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Sobolev Met Poincaré

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Contents

Abstract

There are several generalizations of the classical theory of Sobolev spaces as they are necessary for the applications to Carnot–Carathéodory spaces, subelliptic equations, quasiconformal mappings on Carnot groups and more general Loewner spaces, analysis on topological manifolds, potential theory on infinite graphs, analysis on fractals and the theory of Dirichlet forms.

The aim of this paper is to present a unified approach to the theory of Sobolev spaces that covers applications to many of those areas. The variety of different areas of applications forces a very general setting.

We are given a metric space X equipped with a doubling measure μ. A generalization of a Sobolev function and its gradient is a pair $u \in L^1_{\mathrm{loc}}(X)$, $0 \le g \in L^p(X)$ such that for every ball $B \subset X$ the Poincaré-type inequality

$$\fint_B |u - u_B|\, d\mu \le Cr \left(\fint_{\sigma B} g^p \, d\mu \right)^{1/p}$$

holds, where r is the radius of B and $\sigma \ge 1$, $C > 0$ are fixed constants. Working in the above setting we show that basically all relevant results from the classical theory have their counterparts in our general setting. These include Sobolev-Poincaré type embeddings, Rellich-Kondrachov compact embedding theorem, and even a version of the Sobolev embedding theorem on spheres. The second part of the paper is devoted to examples and applications in the above mentioned areas.

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1 Introduction

The theory of Sobolev spaces is a central analytic tool in the study of various aspects of partial differential equations and calculus of variations. However, the scope of its applications is much wider, including questions in differential geometry, algebraic topology, complex analysis, and in probability theory.

Let us recall a definition of the Sobolev spaces. Let $u \in L^p(\Omega)$, where Ω is an open subset of $\mathbb{R}^n$, and $1 \leq p \leq \infty$. We say that u belongs to the Sobolev space $W^{1,p}(\Omega)$ if the distributional derivatives of the first order belong to $L^p(\Omega)$. This definition easily extends to the setting of Riemannian manifolds, as the gradient is well defined there.

The fundamental results in the theory of Sobolev spaces are the so-called Sobolev embedding theorem and the Rellich–Kondrachov compact embedding theorem. The first theorem states that, for $1 \leq p < n$, $W^{1,p}(\Omega) \subset L^{p^*}(\Omega)$, where $p^* = np/(n-p)$, provided the boundary of Ω is sufficiently nice. The second theorem states that, for every $q < p^*$, the embedding $W^{1,p}(\Omega) \subset L^q(\Omega)$ is compact for such a domain Ω.

Since its introduction the theory and applications of Sobolev spaces have been under intensive study. Recently there have been attempts to generalize Sobolev spaces to the setting of metric spaces equipped with a measure. Let us indicate some of the problems that suggest such a generalization.

1) Study of the Carnot–Carathéodory metric generated by a family of vector fields. 2) Theory of quasiconformal mappings on Carnot groups and more general Loewner spaces. 3) Analysis on topological manifolds. 4) Potential theory on infinite graphs. 5) Analysis on fractals.

Let us briefly discuss the above examples. The Carnot–Carathéodory metric appears in the study of hypoelliptic operators, see Hörmander [128], Fefferman and Phong [69], Jerison [133], Nagel, Stein and Wainger [204], Rotschild and Stein [218], Sánchez-Calle [224].

The Sobolev inequality on balls in Carnot–Carathéodory metric plays a crucial role in the so-called Moser iteration technique, [202], used to obtain Harnack inequalities and Hölder continuity for solutions of various quasilinear degenerate equations. The proof of the Harnack inequality by means of the Moser technique can be reduced to verifying a suitable Sobolev inequality. Conversely, a parabolic Harnack inequality implies a version of the Sobolev inequality as shown by Saloff-Coste, [221]. It seems that the first to use the Moser technique in the setting of the Carnot–Carathéodory metric were Franchi and Lanconelli, [78]. The later work on related questions include the papers by Biroli and Mosco, [8], [9], Buckley, Koskela and Lu, [19], Capogna, Danielli and Garofalo, [27], [28], [29], [30], [31], Chernikov and Vodop'yanov, [38], Danielli, Garofalo, Nhieu, [61], Franchi, [74], Franchi, Gallot and Wheeden, [75], Franchi, Gutiérrez and Wheeden, [76], Franchi and Lanco-

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nelli, [79], Franchi, Lu and Wheeden, [80], [81], Franchi and Serapioni, [83], Garofalo and Lanconelli, [90], Lu, [175], [178], [180], [181], Marchi [189].

The theory of Carnot–Carathéodory metrics and related Sobolev inequalities can be extended to the setting of Dirichlet forms, see Biroli and Mosco [8], Garattini [89], Sturm [239].

For connections to the theory of harmonic maps see the papers by Jost, [142], [143], [144], [145], Jost and Xu, [146], Hajłasz and Strzelecki, [108].

The theory of quasiconformal mappings on Carnot Groups has been studied by Heinonen and Holopainen, [116], Margulis and Mostow, [190], Pansu, [210], Koranyi and Reimann, [159], Heinonen and Koskela, [117], Vodop'yanov and Greshnov, [253]. Results on Sobolev spaces play an important role in this theory. Very recently Heinonen and Koskela, [118], extended the theory to the setting of metric spaces that support a type of a Sobolev inequality.

Semmes, [226], has shown that a large class of topological manifolds admit Sobolev type inequalities, see Section 10. Sobolev type inequalities on a Riemannian manifold are of fundamental importance for heat kernel estimates, see the survey article [55] of Coulhon for a nice exposition.

Discretization of manifolds has lead one to define the gradient on an infinite graph using finite differences and then to investigate the related Sobolev inequalities, see Kanai [149], Auscher and Coulhon [2], Coulhon [52], Coulhon and Grigor'yan [56], Coulhon and Saloff-Coste [59], Delmotte [65], Holopainen and Soardi [126], [127]. These results have applications to the classification of Riemannian manifolds. Also the study of the geometry of finitely generated groups leads to Sobolev inequalities on associated Cayley graphs, see Varopoulos, Saloff-Coste and Coulhon [251], and Section 12 for references.

At last, but not least, the Brownian motion on fractals leads to an associated Laplace operator and Sobolev type functions on fractals, see Barlow and Bass [5], Jonsson [139], Jonsson and Wallin [141], Kozlov [162], Kigami [151], [152], [153], Kigami and Lapidus [154], Lapidus [167], [168], Metz and Sturm [197], Mosco [201].

How does one then generalize the notion of Sobolev space to the setting of a metric space? There are several possible approaches that we briefly describe below.

In general, the concept of a partial derivative is meaningless on a metric space. However, it is natural to call a measurable function $g \geq 0$ an upper gradient of a function u if

$$|u(x) - u(y)| \leq \int_\gamma g\, ds$$

holds for each pair x, y and all rectifiable curves γ joining x, y. Thus, in the Euclidean setting, we consider the length of the gradient of a smooth function instead of the actual gradient. The above definition is due to Heinonen and Koskela, [118].

Assume that the metric space is equipped with a measure μ. Then we can ask if for every pair u, g, where u is continuous and g is an upper gradient of u, the weak version

$$\left(\fint_B |u-u_B|^q\,d\mu\right)^{1/q} \le Cr\left(\fint_{\sigma B} g^p\,d\mu\right)^{1/p} \tag{1}$$

of the Sobolev-Poincaré inequality holds with $q > p \ge 1$ whenever B is a ball of radius r. Here C and $\sigma \ge 1$ are fixed constants, barred integrals over a set A mean integral averages, and u_B is the average value of u over B.

If $q = 1$, then we call (1) a p-Poincaré inequality. It turns out that a p-Poincaré inequality implies a Sobolev-Poincaré inequality, see Section 5.

This approach is however limited to metric spaces that are sufficiently regular. There have to be sufficiently many rectifiable curves, which excludes fractals and graphs. For more information see Section 4, Section 10.2, Section 11.2, Bourdon and Pajot [15], Cheeger [34], Franchi, Hajłasz and Koskela [77], Hanson and Heinonen [111], Heinonen and Koskela [118], [119], Kallunki and Shanmugalingam [148], Laakso [164], Semmes [226], Shanmugalingam [228], Tyson [245].

Recently Hajłasz, [102], introduced a notion of a Sobolev space in the setting of an arbitrary metric space equipped with a Borel measure that we next describe.

One can prove that $u \in W^{1,p}(\Omega)$, $1 < p \le \infty$, where $\Omega \subset \mathbb{R}^n$ is a bounded set with nice boundary if and only if $u \in L^p(\Omega)$ and there is a non-negative function $g \in L^p(\Omega)$ such that

$$|u(x) - u(y)| \le |x-y|(g(x) + g(y)). \tag{2}$$

Since this characterization does not involve the notion of a derivative it can be used to define Sobolev space on an arbitrary metric space, see Hajłasz [102]. These spaces have been investigated or employed in Franchi, Hajłasz and Koskela [77], Franchi, Lu and Wheeden [81], Hajłasz [103], Hajłasz and Kinnunen [104], Hajłasz and Martio [107], Heinonen [115], Heinonen and Koskela [118], Kałamajska [147], Kilpeläinen, Kinnunen and Martio [156], Kinnunen and Martio [157], Koskela and MacManus [161], Shanmugalingam [228].

Another approach is presented in the paper [105] of Hajłasz and Koskela, in which also some of the results from our current work were announced. Given a metric space equipped with a Borel measure we assume that a pair u and g, ($g \ge 0$), of locally integrable functions satisfies the family (1) of Poincaré inequalities with $q = 1$ and a fixed $p \ge 1$ on every ball, that is

$$\fint_B |u-u_B|\,d\mu \le C_P r\left(\fint_{\sigma B} g^p\,d\mu\right)^{1/p}. \tag{3}$$

This family of inequalities is the only relationship between u and g. Then we can ask for the properties of u that follow.

Yet another approach to Sobolev inequalities on metric spaces is presented in the paper [11] by Bobkov and Houdré. However it is much different from the above mentioned setting and it will not be discussed here.

One of the purposes of this paper is to systematically develop the theory of Sobolev spaces from inequality (3). This includes the study of the relationships between (1), (2), and (3). We show that basically all relevant results from the classical theory have their counterparts in our general setting. These include Sobolev-Poincaré type embeddings, Rellich-Kondrachov compact embedding theorem, and even a version of the Sobolev embedding theorem on spheres.

We will work with metric spaces equipped with a doubling measure. Such spaces are often called *spaces of homogeneous type*, but we will call them *doubling spaces*. The reader may find many important examples of spaces of homogeneous type in Christ [41], and Stein [234]. The class of such spaces is pretty large. For example Volberg and Konyagin [255], [256], proved that every compact subset of $\mathbb{R}^n$ supports a doubling measure; see also Wu [257], and Luukkainen and Saksman [182].

Starting from the work of Coifman and Weiss [48], [49], spaces of homogeneous type have become a standard setting for the harmonic analysis related to singular integrals and Hardy spaces, see, e.g., Gatto and Vagi [93], [94], Genebahsvili, Gogatishvili, Kokilashvili and Krbec [95], Han [109], Han and Sawyer [110], Macias and Segovia [183].

However it seems that the development of the theory of Sobolev spaces in such generality did not begin until very recently.

There are some papers on Sobolev inequalities on spaces of homogeneous type related to our work; see Franchi, Lu and Wheeden [81], Franchi, Pérez and Wheeden [82], MacManus and Pérez [184], [185]; the last three papers were motivated by our approach. Also the paper [92] of Garofalo and Nhieu provides a similar approach in the special case of Carnot–Carathéodory spaces.

The reader might wonder why we insist on studying the situation with a fixed exponent p instead of assuming that (3) holds with $p = 1$. There is a simple reason for this. Indeed, for each $p > 1$ one can construct examples of situations where (3) holds for each smooth function u with $g = |\nabla u|$ but where one cannot replace p by any exponent $q < p$. Let us give an example to illustrate the dependence on p. Take two three-dimensional planes in $\mathbb{R}^5$ whose intersection is a line L, and let X be the union of these two planes. The metrics and measures induced from the planes have natural extensions to a metric and a measure on X. If u is a smooth function on X then we define $g(x)$ to be $|\nabla u(x)|$ whenever x does not belong to L, where ∇u is the usual gradient of u in the appropriate plane, and define $g(x)$ to be the sum of the lengths of the two gradients corresponding to the different planes when $x \in L$. One can then check that (3) holds for $p > 2$ but fails for $p \leq 2$.

As we said, we want to develop the theory of Sobolev spaces assuming a family of Poincaré inequalities (3) and the doubling property. Such assumptions have found many applications in the literature in various areas of analysis and geometry. The applications include the above mentioned Carnot–Carathéodory spaces [61], [81], [91], [92], graphs [64], [65], [127], Dirichlet forms [8], [9], [10], [239], and quasiconformal mappings [118].

These assumptions have also found many important applications in Riemannian geometry. The class of open Riemannian manifolds that satisfy both the doubling condition and the p-Poincaré inequality is under intensive investigation, see Colding and Minicozzi [50], [51], Grigor'yan [99], Holopainen [124], Holopainen and Rickman [125], Li and Wang [171], Maheux and Saloff-Coste [186], Rigoli, Salvatori and Vignati [215], Saloff-Coste [221], [222], [223], and Tam [242], where some global properties of manifolds were obtained under the assumption that the Riemannian manifold satisfies a p-Poincaré inequality and the doubling property.

In 1975 Yau, [261], proved that on open Riemannian manifolds of nonnegative Ricci curvature bounded harmonic functions are constant. Some time later he conjectured that for such manifolds the space of harmonic functions with polynomial growth of fixed rate is finite dimensional.

Independently Grigor'yan, [99], and Saloff-Coste, [221], generalized Yau's theorem by proving that bounded harmonic functions are constant provided the manifold satisfies the doubling property and the Poincaré inequality (3) with $g = |\nabla u|$ and $p = 2$. It is known that manifolds with nonnegative Ricci curvature satisfy these two conditions, see Section 10.1. Under the same assumptions the result of Yau has been extended to harmonic mappings, see Li and Wang, [171], and Tam [242].

Very recently, Colding and Minicozzi, [50], [51], answered the conjecture of Yau in the affirmative. Again the assumptions were that the manifold is doubling and that the 2-Poincaré inequality holds.

Many of the above Riemannian results have counterparts in the more general settings of Carnot–Carathéodory spaces, graphs, or Dirichlet forms, and again the main common assumption is the same: doubling and Poincaré.

This common feature was guiding us in our work. The first part of the paper is devoted to general theory and the second part to examples and applications in the areas mentioned above.

