imbedding theorems, *preprint.*

[8] J. HEINONEN and P. KOSKELA, Weighted Sobolev and Poincaré inequalities and quasiregular mappings of polynomial type, *Math. Scand.* (to appear).

[9] A. NAGEL, E. M. STEIN and S. WAINGER, Balls and metrics defined by vector fields I: Basic properties, *Acta Math.,* 155, 1985, pp. 103-147.

[10] P. HAJŁASZ, Sobolev spaces on an arbitrary metric space, *Potential Anal.* (to appear).

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