The paper is organized as follows. In Section 2 we present the setting in which we later on develop the theory of Sobolev inequalities. In Section 3 we discuss the equivalence of various approaches to Sobolev inequalities on metric spaces. Section 4 is devoted to some basic examples and conditions that necessarily hold for spaces that satisfy all p-Poincaré inequalities (1) for pairs of a continuous function and upper gradient. In Section 5 we show that if a pair u, g satisfies a p-Poincaré inequality (3), then $|u - u_B|$ can be estimated by a generalized Riesz potential. This together with a generalization of the Fractional Integration Theorem implies a variant of the Sobolev-Poincaré embedding theorem. In Section 6 we impose the additional condition that the space be connected and improve on one of the inequalities from Section 5, namely we prove a variant of the Trudinger inequality. In Section 7 we prove an embedding theorem on almost all spheres centered at a given point. In Section 8 we generalize the classical Rellich-Kondrachov theorem to the setting of metric spaces. So far all the results are local in nature. In Section 9 we introduce the class of John domains and generalize previous results as global results in John domains. In Section 10 we collect important examples of metric spaces where the theory developed in the paper is applicable (including open Riemannian manifolds, topological manifolds and Loewner spaces). In Section 11 we study the theory of Carnot–Carathéodory spaces that are associated with a family of vector fields, from the point of view of Sobolev inequalities on metric spaces. In Section 12 we discuss Sobolev inequalities on infinite graphs. Section 13 is devoted to applications of the theory to nonlinear potential theory and degenerate elliptic equations. Section 14 is an appendix, where we collect all the results in measure theory and on maximal functions that are needed in the paper.

The exposition is self-contained and the background material needed is the abstract measure theory in metric spaces, some real analysis related to maximal functions and the basic theory of classical Sobolev spaces covered by each of the following references: Evans and Gariepy [66], Gilbarg and Trudinger [96], Malý and Ziemer [187], Ziemer [263].

Some examples and applications that illustrate the theory require slightly more. In Section 11.3-4 some familiarity with Lie groups and commutators of vector fields is needed and in Section 13 we assume basic facts about quasilinear elliptic equations in divergence form. One can, however, skip reading Sections 11.3-4 and 13 and it will not affect understanding of the remaining parts of the paper.

We did make some effort to give comprehensive references to subjects related to our work. We are however sure that many important references are still missing and we want to apologize to those whose contribution is not mentioned.

Notation. Throughout the paper X will be a metric space with a metric d, and a Borel measure μ. The precise assumptions on μ are collected in the appendix. If not otherwise stated, μ will be *doubling* which means that

$$\mu(2B) \le C_d \mu(B) \tag{4}$$

whenever B is a ball and $2B$ is the ball with the same center as B and with radius twice that of B (in the same way we define σB for $\sigma > 0$). We will call such a metric measure space X a *doubling space* and C_d a doubling constant. $\Omega \subseteq X$ will always denote an open subset. Sometimes we will need the doubling property on a subset of X only; we will say that the measure μ is *doubling on* Ω if (4) holds whenever $B = B(x,r)$, $x \in \Omega$ and $r \le 5\,\mathrm{diam}\,\Omega$. By writing $v \in L^q_{\mathrm{loc}}(\Omega)$, we designate that v belongs to the class $L^q(B)$ with respect to μ for each ball $B \subset \Omega$. If $\Omega = X$, we will simply write $v \in L^q_{\mathrm{loc}}$. By $\mathrm{Lip}\,(X)$ we denote the class of Lipschitz functions on the metric space X.

The average value will be denoted by $v_A = ⨍_A v\, d\mu = \mu(A)^{-1}\int_A v\, d\mu$. If $R > 0$ and v is a measurable function, then $M_R v$ stands for the restricted *Hardy-Littlewood maximal function*

$$M_R v(x) = \sup_{0<r\le R} ⨍_{B(x,r)} |v|\, d\mu.$$

If $R = \infty$, then we will simply write Mv. Another version of the maximal function is

$$M_\Omega v(x) = \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{\Omega\cap B(x,r)} |v|\, d\mu,$$

which applies to $v \in L^1_{\mathrm{loc}}(\Omega,\mu)$. It is also clear how to define the restricted maximal function $M_{\Omega,R}v$.

By H^k we denote the k-dimensional Hausdorff measure. The symbol χ_E denotes the characteristic function of a set E. We reserve B to always denote a ball. Observe that in some metric spaces it may happen that the center and the radius of the ball are not uniquely defined. In what follows, when we write B we assume that the center and the radius are fixed. Otherwise σB is not properly defined. By C we will denote a general constant which can change even in a single string of estimates. By writing $C = C(p,q,\lambda)$ we indicate that the constant C depends on p, q and λ only. We write $u \approx v$ to state that there exist two positive constants C_1, and C_2 such that $C_1 u \le v \le C_2 u$.

Some further notation and commonly used results are collected in the appendix.

2 What are Poincaré and Sobolev inequalities?

In this section we describe the general framework and give samples of problems which are treated later on. Until the end of the section we assume that μ is a Borel measure on a metric space X, but we *do not* assume that μ is doubling. As before $\Omega \subset X$ denotes an open set.

Definition. Assume that $u \in L^1_{\mathrm{loc}}(\Omega)$ and a measurable function $g \ge 0$ satisfy the inequality

$$⨍_B |u - u_B|\, d\mu \le C_P r \left(⨍_{\sigma B} g^p\, d\mu \right)^{1/p}, \tag{5}$$

on each ball B with $\sigma B \subset \Omega$, where r is the radius of B and $p > 0$, $\sigma \ge 1$, $C_P > 0$ are fixed constants. We then say that the pair u, g *satisfies a p-Poincaré inequality in* Ω. When $\Omega = X$, we simply say that the pair u, g satisfies *a p-Poincaré inequality.*

Note that if $u \in \mathrm{Lip}\,(\mathbb{R}^n)$, $g = |\nabla u|$ and $p \ge 1$, then (5) is a corollary of the classical Poincaré inequality

$$\left(⨍_B |u - u_B|^p\, dx \right)^{1/p} \le C(n,p) r \left(⨍_B |\nabla u|^p\, dx \right)^{1/p}. \tag{6}$$

Quite often we will call an inequality *weak* if both sides involve a ball and the radius of the ball on the right hand side is greater than the radius of the ball on the left hand side, like in (5).

Unfortunately, as is easy to see, in general, inequality (6) does not hold with $p < 1$ (cf. [16, p. 224]). Nevertheless, there are many important situations where the p-Poincaré inequalities (5) and (6) hold with $p < 1$. For example, they hold when u is a solution to an elliptic equation of a certain type, see Section 13. For this reason we include the case $p < 1$.

It is natural to regard a pair u, g that satisfies a p-Poincaré inequality in Ω as a Sobolev function and its gradient. In this sense we will develop the theory of Sobolev functions on metric spaces with "gradient" in L^p for all $p > 0$.

In the classical approach the Sobolev spaces are defined for $p \geq 1$ only. Moreover, it was expected that there would be no reasonable theory of Sobolev spaces for $0 < p < 1$, see Peetre, [211]. We obtain a rich theory of Sobolev spaces for all $p > 0$. In the Euclidean setting, when $p \geq 1$, our approach is equivalent to the classical one.

In the literature there are a few papers that deal with the Sobolev inequalities for $p < 1$, see Bakry, Coulhon, Ledoux and Saloff-Coste [4], Buckley and Koskela [16], Buckley, Koskela and Lu [19], Calderón and Scott [24], Hajłasz and Koskela [105].

Let us assume that a pair u, g satisfies a p-Poincaré inequality for $p > 0$ in an open set $\Omega \subset X$. We inquire for properties of u that follow from this assumption. A typical question is whether the Sobolev embedding theorem holds i.e., whether the p-Poincaré inequality in Ω implies the global Sobolev-type inequality

$$\inf_{c \in \mathbb{R}} \left(\int_\Omega |u - c|^q \, d\mu \right)^{1/q} \leq C \left(\int_\Omega g^p \, d\mu \right)^{1/p}, \tag{7}$$

with an exponent $q > p$. We suggest the reader to have a look at our earlier paper [105], where a result of this type was obtained by an elementary method. In the current paper we obtain stronger results by more complicated methods.

Note that if $\mu(\Omega) < \infty$ and $q \geq 1$, then the above inequality is equivalent to

$$\left(\int_\Omega |u - u_\Omega|^q \, d\mu \right)^{1/q} \leq C' \left(\int_\Omega g^p \, d\mu \right)^{1/p}, \tag{8}$$

as for $q \geq 1$ and $\mu(\Omega) < \infty$ we have

$$\begin{aligned} \inf_{c \in \mathbb{R}} \left(\int_\Omega |u - c|^q \, d\mu \right)^{1/q} &\leq \left(\int_\Omega |u - u_\Omega|^q \, d\mu \right)^{1/q} \\ &\leq 2 \inf_{c \in \mathbb{R}} \left(\int_\Omega |u - c|^q \, d\mu \right)^{1/q}. \end{aligned} \tag{9}$$

The classical gradient of a Lipschitz function has a very important property: if the function is constant in a set E, then the gradient equals zero a.e. in E. To have a counterpart of this property in the metric setting we introduce the *truncation property*.

Given a function v and $\infty > t_2 > t_1 > 0$, we set

$$v_{t_1}^{t_2} = \min\{\max\{0, v - t_1\}, t_2 - t_1\}.$$

Definition. Let the pair u, g satisfy a p-Poincaré inequality in Ω. Assume that for every $b \in \mathbb{R}$, $\infty > t_2 > t_1 > 0$, and $\varepsilon \in \{-1, 1\}$, the pair $v_{t_1}^{t_2}$, $g\chi_{\{t_1 < v \le t_2\}}$, where $v = \varepsilon(u - b)$, satisfies the p-Poincaré inequality in Ω (with fixed constants C_P, σ). Then we say that the pair u, g has the *truncation property*.

Let $p \ge 1$ and $u \in \mathrm{Lip}\,(\mathbb{R}^n)$. Since $v = \pm(u - b)$ satisfies $|\nabla v_{t_1}^{t_2}| = |\nabla u|\chi_{\{t_1 < v \le t_2\}}$ a.e., the pair u, $|\nabla u|$ has the truncation property. More sophisticated examples are given in Section 10.

We close the section with a result which shows that inequality (7) is equivalent to a weaker inequality provided the pair u, g has the truncation property. The result will be used in the sequel.

Theorem 2.1 *Let $\Omega \subset X$ be an open set with $\mu(\Omega) < \infty$. Fix $\infty > q \ge p > 0$, $C_P > 0$ and $\sigma \ge 1$. Assume that every pair u, g, that satisfies a p-Poincaré inequality in Ω (with given C_P and σ) satisfies also the global Marcinkiewicz–Sobolev inequality*

$$\inf_{c \in \mathbb{R}} \sup_{t \ge 0} \mu(\{x \in \Omega : |u(x) - c| > t\}) t^q \le C_1 \left(\int_\Omega g^p \, d\mu \right)^{q/p}. \tag{10}$$

Then every pair that satisfies the p-Poincaré inequality in Ω (with given C_P and σ) and has the truncation property satisfies also the global Sobolev inequality

$$\inf_{c \in \mathbb{R}} \left(\int_\Omega |u - c|^q \, d\mu \right)^{1/q} \le C_2 \left(\int_\Omega g^p \, d\mu \right)^{1/p} \tag{11}$$

with $C_2 = 8 \cdot (4C_1)^{1/q}$.

Remarks. 1) We call (10) a Marcinkiewicz–Sobolev inequality, because it implies that u belongs to the Marcinkiewicz space L^q_w.

2) The result is surprising even in the Euclidean case: inequality (10) seems much weaker than (11) as the inclusion $L^q \subset L^q_w$ is proper. Similar phenomena have been discovered by V. G. Maz'ya, [194], (cf. [195, Section 2.3.1], [106, Theorem 1]), who proved that a Sobolev embedding is equivalent to a capacitary estimate which is a version of inequality (10). The main idea of Maz'ya was a *truncation method* which is also the key argument in our proof. This method mimics the proof of the equivalence of the Sobolev inequality with the isoperimetric inequality. Inequality (10) plays the role of the relative isoperimetric inequality and the truncation argument provides a discrete counterpart of the co-area formula. The truncation method of Maz'ya has become very useful in proving various versions of the Sobolev embedding theorem with *sharp* exponents in the borderline case where interpolation arguments do not work. To see how the argument works in the case of the classical Sobolev embedding theorem, we refer the reader to the comments after the statement of Theorem 5.1. Recently the truncation method has been employed and even rediscovered by many authors; see Adams and Hedberg [1, Theorem 7.2.1], Bakry, Coulhon, Ledoux and Saloff-Coste [4], Biroli and Mosco [8], [9], Capogna, Danielli

and Garofalo [29], Coulhon [54], Franchi, Gallot and Wheeden [76], Garofalo and Nhieu [92], Heinonen and Koskela [118], [119], Long and Nie [173], Maheux and Saloff-Coste [186], Semmes [226], and Tartar [243].

Proof of Theorem 2.1. Let u, g be a pair that satisfies the p-Poincaré inequality in Ω and that has the truncation property. Choose $b \in \mathbb{R}$ such that

$$\mu(\{u \geq b\}) \geq \frac{\mu(\Omega)}{2} \qquad \text{and} \qquad \mu(\{u \leq b\}) \geq \frac{\mu(\Omega)}{2}\,.$$

Let $v_+ = \max\{u-b, 0\}$, $v_- = -\min\{u-b, 0\}$. We will estimate $\|v_+\|_{L^q}$ and $\|v_-\|_{L^q}$ separately. In what follows v will denote either v_+ or v_-.

Lemma 2.2 *Let ν be a finite measure on a set Y. If $w \geq 0$ is a ν-measurable function such that $\nu(\{w=0\}) \geq \nu(Y)/2$, then for every $t > 0$*

$$\nu(\{w > t\}) \leq 2 \inf_{c \in \mathbb{R}} \nu(\{|w - c| > \frac{t}{2}\}).$$

The proof of the lemma is easy and left to the reader.

By the truncation property the pair $v_{t_1}^{t_2}$, $g\chi_{\{t_1 < v \leq t_2\}}$ satisfies the p-Poincaré inequality and hence it satisfies (10). Moreover, the function $v_{t_1}^{t_2}$ has the property $\mu(\{v_{t_1}^{t_2} = 0\}) \geq \mu(\Omega)/2$. Hence, applying the lemma, we conclude that

$$\begin{aligned}
\sup_{t \geq 0} \mu(\{v_{t_1}^{t_2} > t\}) t^q &\leq 2^{q+1} \inf_{c \in \mathbb{R}} \sup_{t \geq 0} \mu\left(\left\{|v_{t_1}^{t_2} - c| > \frac{t}{2}\right\}\right)\left(\frac{t}{2}\right)^q \\
&\leq 2^{q+1} C_1 \|g\chi_{\{t_1 < v \leq t_2\}}\|_{L^p}^q.
\end{aligned}$$

This yields

$$\begin{aligned}
\int_\Omega v^q \, d\mu &\leq \sum_{k=-\infty}^{\infty} 2^{kq} \mu(\{2^{k-1} < v \leq 2^k\}) \\
&\leq \sum_{k=-\infty}^{\infty} 2^{kq} \mu(\{v \geq 2^{k-1}\}) \\
&= \sum_{k=-\infty}^{\infty} 2^{kq} \mu(\{v_{2^{k-2}}^{2^{k-1}} \geq 2^{k-2}\}) \\
&\leq 2^{3q+1} C_1 \sum_{k=-\infty}^{\infty} \left(\int_\Omega g^p \chi_{\{2^{k-2} < v \leq 2^{k-1}\}} \, d\mu\right)^{q/p} \\
&\leq 2^{3q+1} C_1 \left(\sum_{k=-\infty}^{\infty} \int_\Omega g^p \chi_{\{2^{k-2} < v \leq 2^{k-1}\}} \, d\mu\right)^{q/p} \\
&\leq 2^{3q+1} C_1 \|g\|_{L^p(\Omega)}^q.
\end{aligned}$$

In the second to the last step we used the inequality $q/p \geq 1$. Finally

$$\int_\Omega |u-b|^q = \int_\Omega v_+^q + \int_\Omega v_-^q \leq 2^{3q+2} C_1 \|g\|^q_{L^p(\Omega)}.$$

This completes the proof.

The following theorem is a modification of the above result.

Theorem 2.3 *Let $\infty > q \geq p > 0$, $C_P > 0$ and $\sigma \geq 1$. Assume that every pair u, g that satisfies the p-Poincaré inequality (with given C_P and σ) satisfies also the weak Marcinkiewicz–Sobolev inequality*

$$\inf_{c\in\mathbb{R}} \sup_{t\geq 0} \frac{\mu(\{x\in B : |u(x)-c|>t\})t^q}{\mu(B)} \leq C_1 r^q \left(\fint_{\sigma B} g^p \, d\mu \right)^{q/p}$$

for every ball B, where r denotes the radius of B. Then every pair u, g that satisfies the p-Poincaré inequality (with given C_P and σ) and has the truncation property satisfies also the weak Sobolev inequality

$$\inf_{c\in\mathbb{R}} \left(\fint_B |u-c|^q \, d\mu \right)^{1/q} \leq C_2 r \left(\fint_{\sigma B} g^p \, d\mu \right)^{1/p}$$

for every ball B with $C_2 = 8 \cdot (4C_1)^{1/q}$.

The proof is essentially the same as that for Theorem 2.1 and we leave it to the reader.

3 Poincaré inequalities, pointwise estimates, and Sobolev classes

Our starting point to the theory of Sobolev spaces on metric spaces is to assume that the pair u, g satisfies a p-Poincaré inequality. There are however also other possible approaches. Recently Hajłasz, [102], introduced a notion of a Sobolev space in the setting of metric space equipped with a Borel measure. In this section we will compare this approach to that based on Poincaré inequalities (see Theorem 3.1). The proof is based on pointwise inequalities which are of independent interest and which we state in a more general version than is needed for the sake of the proof (see Theorem 3.2 and Theorem 3.3). Finally we compare the class of L^p-pairs of u, g that satisfy a p-Poincaré inequality to the classical Sobolev space.

For a detailed study on the equivalence of various approaches to Sobolev inequalities on metric spaces, see Franchi, Hajłasz and Koskela [77], and Koskela and MacManus [161]. Results related to those of this section appear also in Franchi, Lu and Wheeden [81], Hajłasz and Kinnunen [104], Heinonen and Koskela [118], Shanmugalingam [228].

Given $p > 0$ and a triple (X, d, μ), where (X, d) is a metric space and μ is a Borel measure (not necessarily doubling), Hajłasz, [102], defines the Sobolev space $M^{1,p}(X, d, \mu)$ as the set of all $u \in L^p(X)$ for which there exists $0 \le g \in L^p(X)$ such that

$$|u(x) - u(y)| \le d(x, y)(g(x) + g(y)) \quad \text{a.e.} \tag{12}$$

When we say that an inequality like (12) holds a.e. we mean that there exists a set $E \subset X$ with $\mu(E) = 0$ such that inequality (12) holds for all $x, y \in X \setminus E$.

If $p \ge 1$, then the space is equipped with a Banach norm $\|u\|_{M^{1,p}} = \|u\|_{L^p} + \inf_g \|g\|_{L^p}$, where the infimum is taken over the set of all $0 \le g \in L^p(X)$ that satisfy (12).

The motivation for the above definition comes from the following result.

If $\Omega = \mathbb{R}^n$ or if $\Omega \subset \mathbb{R}^n$ is a bounded domain with sufficiently regular boundary, $|\cdot|$ is the Euclidean metric, H^n the Lebesgue measure, and $1 < p \le \infty$, then

$$W^{1,p}(\Omega) = M^{1,p}(\Omega, |\cdot|, H^n) \tag{13}$$

as sets and the norms are equivalent, see [102] and also [103], [107], [252]. Here $W^{1,p}(\Omega)$ denotes the classical Sobolev space of L^p-integrable functions with generalized gradient in L^p. If $p = 1$, then the equivalence (13) fails, see [103]. However, for any open set $\Omega \subset \mathbb{R}^n$ and $1 \le p < \infty$, $M^{1,p}(\Omega, |\cdot|, H^n) \subset W^{1,p}(\Omega)$, see [103, Proposition 1], and also [107, Lemma 6].

For the further development and applications of the above approach to Sobolev spaces on metric space, see Franchi, Hajłasz and Koskela [77], Franchi, Lu and Wheeden [81], Hajłasz and Kinnunen [104], Hajłasz and Martio, [107], Heinonen [115], Heinonen and Koskela [118], Kałamajska [147], Kilpeläinen, Kinnunen and Martio [156], Kinnunen and Martio [157], Koskela and MacManus [161], Shanmugalingam [228].

Prior to the work of Hajłasz, Varopoulos, [250], defined a function space on a smooth compact manifold, based on an inequality similar to (12). Recently and independently, Vodop'yanov, [252], used inequality (12) to define a Sobolev space on a Carnot group.

The following result compares the above definition of the Sobolev space with the approach based on Poincaré inequalities.

Theorem 3.1 *Let X be a doubling space. If $1 < p < \infty$, then the following conditions are equivalent.*

1. *$u \in M^{1,p}(X, d, \mu)$.*
2. *$u \in L^p(X)$ and there exist $C > 0$, $\sigma \ge 1$, $0 \le g \in L^p(X)$, and $0 < q < p$ such that the Poincaré inequality*

$$\fint_B |u - u_B|\, d\mu \le C\, r \left(\fint_{\sigma B} g^q\, d\mu \right)^{1/q} \tag{14}$$

holds on every ball B of radius r.

Remarks. 1) In fact we prove the implication 2. $\Rightarrow$ 1. for any $p > 0$. 2) Under much more restrictive assumptions on the measure Theorem 3.1 has been proved by Franchi, Lu and Wheeden, [81], see also [77], [86], [104], [118], [161].

Proof of Theorem 3.1. Integrating inequality (12) over a ball with respect to x and y we obtain

$$\fint_B |u - u_B|\, d\mu \le Cr \fint_B g\, d\mu\,,$$

which proves the implication 1. $\Rightarrow$ 2. The opposite implication follows from Theorem 3.2 and the Maximal Theorem 14.13.

Theorem 3.2 *Let X be a doubling space. Assume that the pair u, g satisfies a p-Poincaré inequality (5), $p > 0$. Then*

$$|u(x) - u(y)| \le Cd(x,y)\left((M_{2\sigma d(x,y)} g^p(x))^{1/p} + (M_{2\sigma d(x,y)} g^p(y))^{1/p} \right) \tag{15}$$

for almost every $x, y \in X$, where $M_R v(x) = \sup_{0<r<R} \fint_{B(x,r)} |v|\, d\mu$.

Before we prove Theorem 3.2 we show how to use it to complete the proof of the implication 2. $\Rightarrow$ 1. Assume that $u, g \in L^p(X)$ satisfy (14). Then inequality (15) holds with p replaced by q. Note that

$$\left(M_{2\sigma d(x,y)} g^q(x)\right)^{1/q} \le (Mg^q(x))^{1/q}\,.$$

Now, $g^q \in L^{p/q}$, $p/q > 1$, and so the Maximal Theorem 14.13 implies $(Mg^q)^{1/q} \in L^p$ and hence the claim follows.

Proof of Theorem 3.2. Let $x, y \in X$ be Lebesgue points of u; by the Lebesgue differentiation theorem (see Theorem 14.15) this is true for almost all points. Write $B_i(x) = B(x, r_i) = B(x, 2^{-i} d(x,y))$ for each nonnegative integer i. Then $u_{B_i(x)} \to u(x)$ as i tends to infinity. Using the triangle inequality, the doubling of μ and the p-Poincaré inequality we conclude that

$$\begin{aligned}
|u(x) - u_{B_0(x)}| &\le \sum_{i=0}^{\infty} |u_{B_i(x)} - u_{B_{i+1}(x)}| \\
&\le \sum_{i=0}^{\infty} \fint_{B_{i+1}(x)} |u - u_{B_i(x)}|\, d\mu \\
&\le C \sum_{i=0}^{\infty} \fint_{B_i(x)} |u - u_{B_i(x)}|\, d\mu \\
&\le C \sum_{i=0}^{\infty} r_i \left(\fint_{\sigma B_i(x)} g^p\, d\mu \right)^{1/p}
\end{aligned}$$

$$
\begin{aligned}
&\leq C\sum_{i=0}^{\infty} r_i (M_{\sigma d(x,y)} g^p(x))^{1/p} \\
&= Cd(x,y)(M_{\sigma d(x,y)} g^p(x))^{1/p}. \qquad (16)
\end{aligned}
$$

Similarly,

$$
|u(y) - u_{B_0(y)}| \leq Cd(x,y)(M_{\sigma d(x,y)} g^p(y))^{1/p}.
$$

Moreover,

$$
\begin{aligned}
|u_{B_0(x)} - u_{B_0(y)}| &\leq |u_{B_0(x)} - u_{2B_0(x)}| + |u_{B_0(y)} - u_{2B_0(x)}| \\
&\leq C \fint_{2B_0(x)} |u - u_{2B_0(x)}|\, d\mu \\
&\leq Cd(x,y) \left(\fint_{2\sigma B_0(x)} g^p\, d\mu \right)^{1/p} \\
&\leq Cd(x,y)(M_{2\sigma d(x,y)} g^p(x))^{1/p}.
\end{aligned}
$$

The claim follows by combining the above three inequalities. This completes the proof of Theorem 3.2 and hence that of Theorem 3.1.

It is interesting to observe that Theorem 3.2 can be converted, see also Heinonen and Koskela [118]. This is the content of the following result.

Theorem 3.3 *Let X be a doubling space and $u \in L^1_{\mathrm{loc}}(X,\mu)$, $0 \leq g \in L^p_{\mathrm{loc}}(X,\mu)$, $1 < p < \infty$. Suppose that the pointwise inequality*

$$
|u(x) - u(y)| \leq Cd(x,y) \left((M_{\sigma d(x,y)} g^p(x))^{1/p} + (M_{\sigma d(x,y)} g^p(y))^{1/p} \right)
$$

holds for almost all $x, y \in X$ with some fixed $\sigma \geq 1$. Then the p-Poincaré inequality

$$
\fint_B |u - u_B|\, d\mu \leq C_P r \left(\fint_{3\sigma B} g^p\, d\mu \right)^{1/p}
$$

holds for all balls B. Here C_P depends only on p, C, C_d.

Proof. Fix a ball B with radius r. Then for almost all $x, y \in B$ we have

$$
|u(x) - u(y)| \leq Cd(x,y) \left((M(g^p \chi_{3\sigma B})(x))^{1/p} + (M(g^p \chi_{3\sigma B})(y))^{1/p} \right).
$$

Fix $t_0 > 0$. Taking an average with respect to x and y, applying Cavalieri's principle (see Theorem 14.10) and the weak type estimate for the maximal function (see Theorem 14.13) we obtain

$$
\begin{aligned}
\fint_B |u - u_B|\, d\mu &\leq Cr \fint_B \left(M(g^p \chi_{3\sigma B}) \right)^{1/p}\, d\mu \\
&= Cr\mu(B)^{-1} \int_0^{\infty} \mu(\{x \in B : M(g^p \chi_{3\sigma B}) > t^p\})\, dt
\end{aligned}
$$

$$\begin{aligned} &\leq \; Cr\mu(B)^{-1}\left(\int_0^{t_0}\mu(B)\,dt+\int_{t_0}^{\infty}\left(\frac{C}{t^p}\int_{3\sigma B}g^p\,d\mu\right)\,dt\right) \\ &= \; Cr\mu(B)^{-1}\left(t_0\mu(B)+Ct_0^{1-p}\int_{3\sigma B}g^p\,d\mu\right). \end{aligned}$$

The claim follows when we choose $t_0=(\mu(B)^{-1}\int_{3\sigma B}g^p\,d\mu)^{1/p}$. The proof is complete.

Note that the argument used above is similar to that used in the proof of Theorem 14.11.

Theorem 3.1 suggests the following question: *Is it true that if a pair $u, g \in L^p(X)$, $1<p<\infty$, satisfies a p-Poincaré inequality in a doubling space X, then there exists $1\leq q<p$ such that the pair u, g satisfies a q-Poincaré inequality?* This seems to be a very delicate question, see the discussion in the remark in Section 4 below.

If the answer to the above question were affirmative, Theorem 3.1 would imply a stronger result: $u\in M^{1,p}$, $p>1$, if and only if $u\in L^p$ and there is $0\leq g\in L^p(X)$ such that the pair u, g satisfies a p-Poincaré inequality.

In the special case when $X=\mathbb{R}^n$, d is the Euclidean metric and μ is the Lebesgue measure, the answer to the above question is in the positive due to the results of Franchi, Hajłasz and Koskela [77], and Koskela and MacManus [161].

The following theorem was proved in [77]. The result is a generalization of some results in [155], [161].

Theorem 3.4 *Let u, $g\in L^p(\mathbb{R}^n)$, $g\geq 0$, $p\geq 1$. Suppose that there exist $\lambda\geq 1$ and C such that*

$$⨍_B |u-u_B|\,dx\leq Cr\left(⨍_{\lambda B}g^p\,dx\right)^{1/p},$$

for all balls $B\subset\mathbb{R}^n$. Then $u\in W^{1,p}(\mathbb{R}^n)$ and $|\nabla u|\leq C_1 g$ a.e. In particular,

$$⨍_B |u-u_B|\,dx\leq C_2 r ⨍_B g\,dx,$$

for all balls $B\subset\mathbb{R}^n$.

Note that it follows from the results stated before Theorem 3.4 that if a pair $u, g\in L^p(\mathbb{R}^n)$ satisfies a p-Poincaré inequality, $p>1$, then $u\in W^{1,q}_{\rm loc}$, for any $1\leq q<p$. Indeed, Theorem 3.2 together with the weak type estimate for the maximal function and the embedding $L^p_w\subset L^q_{\rm loc}$ for all $q<p$ (see Theorem 14.11) imply that for some $h\in L^q_{\rm loc}$ the inequality $|u(x)-u(u)|\leq|x-y|(h(x)+h(y))$ holds a.e. Then the claim follows from (13). This argument, however, does not guarantee that $u\in W^{1,p}(\mathbb{R}^n)$. Thus the proof of Theorem 3.4 requires arguments of a completely different nature.

For far reaching generalizations of Theorem 3.4, see [77] and [161] and also Section 13.

4 Examples and necessary conditions

We first discuss three examples that indicate the dependence of the validity of p-Poincaré inequalities on the exponent p. Notice that if a pair u, g satisfies a p-Poincaré inequality, then it satisfies a q-Poincaré inequality for all $q > p$ by Hölder's inequality. The following examples show that this is not the case for $q < p$.

Example 4.1 Let $X = R^2$ be equipped with the Euclidean metric and let μ be the measure generated by the density $d\mu(x) = |x_2|^t dx$, $t > 0$, where x_2 denotes the second coordinate of x. Then the Poincaré inequality (5) holds for each Lipschitz function u with $g = |\nabla u|$ if and only if $p > t + 1$.

The Poincaré inequality holds as μ is an A_p-weight for the indicated values of p; see [67], [39], [120, Theorem 15.26], [102]. On the other hand, the p-Poincaré inequality fails for $p = 1 + t$ and hence for $1 \le p \le 1 + t$. To see this, let B be the disk of radius 2 and with center $(0, 1)$. Let us consider a sequence u_i of Lipschitz functions that only depend on x_2 and such that $u_i = 1$ if $x \in B$ and $x_2 \le 2^{-i}$, $u_i = 0$ if $x \in B$ and $x_2 \ge 1$, $u_i(x) = -i^{-1}\log_2(x_2)$ if $2^{-i} \le x_2 \le 1$. Then

$$\int_B |\nabla u_i|^{1+t}\, d\mu \le 2(i \log 2)^{-(1+t)} \int_{2^{-i}}^1 \frac{ds}{s}$$

which tends to zero as i approaches infinity. On the other hand, $|u_i(x) - u_{iB}| \ge 1/2$ for all x either in the part of B above the line $x_2 = 1$ or in the part below the line $x_2 = 0$. Hence the integral of $|u_i - u_{iB}|$ over B is bounded away from zero independently of i, and so the $(1+t)$-Poincaré inequality cannot hold for all Lipschitz functions. Using a standard regularization argument we can then assume that functions u_i in the above example are C^∞ smooth, so the $(1+t)$-Poincaré inequality cannot hold for all C^∞ smooth functions either.

Example 4.2 Let $X = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \,:\, x_1^2 + \cdots + x_{n-1}^2 \le x_n^2\}$ be equipped with the Euclidean metric of $\mathbb{R}^n$ and with the Lebesgue measure. The set X consists of two infinite closed cones with a common vertex. Denote the upper cone by X_+ and the lower one by X_-.

We will prove that the p-Poincaré inequality (5) holds in X for every pair u, g where u is a continuous function and g an upper gradient of u if and only if $p > n$. (For more information about upper gradients, see Section 10.2.)

First we prove that the inequality fails for $p = n$ (and hence for $p < n$). Fix $\varepsilon > 0$. Since $\varphi(x) = \log|\log|x||$ satisfies $\varphi \in W^{1,n}(B^n(0, 1/2))$ and $\varphi(x) \to \infty$ as $x \to 0$, we can truncate it to obtain a continuous function $u_\varepsilon \in W^{1,n}(X_+)$ such that $u_\varepsilon(0) = 1$, $u_\varepsilon(x) = 0$ for $|x| \ge \varepsilon$ and $\|\nabla u_\varepsilon\|_{L^n(X_+)} < \varepsilon$. We extend this function to the lower cone as the constant 1. Fix a ball B centered at the origin. Then $\|\nabla u_\varepsilon\|_{L^n(B)} < \varepsilon$ while $\|u_\varepsilon - u_{\varepsilon B}\|_{L^1(B)} > C$ uniformly with respect to ε, and thus the n-Poincaré inequality cannot hold.

It remains to prove the inequality for $p > n$. Since $|\nabla u| \le g$ for an upper gradient of u (Proposition 10.1), it suffices to prove the p-Poincaré inequality for the pair u, $|\nabla u|$. By Theorem 3.3 it suffices to verify the pointwise estimate

$$|u(x)-u(y)| \le C|x-y|(M_{2|x-y|}|\nabla u|^p(x) + M_{2|x-y|}|\nabla u|^p(y))^{1/p}.$$

We can assume that x and y belong to different cones as the p-Poincaré inequality holds in each of those two cones. Then, by the triangle inequality, either $|u(x)-u(0)| \ge |u(x)-u(y)|/2$ or $|u(y)-u(0)| \ge |u(x)-u(y)|/2$. Assume that the first inequality holds. Let $\Delta = X_+ \cap \overline{B}(x,|x|)$. Then by the embedding into Hölder continuous functions

$$|u(x)-u(0)| \le C|x-0|^{1-n/p}\left(\int_\Delta |\nabla u|^p\right)^{1/p} \le C|x-y|(M_{2|x-y|}|\nabla u|^p(x))^{1/p}.$$

This ends the proof of the claim.

Modifying the argument used above one can construct many other examples. Let for example X be the union of two 3-dimensional planes in $\mathbb{R}^5$ whose intersection is a line. Equip X with the 3-dimensional Lebesgue measure and with the metric induced by the Euclidean metrics of the planes. Then the p-Poincaré inequality holds in X for all pairs u, g, where u is a continuous function in X and g an upper gradient of u, if and only if $p > 2$.

A much more general result that allows one to build similar examples in the setting of metric spaces was proven by Heinonen and Koskela, [118, Theorem 6.15].

Example 4.3 For each $1 < p \le n$ there is an open set $X \subset \mathbb{R}^n$ equipped with the Euclidean metric and the Lebesgue measure, such that the p-Poincaré inequality (5) holds for each smooth function u with $g = |\nabla u|$ but no Poincaré inequality holds for smaller exponents for all smooth functions.

Such an example was constructed by Koskela, [160]. We will recall the idea of the example following [160]. Let $E \subset \mathbb{R}^n$ be a compact set such that $W^{1,p}(\mathbb{R}^n \setminus E) = W^{1,p}(\mathbb{R}^n)$, and $W^{1,q}(\mathbb{R}^n)$ is a proper subset of $W^{1,q}(\mathbb{R}^n \setminus E)$ for all $1 \le q < p$. In other words, the set E is $W^{1,p}$-removable but it is not $W^{1,q}$-removable for any $1 \le q < p$. Such sets were explicitly constructed in [160]. In fact there is a smooth function u in $\mathbb{R}^n \setminus E$ such that $|\nabla u| \in L^q(\mathbb{R}^n)$, for all $q < p$, but the pair u, $|\nabla u|$ does not satisfy a q-Poincaré inequality in $\mathbb{R}^n \setminus E$ for any $q < p$. The pair satisfies the p-Poincaré inequality but $|\nabla u| \notin L^p(\mathbb{R}^n \setminus E)$ as otherwise we would have $u \in W^{1,p}(\mathbb{R}^n \setminus E) = W^{1,p}(\mathbb{R}^n)$ and hence it would even satisfy the 1-Poincaré inequality. Thus this example does not solve the question posed after Theorem 3.3.

Remark. The last example shows that it may happen that a p-Poincaré inequality holds for all smooth pairs u, $|\nabla u|$ in X and there is a smooth function u in X such that no q-Poincaré inequality holds for any $q < p$ for the pair u, $|\nabla u|$. However, in this example $|\nabla u| \notin L^p(X)$ and hence we do not know if a p-Poincaré inequality for the pair u, g with $g \in L^p(X)$ implies a q-Poincaré inequality for some $q < p$. As Example 4.1 shows, a q-Poincaré inequality cannot hold for all $q < p$ in general, but we do not know if it can hold for some $q < p$ sufficiently close to p.

Let us next describe some necessary geometric conditions for the validity of Poincaré inequalities. The first result says that a Poincaré inequality implies that there are short rectifiable curves. We would like to thank Anton Petrunin for his very clever argument with a "broken length" that we use in the proof of Proposition 4.4. Our original version of the proposition was proven under more restrictive assumptions on X.

Definition. If each pair x_1, x_2 of points in a metric space X can be joined by a curve whose length is no more than $Cd(x_1, x_2)$, then we say that X is *quasiconvex*.

Recall that X is *proper* if closed balls are compact. Observe that this is a stronger condition than being locally compact as the example $X = \mathbb{R}^n \setminus \{0\}$ shows.

Proposition 4.4 *Suppose that X is proper, path connected and doubling. Let $p \geq 1$. If each pair u, g of a continuous function and its upper gradient satisfies a p-Poincaré inequality (with fixed σ, C_P), then X is quasiconvex.*

Proof. Fix $k \in \mathbb{N}$. Let $\gamma : [0,1] \to X$ be a continuous path. For any partition τ: $0 = t_0 < t_1 < \ldots < t_n = 1$ consider the sum

$$s_\tau = \sum_{i=0}^{n-1} \min\{l(\gamma|_{[t_i,t_{i+1}]}), kd(\gamma(t_i), \gamma(t_{i+1}))\}\,,$$

where l denotes the length of a curve, and then define

$$l_k(\gamma) = \inf_\tau s_\tau\,,$$

where the infimum is taken over all possible partitions of $[0,1]$. We do not require that γ is rectifiable. Indeed, if no part of γ is rectifiable, then $l_k(\gamma) = kd(\gamma(0), \gamma(1))$.

Fix $x_0 \in X$ and define

$$u_k(x) = \inf_\gamma l_k(\gamma)$$

for $x \in X$, where the infimum is taken over all curves that join x to x_0.

It is easy to verify that the function u_k is k-Lipschitz and that the function $g \equiv 1$ is an upper gradient of u_k. Thus the pair (u_k, g) satisfies the p-Poincaré inequality and hence by Theorem 3.2

$$|u_k(x) - u_k(y)| \leq Cd(x,y)\left((M_{2\sigma d(x,y)} g^p(x))^{1/p} + (M_{2\sigma d(x,y)} g^p(y))^{1/p}\right) = 2Cd(x,y)\,.$$

In particular, for $y = x_0$, we obtain

$$|u_k(x)| \leq Cd(x, x_0)\,.$$

Let γ_k be a curve that joins x to x_0 and such that $l_k(\gamma_k) \leq 2u_k(x)$. Then let τ_k be a partition of $[0,1]$ such that $s_{\tau_k} \leq 3u_k(x)$.

We do this for all $k \geq 1$. We would like to show that a subsequence γ_{k_i} converges, in some sense, to a rectifiable curve γ that joins x to x_0 and such that

$$l(\gamma) \leq \liminf_{i\to\infty} l_{k_i}(\gamma_{k_i}) \leq Cd(x, x_0)\,.$$

This would end the proof.

Recall the following classical argument (cf. Lemma 9.4). Let $\{\delta_k\}_{k=1}^{\infty}$ be a sequence of rectifiable curves in a proper metric space Y. Assume that all the curves δ_k join given two points $x, y \in Y$ and that $\sup_k l(\delta_k) < \infty$. Parametrize each δ_k by arclength. Scaling the arclength parametrizations we may assume that all curves are defined on the interval $[0,1]$. Now it easily follows that the family $\{\delta_k\}$ is equicontinuous (because of the good parametrization and the fact that $\sup_k l(\delta_k) < \infty$). By a standard diagonal method we can find a subsequence δ_{k_i} which converges on a dense subset of $[0,1]$. The equicontinuity implies the uniform convergence on the whole interval to a curve δ. Then it is easy to show that $l(\delta) \leq \liminf_{i\to\infty} l(\delta_{k_i})$.

Unfortunately the above argument does not apply to our situation as $l_k(\gamma_k)$ is not the length of γ_k. We do not even know whether the curves γ_k are rectifiable.

There is however a trick that makes the above argument work: we modify our metric space by attaching to it infinitely many segments and then we modify the family of curves. In this new framework the above argument will work perfectly.

We construct a new metric space $\widehat{X}$ by attaching to the space X infinitely many Euclidean segments. We do this as follows. Let τ_k: $0 = t_0 < t_1 < t_2 < \ldots < t_n = 1$ be a partition associated with γ_k chosen as above. If

$$l(\gamma_k|_{[t_i,t_{i+1}]}) \geq k d(\gamma_k(t_i), \gamma_k(t_{i+1}))\,, \tag{17}$$

then we glue to the space X a straight Euclidean segment $I_{k,i}$ of the length $d(\gamma_k(t_i), \gamma_k(t_{i+1}))$ (i.e. $I_{k,i}$ is isometric with $[0, d(\gamma_k(t_i), \gamma_k(t_{i+1}))]$) in such a way that ends of the segment are attached to the space at points $\gamma_k(t_i)$ and $\gamma_k(t_{i+1})$. We do this for every curve γ_k and every i such that (17) is true.

The space $\widehat{X}$ is equipped with a natural metric which is induced from the Euclidean metric in each segment and the metric d in X. We denote the metric in $\widehat{X}$ by $\widehat{d}$.

Now denote by $\widehat{\gamma}_k$ the new curve which is obtained for γ_k in the following way. Let $0 \leq i \leq n-1$. If (17) does not hold, then

$$\widehat{\gamma}_k|_{[t_i,t_{i+1}]} = \gamma_k|_{[t_i,t_{i+1}]}\,.$$

If (17) is true, then $\widehat{\gamma}_k|_{[t_i,t_{i+1}]}$ is the obvious linear function with values in the segment $I_{k,i}$.

This means that if the part $\gamma_k|_{[t_i,t_{i+1}]}$ of the curve γ_k is too long in the sense of (17), then we replace this part by a shortcut going through the attached segment.

Since $\sup_k s_{\tau_k} \leq C d(x, x_0)$ we conclude that $l(I_{k,i}) < C d(x, x_0)/k$. Thus, in the worst case, we attach to the space X infinitely many segments $I_{k,i}$ of length converging to zero. This easily implies that the space $\widehat{X}$ is proper as well. Now $\widehat{l}(\widehat{\gamma}_k) \leq s_{\tau_k} \leq C d(x, x_0)$, where $\widehat{l}$ denotes the length in the new metric space $\widehat{X}$.

Now we are in a situation where the above classical argument applies. Choosing a good parametrization of $\widehat{\gamma}_k$ we may subtract a subsequence $\widehat{\gamma}_{k_i}$ that converges uniformly to a curve γ such that

$$\widehat{l}(\gamma) \le \liminf_{i\to\infty} \widehat{l}(\widehat{\gamma}_{k_i}) \le Cd(x,x_0)\,.$$

Since the lengths of $I_{k,i}$ converge to zero as $k \to \infty$ we conclude that the values of γ belong to X and hence one can easily show that $\widehat{l}(\gamma) = l(\gamma)$. The proof is complete.

Thus the validity of a p-Poincaré inequality guarantees the existence of short curves. If the doubling measure μ behaves as the Euclidean volume and the exponent p is no more than the growth order of the volume, then X cannot have narrow parts. This conclusion is a consequence of Proposition 4.5 below. Under the additional assumption that the space satisfies a weak local version of quasiconvexity, this result can be deduced from the results in [118].

Proposition 4.5 *Suppose that X is proper, path connected and that $\mu(B(x,r)) \approx r^s$ with $s > 1$ for each x and all r. Assume that each pair u, g of a continuous function and its upper gradient satisfies an s-Poincaré inequality (with fixed σ, C_P). If $x_0 \in X$, $r > 0$, and $x_1, x_2 \in B(x_0,r) \setminus B(x_0,r/2)$, then x_1, x_2 can be joined in $B(x_0,Cr) \setminus B(x_0,r/C)$ by a curve whose length does not exceed $Cd(x_1,x_2)$.*

Notice that the claim of the proposition would still be true if we replaced the s-Poincaré inequality by a p-Poincaré inequality, $p < s$, as a p-Poincaré inequality implies an s-Poincaré inequality, by means of the Hölder inequality. However we cannot replace the s-Poincaré inequality by a p-Poincaré for any $p > s$ as follows from Example 4.2.

Proof. The proof is very similar to the arguments used in the proof of [118, Corollary 5.8] and in the proof of Proposition 4.4. Throughout $C \ge 1$ denotes a constant whose value can change from line to line but that only depends on the given data. By Proposition 4.4 we may assume that $d(x_1,x_2) \ge C^{-1}r$ as x_1 and x_2 can be connected by a curve of length comparable to $d(x_1,x_2)$. Cutting pieces of the curve near x_1 and x_2 we obtain rectifiable curves $F_1, F_2 \subset B(x_0,2r) \setminus B(x_0,r/4)$, both of length comparable to r and such that $\operatorname{dist}(F_1,F_2)$ is comparable to r as well. It suffices to show that F_1, F_2 can be joined by a curve of length less than Cr inside $B(x_0,Cr) \setminus B(x_0,r/C)$ for sufficiently large C.

If follows from the s-Poincaré inequality and from the volume growth condition that

$$\int_{B(x_0,6\sigma r)} g^s\, d\mu \ge C^{-1} \tag{18}$$

for any upper gradient g of any continuous function u that takes on the constant value 0 in F_1 and takes on a value greater than or equal to 1 at each point of F_2. Indeed, assume first that $|u_{B(x_0,2r)}| \le 1/2$. Then slightly modifying the proof of (16) we get for all $x \in F_2$

$$\frac{1}{2} \le |u(x) - u_{B(x_0,2r)}| \le Cr^{1/s} \sup_{R<4\sigma r} \left(R^{-1} \int_{B(x,R)} g^s\, d\mu \right)^{1/s} .$$

Thus for some $R_x < 4\sigma r$

$$C^{-1}R_x/r \le \int_{B(x,R_x)} g^s\, d\mu\,.$$

Now inequality (18) follows from the covering lemma (Theorem 14.12) and the fact that if $F_2 \subset \bigcup B_i(r_i)$, then $\sum_i r_i \ge C^{-1}r$. If $|u_{B(x_0,2r)}| \ge 1/2$, then inequality (18) follows by a symmetric argument. The proof of (18) is complete.

Fix $\tau > 4$. Set $g_1(x) = (\log(\frac{\tau}{4}))^{-1}d(x,x_0)^{-1}$ in $B(x_0,\tau r)\setminus B(x_0,\tau^{-1}r)$ and extend g_1 as zero to the rest of X. Suppose that F_1, F_2 cannot be joined in $B(x_0,\tau r)\setminus B(x_0,\tau^{-1}r)$ by a rectifiable curve. Define $u_1(x) = \inf_{\gamma_x}\int_{\gamma_x} g_1\, ds$ where the infimum is taken over all rectifiable curves that join x to F_1. Then g_1 is an upper gradient of u_1, the restriction of u to F_1 is zero and $u(x) \ge 1$ at each point of F_2. By the preceding paragraph, we see that the integral of g_1^s over $B(x_0,6\sigma r)$ is bounded away from zero. On the other hand a computation using the volume growth condition and Cavalieri's principle (Theorem 14.10) shows that the integral of g_1^s over $B(x_0,6\sigma r)$ goes to 0 as τ goes to infinity. Hence τ is bounded from above. Thus we can fix τ large enough so that F_1, F_2 can be joined by a rectifiable curve in $B(x_0,\tau r)\setminus B(x_0,\tau^{-1}r)$. It remains to find such a curve with a length comparable to r.

Set $a = \inf_\gamma l(\gamma)$, where the infimum is taken over all rectifiable curves that join F_1 to F_2 in $B(x_0,\tau r)\setminus B(x_0,\tau^{-1}r)$. We define a function u_2 similarly as we defined u_1 above using g_1+g_2, where $g_2(x) = a^{-1}\chi_U(x)$, and χ_U is the characteristic function of $B(x_0,\tau r)\setminus B(x_0,\tau^{-1}r)$. Observe that $u_2|_{F_1} \equiv 0$ and $u_2|_{F_2} \ge 1$ independently of τ and hence (18) holds with $g = g_1+g_2$. As we can make the integral of g_1^s over the ball $B(x_0,6\sigma r)$ as small as we wish by choosing the constant τ large enough, we obtain that the integral of g_2^s over $B(x_0,6\sigma r)$ must be bounded away from zero, and thus the volume growth condition implies that $a \le Cr$, as desired.

5 Sobolev type inequalities by means of Riesz potentials

As it was pointed out in Section 2, one of the aims of this paper is to prove a global Sobolev inequality

$$\inf_{c\in\mathbb{R}}\left(\int_\Omega |u-c|^q\, d\mu\right)^{1/q} \le C\left(\int_\Omega g^p\, d\mu\right)^{1/p}, \tag{19}$$

where $q > p$, or at least a weak local Sobolev inequality

$$\inf_{c\in\mathbb{R}}\left(-\!\!\!\!\!\!\int_B |u-c|^q\, d\mu\right)^{1/q} \le Cr\left(-\!\!\!\!\!\!\int_{5\sigma B} g^p\, d\mu\right)^{1/p}, \tag{20}$$

where $\sigma \ge 1$, and B is any ball of radius r, assuming only that the pair u, g satisfies a p-Poincaré inequality.

Inequality (19) requires some additional information on Ω, while (20) turns out to be true in a very general setting.

Another question we deal with is how to determine the best possible Sobolev exponent q in inequalities (19) and (20).

In the remaining part of the section we will be consider inequalities of the type (20). The case of the global Sobolev inequality (19) will be treated in Section 9.

Let X be a doubling space. Beside the doubling condition we will sometimes require that

$$\frac{\mu(B)}{\mu(B_0)} \geq C_b \left(\frac{r}{r_0}\right)^s \tag{21}$$

whenever B_0 is an arbitrary ball of radius r_0 and $B = B(x,r)$, $x \in B_0$, $r \leq r_0$.

Notice that the doubling condition on μ always implies (21) for some exponent s that only depends on the doubling constant of μ. This follows by a standard iteration of the doubling condition, see Lemma 14.6 in the appendix. Inequality (21) could well hold with exponents smaller than the one following from the doubling condition and in the following results s refers to any exponent for which (21) is valid.

Theorem 5.1 *Assume that the pair u, g satisfies a p-Poincaré inequality (5), $p > 0$, in a doubling space X. Assume that the measure μ satisfies condition (21).*

1. *If $p < s$, then*

$$\frac{\mu(\{x \in B \,:\, |u(x) - u_B| > t\})t^{p^*}}{\mu(B)} \leq C r^{p^*} \left(\fint_{5\sigma B} g^p \, d\mu \right)^{1/p},$$

where $p^* = sp/(s-p)$ and B is any ball of radius r. Hence for every $0 < h < p^*$

$$\left(\fint_B |u - u_B|^h \, d\mu \right)^{1/h} \leq Cr \left(\fint_{5\sigma B} g^p \, d\mu \right)^{1/p}.$$

Moreover for every q with $p < q < s$

$$\left(\fint_B |u - u_B|^{q^*} \, d\mu \right)^{1/q^*} \leq Cr \left(\fint_{5\sigma B} g^q \, d\mu \right)^{1/q},$$

where $q^* = sq/(s-q)$ and B is any ball of radius r. If, in addition, the pair u, g has the truncation property, then

$$\left(\fint_B |u - u_B|^{p^*} \, d\mu \right)^{1/p^*} \leq Cr \left(\fint_{5\sigma B} g^p \, d\mu \right)^{1/p}. \tag{22}$$

2. *If $p = s$, then*

$$\fint_B \exp\left(\frac{C_1 \mu(B)^{1/s} |u - u_B|}{r \|g\|_{L^s(5\sigma B)}} \right) d\mu \leq C_2. \tag{23}$$

3. If $p > s$, then u (after redefinition in a set of measure zero) is locally Hölder continuous and

$$\sup_{x\in B} |u(x) - u_B| \leq Cr \left(\fint_{5\sigma B} g^p \, d\mu \right)^{1/p} . \tag{24}$$

In particular

$$|u(x) - u(y)| \leq C r_0^{s/p} d(x,y)^{1-s/p} \left(\fint_{5\sigma B_0} g^p \, d\mu \right)^{1/p} \tag{25}$$

for all $x, y \in B_0$, where B_0 is an arbitrary ball of radius r_0.

The constants in the theorem depend on p, q, h, s, C_d, σ, C_P, and C_b only.

Remarks. 1) Inequality (22) holds also for functions on graphs, see Theorem 12.2.
2) Assuming that the space is connected we can improve on inequality (23); see Section 6.
3) Instead of assuming that X be doubling we could assume, for instance, that the doubling condition holds on all balls with radii bounded from above by r_0, (such a situation occurs for example on Riemannian manifolds with a lower bound on the Ricci curvature, see Section 10) or that it holds on a given open set. Then the inequalities of the theorem would hold on balls with radii bounded from above or on small balls centered at the open set. We leave it to the reader to check that the proof of Theorem 5.1 gives such a statement.
4) A modification of the proof shows that the ball $5\sigma B$ can be replaced by $(1+\varepsilon)\sigma B$; the details are left to the reader.
5) We present only one of the possible proofs of the above theorem. The proof can also be based on the embedding theorem for Sobolev spaces on metric spaces from Hajłasz, [102]. This approach uses the observation that a family of Poincaré inequalities leads to pointwise inequalities (15); we do not provide the details here.

Since the proof of the theorem is rather complicated, we begin with some comments that will explain the idea.

In one of the proofs of the classical Sobolev embedding $W^{1,p}(B) \subset L^{p^*}(B)$, where $1 \leq p < n$, $p^* = np/(n-p)$, and B is an n-dimensional Euclidean ball, one first proves the inequality

$$|u(x) - u_B| \leq C I_1^B |\nabla u|(x), \tag{26}$$

where $I_1^B g(x) = \int_B g(z)|x-z|^{1-n} \, dz$ and then applies the Fractional Integration Theorem which states that

$$I_1^B : L^p(B) \longrightarrow L^{p^*}(B) \tag{27}$$

is a bounded operator for $1 < p < n$. If $p = 1$ one only gets a weak type estimate

$$|\{x \in B : I_1^B g(x) > t\}| t^{\frac{n}{n-1}} \leq C \left(\int_B |g(z)| \, dz \right)^{\frac{n-1}{n}}$$

in place of (27), which, in turn, leads to the embedding $W^{1,1}(B) \subset L_w^{n/(n-1)}(B)$. Then the embedding $W^{1,1}(B) \subset L^{n/(n-1)}(B)$ follows from Theorem 2.1.

The main idea of our proof of inequalities like (19) or (20) is to mimic the above argument. Thus the proof splits into two steps.

Assume that a pair u, g satisfies a p-Poincaré inequality in a given doubling space. In the first step we prove the inequality

$$|u - u_B| \le C J_{1,p}^{\sigma,B} g, \tag{28}$$

where $J_{1,p}^{\sigma,B}$ is a suitable generalization of the Riesz potential I_1^B and then, in the second step, we prove a version of the Fractional Integration Theorem for the operator $J_{1,p}^{\sigma,B}$. This will complete the proof of (20). The proof of (19) will require a more sophisticated version of the inequality (28); the details will be completed in Section 9 where we introduce an appropriate class of domains Ω for the Sobolev-Poincaré embedding (19).

Any inequality of the type (28) will be called a *representation formula*.

Before we define $J_{\alpha,p}^{\sigma,B}$ we continue with a discussion on Riesz potentials to explain the motivation. The classical Riesz potential is defined as

$$I_\alpha g(x) = \gamma_{\alpha,n} \int_{\mathbb{R}^n} \frac{g(y)}{|x-y|^{n-\alpha}}\, dy, \tag{29}$$

where $0 < \alpha < n$ and $\gamma_{\alpha,n}$ is a suitable constant. In this paper the exact value of the constant $\gamma_{\alpha,n}$ is irrelevant to us. Moreover, for our purposes, any operator J such that

$$C_1 I_\alpha g \le Jg \le C_2 I_\alpha g \quad \text{for} \quad g \ge 0 \tag{30}$$

is as good as I_α.

A natural generalization of the Riesz potential to the setting of doubling spaces is

$$I_\alpha g(x) = \int_X \frac{g(y) d^\alpha(x,y)}{\mu(B(x, d(x,y)))}\, d\mu(y)\,,$$

or its local version

$$I_\alpha^\Omega g(x) = \int_\Omega \frac{g(y) d^\alpha(x,y)}{\mu(B(x, d(x,y)))}\, d\mu(y)\,. \tag{31}$$

We would like to estimate $|u - u_B|$ by $CI_1^\Omega g$, but, in general, this is not possible. Instead of that we have to consider a potential which is strictly larger than $I_1^\Omega g$.

Observe that the potential defined by

$$\widetilde{I}_\alpha^\Omega g(x) = \sum_{i=-\infty}^{\infty} 2^{i\alpha} \left(\mu(B_i(x))^{-1} \int_{A_i(x)\cap\Omega} |g(y)|\, d\mu(y) \right), \tag{32}$$

where $A_i(x) = B_i(x) \setminus B_{i-1}(x) = B(x, 2^i) \setminus B(x, 2^{i-1})$, is equivalent to $I_\alpha^\Omega g$ in the sense of (30). Note that if $2^{i-1} > \operatorname{diam}\Omega$, then $A_i(x) \cap \Omega = \emptyset$, so all the summands in (32) for $2^i > 2\operatorname{diam}\Omega$ vanish. Thus, replacing the integral over $A_i(x)\cap\Omega$ by the integral over $B_i(x)$ and then taking the sum over $2^i \le 2\operatorname{diam}\Omega$, we obtain the new potential

$$J_\alpha^\Omega g(x) = \sum_{2^i \le 2\operatorname{diam}\Omega} 2^{i\alpha}\left(\fint_{B_i(x)} |g|\, d\mu\right),$$

which satisfies $\widetilde{I}_\alpha^\Omega g \le J_\alpha^\Omega g$. Now we define

$$J_{\alpha,p}^{\sigma,\Omega} g(x) = \sum_{2^i \le 2\sigma\operatorname{diam}\Omega} 2^{i\alpha}\left(\fint_{B_i(x)} |g|^p\, d\mu\right)^{1/p},$$

where $\sigma \ge 1$, $p>0$ and $\alpha>0$ are fixed constants.

Another generalization is

$$I_{\alpha,p}^\Omega g(x) = \sum_{i=-\infty}^{\infty}\left(\int_{A_i(x)\cap\Omega} \frac{|g(y)|^p d^{\alpha p}(x,y)}{\mu(B(x,d(x,y)))}\, d\mu(y)\right)^{1/p}.$$

Observe that $I_{\alpha,1}^\Omega g = I_\alpha^\Omega|g|$, and $I_{\alpha,p}^\Omega g \le C J_{\alpha,p}^{1,\Omega} g$ a.e. Thus once we prove the fractional integration theorem for $J_{\alpha,p}^{\sigma,\Omega} g$ it is true for $I_{\alpha,p}^\Omega g$ as well.

In Section 9 we will obtain a version of the representation formula (28) with $I_{1,p}^B g$ in place of $J_{1,p}^{\sigma,B}$; see Theorem 9.10.

Theorem 5.2 *Let the pair u, g satisfy a p-Poincaré inequality in a doubling space X. Then for every ball $B\subset X$ the representation formula*

$$|u(x)-u_B| \le C J_{1,p}^{\sigma,B} g(x) \tag{33}$$

holds almost everywhere in B.

This representation formula together with a suitable Fractional Integration Theorem (see Theorem 5.3) will lead to embedding (20).

Proof. The argument is very similar to that used in the proof of inequality (16). Let $x\in B$ be a Lebesgue point of u. Put $D_i(x) = B(x, 2^i\sigma^{-1})$. Let i_0 be the least integer such that $2^{i_0} \ge \sigma\operatorname{diam} B$. Then $B\subset D_{i_0}(x)$. Since $u_{D_i(x)} \to u(x)$ as $i\to-\infty$ we obtain

$$\begin{aligned}
|u(x)-u_B| &\le |u_B - u_{D_{i_0}(x)}| + \sum_{i=-\infty}^{i_0} |u_{D_i(x)} - u_{D_{i-1}(x)}| \\
&\le C\sum_{i=-\infty}^{i_0} \sigma^{-1}2^i\left(\fint_{\sigma D_i(x)} g^p\, d\mu\right)^{1/p} \\
&\le C J_{1,p}^{\sigma,B} g(x).
\end{aligned}$$

Theorem 5.3 *Let $\Omega \subset X$ be an open and bounded set and let $0 < p < \infty$, $1 \leq \sigma < \infty$. Assume that the measure μ is doubling on $V = \{x \in X \,:\, \mathrm{dist}\,(x, \Omega) < 2\sigma \mathrm{diam}\,\Omega\}$. Moreover, assume that for some constants $C_b, s > 0$*

$$\mu(B(x,r)) \geq C_b \left(\frac{r}{\mathrm{diam}\,\Omega}\right)^s \mu(\Omega)$$

whenever $x \in \Omega$ and $r \leq \sigma \mathrm{diam}\,\Omega$ and that $g \in L^p(V, \mu)$.

1. *If $\alpha p < s$, then $J^{\sigma,\Omega}_{\alpha,p} g \in L^{p^*}_w(\Omega)$ where $p^* = sp/(s - \alpha p)$. Moreover*

$$\mu(\{x \in \Omega \,:\, J^{\sigma,\Omega}_{\alpha,p} g > t\}) \leq C_1 t^{-p^*} (\mathrm{diam}\,\Omega)^{\alpha p^*} \mu(\Omega)^{1-p^*/p} \|g\|^{p^*}_{L^p(V,\mu)} \tag{34}$$

   for $t > 0$, and hence for every $0 < r < p^*$

$$\|J^{\sigma,\Omega}_{\alpha,p} g\|_{L^r(\Omega,\mu)} \leq C_2 (\mathrm{diam}\,\Omega)^\alpha \mu(\Omega)^{1/r - 1/p} \|g\|_{L^p(V,\mu)}. \tag{35}$$

   Here the constants C_1 and C_2 depend on α, σ, p, C_b, s and C_d only.

2. *If $p < q$ and $\alpha q < s$, then*

$$\|J^{\sigma,\Omega}_{\alpha,p} g\|_{L^{q^*}(\Omega,\mu)} \leq C (\mathrm{diam}\,\Omega)^\alpha \mu(\Omega)^{-\alpha/s} \|g\|_{L^q(V,\mu)}, \tag{36}$$

   where $q^* = sq/(s - \alpha q)$ and $C = C(\alpha, \sigma, p, q, b, s, C_d)$.

3. *If $\alpha p = s$, then*

$$\fint_\Omega \exp\left(\frac{C_1 \mu(B)^{1/s} J^{\sigma,\Omega}_{\alpha,p} g}{(\mathrm{diam}\,\Omega)^\alpha \|g\|_{L^s(V)}}\right) d\mu \leq C_2,$$

   where $C_i = C(\alpha, \sigma, p, b, s, C_d)$, $i = 1, 2$.

4. *If $\alpha p > s$, then $J^{\sigma,\Omega}_{\alpha,p} \in L^\infty(\Omega, \mu)$ and*

$$\|J^{\sigma,\Omega}_{\alpha,p} g\|_{L^\infty(\Omega,\mu)} \leq C (\mathrm{diam}\,\Omega)^\alpha \mu(\Omega)^{-1/p} \|g\|_{L^p(V,\mu)},$$

   where $C = C(\alpha, \sigma, p, b, s, C_d)$.

Proof of Theorem 5.3. We modify a standard proof for the the case of usual Riesz potentials. All the constants C appearing in the proof depend on α, σ, p, q, b, s, and C_d only.

Case $\alpha p < s$. Take arbitrary $q \geq p$ such that $\alpha q < s$. Fix $0 < r \leq 2\sigma \mathrm{diam}\,\Omega$. Decompose the sum which defines $J^{\sigma,\Omega}_{\alpha,p} g$ into $J_r g + J^r g$, where $J_r g = \sum_{2^i \leq r}$ and $J^r g = \sum_{r < 2^i \leq 2\sigma \mathrm{diam}\,\Omega}$. For $x \in \Omega$ we have

$$J_r g(x) \leq \left(\sum_{2^i \leq r} 2^{i\alpha}\right) (M_V |g|^p(x))^{1/p} \approx r^\alpha \left(M_V |g|^p(x)\right)^{1/p}.$$

Here $M_V h$ denotes the maximal function relative to the open set V.

To estimate $J^r g$, we apply the lower bound on μ :

$$\begin{aligned} J^r g(x) &= \sum_{r<2^i\leq 2\sigma \operatorname{diam}\Omega} 2^{i\alpha}\left(\fint_{B_i(x)} |g|^p\,d\mu\right)^{1/p} \\ &\leq \sum_{r<2^i\leq 2\sigma \operatorname{diam}\Omega} 2^{i\alpha}\mu(B_i(x))^{-1/q}\left(\int_{B_i(x)} |g|^q\,d\mu\right)^{1/q} \\ &\leq C\sum_{r<2^i\leq 2\sigma \operatorname{diam}\Omega} 2^{i(\alpha-s/q)}(\operatorname{diam}\Omega)^{s/q}\mu(\Omega)^{-1/q}\left(\int_V |g|^q\,d\mu\right)^{1/q} \\ &\leq Cr^{(\alpha-s/q)}(\operatorname{diam}\Omega)^{s/q}\mu(\Omega)^{-1/q}\left(\int_V |g|^q\,d\mu\right)^{1/q}. \end{aligned}$$

In the last step we used the fact $\alpha - s/q < 0$ to estimate the sum of the series by its first summand. Now

$$J^{\sigma,\Omega}_{\alpha,p} g(x) \leq C\left(r^\alpha\left(M_V|g|^p\right)^{1/p} + r^{(\alpha-s/q)}(\operatorname{diam}\Omega)^{s/q}\mu(\Omega)^{-1/q}\left(\int_V |g|^q\,d\mu\right)^{1/q}\right).$$

Note that

$$r^\alpha\left(M_V|g|^p\right)^{1/p} \leq r^{(\alpha-s/q)}(\operatorname{diam}\Omega)^{s/q}\mu(\Omega)^{-1/q}\left(\int_V |g|^q\,d\mu\right)^{1/q}$$

if and only if

$$r \leq (\operatorname{diam}\Omega)\mu(\Omega)^{-1/s}\left(\|g\|_{L^q(V)}/(M_V|g|^p)^{1/p}\right)^{q/s}. \tag{37}$$

If the RHS in (37) does not exceed $\sigma\operatorname{diam}\Omega$, then we take r equal to the RHS. In this case we get

$$J^{\sigma,\Omega}_{\alpha,p} g(x) \leq C(\operatorname{diam}\Omega)^\alpha\mu(\Omega)^{-\alpha/s}\|g\|^{\alpha q/s}_{L^q(V)}(M_V|g|^p(x))^{(s-\alpha q)/sp}, \tag{38}$$

and hence

$$\begin{aligned} J^{\sigma,\Omega}_{\alpha,p} g(x)^{sp/(s-\alpha q)} \leq\ & C(\operatorname{diam}\Omega)^{\alpha sp/(s-\alpha q)}\mu(\Omega)^{-\alpha p/(s-\alpha q)} \\ & \times \|g\|^{\alpha qp/(s-\alpha q)}_{L^q(V)} M_V|g|^p(x). \end{aligned} \tag{39}$$

If the RHS in (37) is greater than $\sigma\operatorname{diam}\Omega$, then we take $r = \sigma\operatorname{diam}\Omega$. Then

$$J^{\sigma,\Omega}_{\alpha,p} g(x) \leq C(\operatorname{diam}\Omega)^\alpha\mu(\Omega)^{-1/q}\|g\|_{L^q(V)}. \tag{40}$$

Let A_1 denote the set of points in Ω for which (38) holds and let A_2 consist of those points in Ω that satisfy (40). Write $\Omega_t = \{x\in\Omega : J^{\sigma,\Omega}_{\alpha,p} g > t\}$. Then

$$\mu(\Omega_t) \leq \mu(A_1\cap\Omega_t) + \mu(A_2\cap\Omega_t). \tag{41}$$

If we take $q = p$, then inequality (34) follows from estimates (39), (40), and (41): the weak type estimate for the maximal function $M_V|g|^p$ (see Theorem 14.13) gives

$$\begin{aligned}\mu(A_1 \cap \Omega_t) &\leq \mu\left(\left\{CD\|g\|_{L^p(V)}^{\alpha p^2/(s-\alpha p)} M_V|g|^p > t^{sp/(s-\alpha p)}\right\}\right)\\ &\leq CDt^{-p^*}\|g\|_{L^p(V)}^{\alpha p^2/(s-\alpha p)}\|g\|_{L^p(V)}^p\\ &= CDt^{-p^*}\|g\|_{L^p(V)}^{p^*}\end{aligned}$$

with $D = (\operatorname{diam}\Omega)^{\alpha sp/(s-\alpha p)}\mu(\Omega)^{-\alpha p/(s-\alpha p)}$, and from (40) we obtain $A_2 \cap \Omega_t = \emptyset$ when $t \geq C(\operatorname{diam}\Omega)^\alpha\mu(\Omega)^{-1/p}\|g\|_{L^p(V)}$ and for all smaller t

$$\mu(A_2 \cap \Omega_t) \leq \mu(\Omega) \leq C(\operatorname{diam}\Omega)^{\alpha p^*}\mu(\Omega)^{1-p^*/p}t^{-p^*}\|g\|_{L^q(\Omega)}^{p^*}.$$

This completes the proof of inequality (34). Inequality (35) follows from Theorem 14.11.

To prove inequality (36) we take $L^{q/p}$-norms on both sides of inequalities (39) and (40) and after that we apply Maximal Theorem 14.13; we use the fact that $q/p > 1$.

Case $\alpha p = s$. Notice first that

$$\exp(t) = \sum_{k\geq 0}\frac{t^k}{k!}.$$

Secondly, (35) and the Hölder inequality give for each integer $k \geq 1$ the estimate

$$\|J_{\alpha,p}^{\sigma,\Omega}g\|_{L^k(\Omega,\mu)} \leq C(\operatorname{diam}\Omega)^\alpha\mu(\Omega)^{1/k-1/s}\|g\|_{L^s(V,\mu)}.$$

By keeping good track of the constants appearing in the proof of (35), one can check that $C = C_0(\sigma, s, b, C_d)k$. The desired inequality follows by summing over k. We leave the details to the reader as we prove a better estimate in the next section under slightly stronger assumptions.

Case $\alpha p > s$. The lower bound on μ gives

$$\begin{aligned}J_{\alpha,p}^{\sigma,\Omega}g(x) &= \sum_{2^i\leq 2\sigma\operatorname{diam}\Omega} 2^{i\alpha}\mu(B_i(x))^{-1/p}\left(\int_{B_i(x)}|g|^p\,d\mu\right)^{1/p}\\ &\leq C\sum_{2^i\leq 2\sigma\operatorname{diam}\Omega} 2^{i(\alpha-s/p)}(\operatorname{diam}\Omega)^{s/p}\mu(\Omega)^{-1/p}\|g\|_{L^p(V)}\\ &\approx C(\operatorname{diam}\Omega)^\alpha\mu(\Omega)^{-1/p}\|g\|_{L^p(V)}.\end{aligned}$$

The proof of Theorem 5.3 is complete.

Proof of Theorem 5.1. All the inequalities but (22) and (25) follow directly from Theorem 5.2 and Theorem 5.3. The Hölder continuity estimate (25) follows using (24) and the lower bound (21). If $p^* < 1$, then inequality (22) is trivial as it is weaker than the p-Poincaré inequality. If $p^* \geq 1$, (22) follows from Theorem 2.3 and from the first inequality in Theorem 5.1. The proof is complete.

6 Trudinger inequality

When $X = \mathbb{R}^n$ and u belongs to the Sobolev class $W^{1,n}(\Omega)$ for a ball Ω, one has the following Trudinger inequality [241]:

$$\fint_\Omega \exp\left(\frac{C_1|u-u_\Omega|}{||\nabla u||_{L^n(\Omega)}}\right)^{n/(n-1)} dx \le C_2.$$

Here C_1 and C_2 depend only on the dimension n. As in case of the Poincaré inequality, the exact value of u_Ω is not crucial. In fact, it is easy to see that we may replace it by the average of u over some fixed ball $B \subset\subset \Omega$. In the previous section we observed that an s-Poincaré inequality with s not exceeding the lower order of the doubling measure results in exponential integrability. We do not know if one could get an analog of the Trudinger inequality in such a general setting but we doubt it.

In this section we verify an analog of the Trudinger inequality for connected doubling spaces. Thus the only assumption we need to add is that X be connected. For related results, see Bakry, Coulhon, Ledoux and Saloff-Coste [4], Buckley and O'Shea [21], Coulhon [54], and MacManus and Pérez [185].

Theorem 6.1 *Assume that X is a connected doubling space and that the measure μ satisfies condition (21) with $s > 1$. Suppose that the pair u, g satisfies an s-Poincaré inequality. Then there are constants C_1 and C_2 such that*

$$\fint_B \exp\left(\frac{C_1\mu(B)^{1/s}|u-u_B|}{\operatorname{diam}(B)\|g\|_{L^s(5\sigma B)}}\right)^{s/(s-1)} d\mu \le C_2 \tag{42}$$

for any ball $B \subset X$.

Remarks. 1) It is easy to deduce from the connectivity of the space that condition (21) cannot hold with $s < 1$. We leave the details to the reader. 2) The argument employed in the proof actually shows that the inequality holds with $5\sigma B$ replaced by $(1+\varepsilon)\sigma B$.

For the proof of this theorem we need a chain condition, a version of which will also be used later on.

We say that X satisfies a *chain condition* if for every $\lambda \ge 1$ there is a constant M such that for each $x \in X$ and all $0 < r < R < \operatorname{diam}(X)/4$ there is a sequence of balls $B_0, B_1, B_2, \ldots, B_k$ for some integer k with

1. $B_0 \subset X \setminus B(x,R)$ and $B_k \subset B(x,r)$,
2. $M^{-1}\operatorname{diam}(B_i) \le \operatorname{dist}(x, B_i) \le M\operatorname{diam}(B_i)$ for $i = 0, 1, 2, \ldots, k$,
3. there is a ball $R_i \subset B_i \cap B_{i+1}$, such that $B_i \cup B_{i+1} \subset MR_i$ for $i = 0, 1, 2, \ldots, k$,
4. no point of X belongs to more than M balls λB_i.

The sequence $\{B_i\}$ will be called a *chain associated with* x, r, R.

The existence of a doubling measure on X does not guarantee a chain condition. In fact, such a space can be badly disconnected, whereas a space with a chain condition cannot have "large gaps".

Let us show that each connected doubling space satisfies a chain condition. Fix ε. Write $A_j(x) = B(x, 2^j) \setminus B(x, 2^{j-1})$ for $r/4 \leq 2^j \leq 2R$. As μ is doubling we can cover each annulus $\overline{A}_j(x)$ by at most N balls of radii equal to $\varepsilon 2^j$ with N independent of x, j. Naturally, N depends on ε, and the smaller the ε, the larger the number N. Consider the collection of all these balls when $r/4 \leq 2^j \leq 2R$. When ε is sufficiently small, depending only on λ, the balls $2\lambda B$ with B corresponding to $A_j(x)$ and $2\lambda B'$ with B' corresponding to $A_i(x)$ do not intersect provided $|i-j| \geq 2$. The balls B corresponding to the annuli $A_j(x)$ together with $B(x, r/2)$, $X \setminus B(x, 2R)$ form an open cover of X. As X is connected and contains a point inside $B(x, r/2)$ and another point outside $B(x, 2R)$, we can pick a chain of these balls B that joins $B(x, r/2)$ to $X \setminus B(x, 2R)$. The required chain is then obtained as the collection of the balls $2B$ from the balls B different from $B(x, r/2)$ in this chain.

Lemma 6.2 *Assume that X satisfies a chain condition and suppose that a pair u, g satisfies an s-Poincaré inequality for all balls in X. Then the following holds for almost every x. Let $0 < R < \operatorname{diam}(X)/4$. There is r and a chain $\{B_i\}$ corresponding to x, r, R with $\lambda = \sigma$, such that*

$$|u(x) - u_{B_0}| \leq C \sum_{i=0}^{k} r_i \left(\fint_{\sigma B_i} g^s \, d\mu \right)^{1/s}. \tag{43}$$

Proof. As λ is fixed, conditions 1 and 2 of the definition of the chain and the Lebesgue differentiation theorem (see Theorem 14.15) guarantee that $u_{B_k} \to u(x)$ for almost all x when r tends to zero (here $k = k_r$). For such a point we have for appropriate r and corresponding k

$$\begin{aligned}
|u(x) - u_{B_0}| &\leq 2 \sum_{i=0}^{k} |u_{B_i} - u_{B_{i+1}}| \\
&\leq 2 \sum_{i=0}^{k} (|u_{B_i} - u_{R_i}| + |u_{B_{i+1}} - u_{R_i}|) \\
&\leq 2 \sum_{i=0}^{k} \left(\fint_{R_i} |u - u_{B_i}| \, d\mu + \fint_{R_i} |u - u_{B_{i+1}}| \, d\mu \right) \\
&\leq C \sum_{i=0}^{k} \fint_{B_i} |u - u_{B_i}| \, d\mu \\
&\leq C \sum_{i=0}^{k} r_i \left(\fint_{\sigma B_i} g^s \, d\mu \right)^{1/s}.
\end{aligned}$$

The proof is complete.

Proof of Theorem 6.1. By the discussion preceding the previous lemma we know that X satisfies a chain condition. Thus we may assume that the pointwise inequality (43) holds for a given point x. Write r for the radius of the fixed ball B. We may assume that $\operatorname{diam} B_0 \geq r/C$ and that $B_i \subset 5B$, $\sigma B_i \subset 5\sigma B$ for each i.

Fix $q > \max\{s, s/(s-1)\}$. For $0 < \varepsilon < q^{-1}$ we have that

$$\begin{aligned} |u(x) - u_{B_0}| &\leq C \sum_{i=0}^{k} r_i \left(\fint_{\sigma B_i} g^s \, d\mu \right)^{1/s} \\ &= C \sum_i r_i^{1-\varepsilon} \mu(\sigma B_i)^{1/q-1/s} \left(r_i^{q\varepsilon} \fint_{\sigma B_i} g^s \, d\mu \right)^{1/q} \left(\int_{\sigma B_i} g^s \, d\mu \right)^{1/s-1/q} . \end{aligned}$$

As $(s-1)/s + 1/q + (1/s - 1/q) = 1$, we can use Hölder's inequality to obtain the estimate

$$\begin{aligned} |u(x) - u_{B_0}| \leq\ & C \left(\sum_i \left(r_i^{1-\varepsilon} \mu(\sigma B_i)^{1/q-1/s} \right)^{\frac{s}{s-1}} \right)^{\frac{s-1}{s}} \\ & \times \left(\sum_i r_i^{q\varepsilon} M_{5\sigma B} g^s(x) \right)^{1/q} \|g\|_{L^s(5\sigma B)}^{1-s/q}; \end{aligned}$$

here we replaced $\fint_{\sigma B_i} g^s \, d\mu$ by $C M_{5\sigma B} g^s(x)$ and used the bounded overlap of the balls σB_i to replace the sum of the integrals of g^s over these balls by the integral of g^s over $5\sigma B$.

To estimate the second term in the product, we sum over the balls B_i that correspond to an annulus A_j; let us write $I_{i,j}$ for the set of indices i corresponding to A_j. By the construction of the chain we know that we have at most N balls for each fixed j. Moreover the radii of balls corresponding to different A_j form a geometric sequence and hence

$$\begin{aligned} \sum_i r_i^{q\varepsilon} M_{5\sigma B} g^s(x) &= M_{5\sigma B} g^s(x) \sum_j \sum_{I_{i,j}} r_i^{q\varepsilon} \\ &\leq C M_{5\sigma B} g^s(x) r^{q\varepsilon} (1 - 2^{-q\varepsilon})^{-1} \\ &\leq C (q\varepsilon)^{-1} r^{q\varepsilon} M_{5\sigma B} g^s(x), \end{aligned}$$

where C is independent of q, ε. In the last inequality we employed the fact that $q\varepsilon < 1$.

For the first term, we use the lower bound on $\mu(B_i)$ and argue as above:

$$\begin{aligned} \sum_i \left(r_i^{1-\varepsilon} \mu(\sigma B_i)^{1/q-1/s} \right)^{s'} &\leq C \left(r^{1-s/q} \mu(B)^{1/q-1/s} \right)^{s'} \sum_i r_i^{(s/q-\varepsilon)s'} \\ &\leq C q (s - \varepsilon q)^{-1} \left(r^{1-\varepsilon} \mu(B)^{1/q-1/s} \right)^{s'}, \end{aligned}$$

where $s' = s/(s-1)$ and C is an absolute constant.

If we let $\varepsilon = sq^{-2}$, then $q(s-\varepsilon q)^{-1} = q(s-s/q)^{-1} \le q$, as $q \ge s/(s-1)$. Hence

$$|u(x) - u_{B_0}| \le C\|g\|_{L^s(5\sigma B)}^{1-s/q}\mu(B)^{1/q-1/s}q^{1/q+(s-1)/s}r\left(M_{5\sigma B}g^s(x)\right)^{1/q};$$

here C is an absolute constant.

We proceed to estimate the integrals of $|u-u_B|$. By the triangle inequality

$$|u-u_B| \le |u-u_{B_0}| + |u_{B_0} - u_B|\,.$$

By controlling the second term by the Poincaré inequality and using the above pointwise estimate for the first term we arrive at

$$\begin{aligned}\int_B |u-u_B|^{q/2}\,d\mu \quad &\le \quad C^q q^{1/2+(s-1)q/2s}\mu(B)^{1/2-q/2s}\|g\|_{L^s(5\sigma B)}^{q/2-s/2}r^{q/2} \\ &\qquad\times\int_B \left(M_{5\sigma B}g^s\right)^{1/2}\,d\mu + C^q r^{q/2}\mu(B)^{1-q/2s}\|g\|_{L^s(5\sigma B)}^{q/2}\,.\end{aligned}$$

By the Maximal Theorem (see Theorem 14.13) and Theorem 14.11 in the appendix

$$\int_B \left(M_{\sigma r}g^s\right)^{1/2}\,d\mu \le C\left(\mu(B)\int_{5\sigma B} g^s\,d\mu\right)^{1/2}$$

and hence we conclude that

$$\fint_B |u-u_B|^{q/2}\,d\mu \le C^q q^{1/2+(s-1)q/2s}\left(r^s\fint_{5\sigma B} g^s\,d\mu\right)^{q/2s}\,,$$

where C does not depend on q. Notice that this estimate holds as well for $q \le \max\{s, s/(s-1)\}$ by Theorem 5.1.

Now

$$\exp\left(t|u(x)-u_B|\right)^{s/(s-1)} = \sum_{k\ge 0}\frac{(t|u(x)-u_B|)^{ks/(s-1)}}{k!}\,.$$

Integrating over B and using the above estimate we obtain

$$\begin{aligned}\fint_B \exp\left(t|u(x)-u_B|\right)^{s/(s-1)}\,d\mu \quad &\le \quad 1 + \sum_{k>0}\Big((Ck)^{1/2+k}(k!)^{-1} \\ &\qquad\times\left(tr\mu(5\sigma B)^{-1/s}\|g\|_{L^s(5\sigma B)}\right)^{ks/(s-1)}\Big)\,.\end{aligned}$$

This series converges when $tr\mu(5\sigma B)^{-1/s}\|g\|_{L^s(5\sigma B)} \le C_0$, where C_0 depends only on C, s, and the claim follows.

7 A version of the Sobolev embedding theorem on spheres

In order to state our version of the Sobolev embedding theorem on spheres we first have to deal with the problem that u is only defined almost everywhere. To take care of this matter we define $u(x)$ *everywhere* by the formula

$$u(x) := \limsup_{r\to 0} \fint_{B(x,r)} u(z)\, d\mu(z). \tag{44}$$

As almost every point is a Lebesgue point, we have only modified u in a set of measure zero. This redefinition of u essentially corresponds to picking a representative of u with nice continuity properties; for related results see Hajłasz and Kinnunen [104], and Kinnunen and Martio [157].

We again assume that X is a doubling space and

$$\mu(B(x,r)) \geq C_b \mu(B_0) \left(\frac{r}{r_0}\right)^s$$

whenever $B(x,r) \subset B_0 = B(x_0,r_0)$. Recall that such an estimate follows from the doubling condition.

Theorem 7.1 *Suppose that the pair u, g satisfies a p-Poincaré inequality and that $p > s-1$. Then the restriction of u to the set $\{x : d(x,x_0) = r\}$ is uniformly Hölder continuous with exponent $1-(s-1)/p$ for almost every $0 < r < r_0$. In particular, there is a constant C_1 and a radius $r_0/2 < r < r_0$ such that*

$$|u(x) - u(y)| \leq C_1 d(x,y)^{1-(s-1)/p} r_0^{(s-1)/p} \left(\fint_{5\sigma B_0} g^p\, d\mu\right)^{1/p}$$

whenever $d(x,x_0) = d(y,x_0) = r$. The constant C_1 only depends on p, s, C_P, C_b, C_d.

In the case of Carnot groups a related result has been independently obtained by Vodop'yanov, [252].

The usual Sobolev embedding theorem on spheres (cf. [187, Lemma 2.10]) is based on showing that the trace of a Sobolev function belongs to a Sobolev class on almost all spheres. One then uses the Sobolev embedding on the sphere that is lower dimensional than the ball. In our situation a sphere can be very wild and this approach cannot be used. We prove the above result by using a maximal function argument.

The reader may wonder why the integration is taken over all of $5\sigma B_0$ and not over an annulus. The reason for this is that points on the sphere cannot necessarily be joined inside an annulus centered at x_0. For a trivial example, let X be the real axis. If we assume that X has reasonable connectivity properties, we obtain a stronger conclusion.

Theorem 7.2 *Suppose that the pair u, g satisfies a p-Poincaré inequality and that $p > s-1$. Assume that any pair of points in $B_0 \setminus \frac{1}{2}B_0$ can be joined by a continuum F in $CB_0 \setminus C^{-1}B_0$ with $\operatorname{diam} F \le Cd(x,y)$. Then there is a constant C_1 and a radius $r_0/2 < r < r_0$ such that*

$$|u(x)-u(y)| \le C_1 d(x,y)^{1-(s-1)/p} r_0^{(s-1)/p} \left(\fint_{C_1B_0 \setminus C_1^{-1}B_0} g^p \, d\mu \right)^{1/p}$$

whenever $d(x,x_0) = d(y,x_0) = r$. The constant C_1 only depends on p, s, C, C_P, C_b, C_d.

By combining Proposition 4.5 and Theorem 7.2 we obtain the following corollary (recall that a p-Poincaré inequality guarantees a q-Poincaré inequality when $q > p$).

Corollary 7.3 *Suppose that $C^{-1}r^s \le \mu(B(x,r)) \le Cr^s$ with $s > 1$ for each x and all r. Let $s-1 < p \le s$. Assume that for each pair u, g of a continuous function and its upper gradient we have a p-Poincaré inequality. Then there is a constant C_1 and a radius $r_0/2 < r < r_0$ such that*

$$|u(x)-u(y)| \le C_1 d(x,y)^{1-(s-1)/p} r_0^{(s-1)/p} \left(\fint_{C_1B_0 \setminus C_1^{-1}B_0} g^p \, d\mu \right)^{1/p}$$

whenever the pair u, g satisfies the p-Poincaré inequality and $d(x,x_0) = d(y,x_0) = r$. The constant C_1 only depends on p, s, C, C_P.

In the preceding corollary we assumed that $s > 1$ and that $p \le s$. Both of these assumptions are necessary. Indeed, the 1-Poincaré inequality holds for the real axis, but one needs to integrate over balls instead of annuli. The union of the two closed cones in $\mathbb{R}^n$ with a common vertex of Example 4.2 supports a p-Poincaré inequality for all $p > n$ and $\mu(B(x,r)) \approx r^n$ for each x and all r. One again needs to integrate over balls instead of annuli.

Before proceeding with the proofs of Theorems 7.1 and 7.2 let us discuss one more application. We say that u is monotone if

$$\sup_{y,w \in B(x,r)} |u(x)-u(y)| \le \sup \{ |u(y)-u(w)| : \; d(x,y) = d(x,w) = r \}$$

for each ball $B(x,r)$. Suppose that u is monotone, u has an upper gradient in L^s and the assumptions of the previous corollary hold. Then u is continuous and

$$|u(x)-u(y)| \le C \left(\log \frac{C_1 M}{d(x,y)} \right)^{-1/s} \|g\|_{s, B(x, C_1 M)}$$

for all x, y with $d(x,y) \le M$. This estimate is commonly used in the Euclidean setting. We leave it to the reader to deduce this conclusion from the above corollary.

Proof of Theorem 7.1. Fix $x, y \in X$ and $0 \le \alpha < 1$. Set $B_0 = B(x, d(x,y))$, and define $B_i = B(x, 2^i d(x,y))$ when $i \le 1$ and $B_i = B(y, 2^{-i} d(x,y))$ when $i > 1$.

Then, using the Poincaré inequality and the triangle inequality as in the proof of Theorem 3.2, we have that

$$\begin{aligned}
|u(x)-u(y)| &\le C \sum_{i=-\infty}^{\infty} \fint_{B_i} |u - u_{B_i}|\, d\mu \\
&\le C \sum_{i=-\infty}^{\infty} r_i \left(\fint_{2\sigma B_i} g^p \right)^{1/p} \\
&= C \sum_{i=-\infty}^{\infty} r_i^{1-\alpha} r_i^{\alpha} \left(\fint_{2\sigma B_i} g^p \right)^{1/p} \\
&\le C d(x,y)^{1-\alpha} \left(M_{2\sigma d(x,y),p,\alpha} g(x) + M_{2\sigma d(x,y),p,\alpha} g(y) \right),
\end{aligned}$$

where

$$M_{R,p,\alpha} g(x) = \sup_{r<R} r^{\alpha} \left(\fint_{B(x,r)} g^p \, d\mu \right)^{1/p}.$$

Observe that, according to (44), the above inequality holds *everywhere* (cf. [104]).

Write $G_t = \{x \in B_0 : M_{4\sigma r_0,p,\alpha} g(x) < t\}$. Then

$$|u(x)-u(y)| \le Ct d(x,y)^{1-\alpha} \tag{45}$$

for all $x, y \in G_t$.

By the covering Lemma 14.12 and the lower bound on $\mu(B(x,r))$ we have

$$H_\infty^{s-\alpha p}(B_0 \setminus G_t) \le C t^{-p} r_0^s \fint_{5\sigma B_0} g^p d\mu\,.$$

Recall that the *Hausdorff content* $H_\infty^\gamma(E)$, $\gamma \ge 0$, is defined as the infimum of $\sum_i r_i^\gamma$, where the infimum is taken over the set of all countable coverings of the set E by balls with radii r_i.

Define $v : B_0 \to [0, r_0)$ by the formula $v(x) = d(x, x_0)$. Then v is Lipschitz continuous with constant 1, and hence, for each set $E \subset B_0$,

$$H_\infty^{s-\alpha p}(v(E)) \le H_\infty^{s-\alpha p}(E).$$

Let $\alpha = (s-1)/p$. Then

$$H_\infty^1(v(B_0 \setminus G_t)) \le C t^{-p} r_0^s \fint_{5\sigma B_0} g^p \, d\mu\,.$$

This implies that the length of the set $v(B_0 \setminus G_t) \subset [0, r_0)$ goes to 0 as t goes to ∞. Now the theorem follows from the observation that for $r \in [0, r_0) \setminus v(B_0 \setminus G_t)$ the "sphere" $\{x : d(x,x_0) = r\}$ is contained in G_t and hence inequality (45) applies. The proof is complete.

Proof of Theorem 7.2. Join the points x, y by a continuum F in $A = CB_0 \setminus B_0/C$ with $\operatorname{diam}(F) \leq Cd(x,y)$. Let $r = (100C\sigma)^{-1}d(x,y)$, and consider the collection of all balls $B(w,r)$ with $w \in A \cap B(x, 2Cd(x,y))$. As μ is doubling we find a cover of $A \cap B(x, 2Cd(x,y))$ consisting of k of these balls with k depending only on C, C_d, σ. Pick those balls from this cover that intersect F and order them into a chain. That is, denoting the balls by V_i, $V_i \cap V_{i+1} \neq \emptyset$ for $i = 1, ..., l-1$, and $x \in V_1, y \in V_l$, assuming that we have l balls. The claim of Theorem 7.2 follows repeating the proof of Theorem 7.1 with the following modification: we define $B_i = V_i$ for $i = 1, ..., l$, $B_i = B(x, 2^i(100C\sigma)^{-1}d(x,y))$ for $i < 0$ and $B_i = B(y, 2^{-i+l}(100C\sigma)^{-1}d(x,y))$ for $i > l$. It is helpful to notice here that the balls B_i, for $1 \leq i \leq l$, have uniformly bounded overlap as $l \leq k$.

8 Rellich-Kondrachov

The classical Rellich–Kondrachov embedding theorem states that, given a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary, the Sobolev space $W^{1,p}(\Omega)$, $1 \leq p < \infty$, is compactly embedded into $L^q(\Omega)$, where $q \geq 1$ is any finite exponent when $p \geq n$ and any exponent strictly less that $np/(n-p)$ when $p < n$. Of course, here, the Sobolev space $W^{1,p}(\Omega)$ is defined in the classical way.

In the case of Sobolev spaces associated with vector fields, some compact embedding theorems have been obtained by Danielli, [60], Franchi, Serapioni and Serra Cassano, [85], Garofalo and Lanconelli, [90], Garofalo and Nhieu, [92], Lu, [176], Manfredini, [188], Rothschild and Stein, [218].

In this section we extend the Rellich–Kondrachov theorem to the setting of metric spaces. As we will see in Section 11, Sobolev inequalities for vector fields are special cases of Sobolev inequalities on metric spaces. Hence our result covers many of the above results. It extends also an earlier result of Hajłasz and Koskela, [106], from the Euclidean setting. In the case of Sobolev spaces on metric spaces introduced by Hajłasz, [102], a related compactness theorem has been proved independently by Kałamajska, [147].

Let μ be a Borel measure on X, doubling on Ω. As usual, $\Omega \subset X$ denotes an open subset of a metric space. In order to prove the compactness theorem for Sobolev functions on Ω, we need to assume that a kind of embedding theorem holds on Ω. Thus, until the end of the section, we make the following assumption:

The open set $\Omega \subset X$ satisfies $\mu(\Omega) < \infty$ and there exist exponents $p > 0$ and $q > 1$ such that every pair u, g which satisfies a p-Poincaré inequality (5) in Ω (with given constants C_P, σ) satisfies also the global Sobolev inequality

$$\left(\int_\Omega |u|^q \, d\mu\right)^{1/q} \leq C\left(\int_\Omega |u| \, d\mu + \left(\int_\Omega g^p \, d\mu\right)^{1/p}\right). \tag{46}$$

Observe that (46) follows from the Sobolev–Poincaré inequality (7).

Theorem 8.1 *Let X, Ω, μ, $p > 0$ and $q > 1$ be as above. Let $\{u_i, g_i\}$ be a sequence of pairs, all of which satisfy the p-Poincaré inequality (5) in Ω with given constants C_P, σ. If the sequence $\|u_i\|_{L^1(\Omega)} + \|g_i\|_{L^p(\Omega)}$ is bounded, then $\{u_i\}$ contains a subsequence that converges in $L^\alpha(\Omega)$ for any $1 \le \alpha < q$ to some $u \in L^q(\Omega)$.*

Proof. Let $\{u_i, g_i\}$ be a sequence satisfying the assumptions of the theorem. Since the sequence $\{u_i\}$ is bounded in $L^q(\Omega)$, we can select a subsequence (still denoted by $\{u_i\}$) weakly convergent in $L^q(\Omega)$ to some $u \in L^q(\Omega)$. It remains to prove that this sequence converges to u in the norm of $L^\alpha(\Omega)$ for every $1 \le \alpha < q$.

Lemma 8.2 *Let Y be a set equipped with a finite measure ν. Assume that $\{v_i\} \subset L^q(Y)$, $1 < q < \infty$, is a bounded sequence. If v_i converges in measure to $v \in L^q(Y)$, then v_i converges to v in the norm of $L^\alpha(Y)$ for every $1 \le \alpha < q$.*

The lemma is a variant of Proposition 14.9. We postpone the proof of the lemma for a moment and we show how to use it to complete the proof of the theorem. According to the lemma it remains to prove that the functions u_i converge to u in measure.

Assume that $\Omega^c \neq \emptyset$; otherwise the proof is even simpler. For $t > 0$ set $\Omega_t = \{x \in \Omega : \operatorname{dist}(x, \Omega^c) > t\}$. Fix $\varepsilon > 0$ and $t > 0$. For $h < t/\sigma$ (recall that σ appears in (5)) and $x \in \Omega_t$ we set

$$u_h(x) = \fint_{B(x,h)} u \, d\mu \qquad \text{and} \qquad u_{i,h}(x) = \fint_{B(x,h)} u_i \, d\mu.$$

We have

$$\begin{aligned}
\mu(\{x \in \Omega_t : |u_i - u| > \varepsilon\}) &\le \mu(\{x \in \Omega_t : |u_i - u_{i,h}| > \varepsilon/3\}) \\
&+ \mu(\{x \in \Omega_t : |u_{i,h} - u_h| > \varepsilon/3\}) \\
&+ \mu(\{x \in \Omega_t : |u - u_h| > \varepsilon/3\}) \\
&= A_{i,h} + B_{i,h} + C_h.
\end{aligned}$$

Note first that

$$|u_i(x) - u_{i,h}(x)| \le Ch \left(M_{\sigma h} g_i^p(x)\right)^{1/p} \le Ch \left(M_\Omega g_i^p(x)\right)^{1/p}$$

by (16) for almost every $x \in \Omega_t$. Thus the maximal theorem (see Theorem 14.13) gives

$$A_{i,h} \le \mu\left(\left\{M_\Omega g_i^p > C\left(\frac{\varepsilon}{h}\right)^p\right\}\right) \le C\left(\frac{h}{\varepsilon}\right)^p \int_\Omega g_i^p \, d\mu \xrightarrow{h \to 0} 0.$$

This convergence is uniform with respect to i as the sequence $\|g_i\|_{L^p(\Omega)}$ is bounded. It follows from the definition of the weak convergence in $L^q(\Omega)$ that for every $x \in \Omega$, $u_{i,h}(x) \to u_h(x)$ as $i \to \infty$, so $B_{i,h} \to 0$ as $i \to \infty$. Finally $C_h \to 0$ by the Lebesgue differentiation theorem (see Theorem 14.15). Now it easily follows that $u_i \to u$ in measure. Thus the proof is completed provided we prove the lemma.

Proof of the lemma. Fix $1 \leq \alpha < q$. It suffices to prove that every subsequence of $\{v_i\}$ contains a subsequence convergent to v in $L^\alpha(Y)$. In what follows all the subsequences of $\{v_i\}$ will be simply denoted by $\{v_i\}$. Take an arbitrary subsequence of $\{v_i\}$. The convergence in measure implies that this subsequence contains a subsequence which is convergent almost everywhere. Then, by Egorov's theorem, for any $\varepsilon > 0$ there exists a measurable set $E \subset Y$ with the property that $\nu(Y \setminus E) < \varepsilon$ and v_i converges to v uniformly on E. Hence

$$\begin{aligned}\left(\int_Y |v_k - v_j|^\alpha \, d\nu\right)^{1/\alpha} &\leq \nu(Y \setminus E)^{1/\alpha - 1/q} \left(\int_{Y \setminus E} |v_k - v_j|^q \, d\nu\right)^{1/q} \\ &+ \left(\int_E |v_k - v_j|^\alpha \, d\nu\right)^{1/\alpha} \\ &\leq C\varepsilon^{1/\alpha - 1/q} + \left(\int_E |v_k - v_j|^\alpha \, d\nu\right)^{1/\alpha},\end{aligned}$$

which gives $\limsup_{j,k\to\infty} \|v_k - v_j\|_{L^\alpha(Y)} \leq C\varepsilon^{1/\alpha - 1/q}$. Since $\varepsilon > 0$ was arbitrary, the subsequence $\{v_i\}$ is a Cauchy sequence in $L^\alpha(Y)$ and hence the lemma follows. This completes also the proof of the theorem.

Below we state another version of the compactness theorem. The proof follows by some obvious modifications to the above proof.

Theorem 8.3 *Let X be a doubling space and let s be the lower decay order of the measure from (21). Suppose that all the pairs u_i, g_i satisfy a p-Poincaré inequality in X (with fixed constants C_P, σ). Fix a ball B and assume that the sequence $\|u_i\|_{L^1(B)} + \|g_i\|_{L^p(5\sigma B)}$ is bounded. Then there is a subsequence of $\{u_i\}$ that converges in $L^q(B)$ for each $1 \leq q < ps/(s-p)$, when $p < s$ and for each $q \geq 1$ when $p \geq s$.*

Notice that this theorem gives compactness in the entire space provided the space has finite diameter.

We would like to thank Agnieszka Kałamajska for an argument that simplified our original proof of the compactness theorem.

9 Sobolev classes in John domains

In the p-Poincaré inequality (5) we have allowed g to be integrated over a larger ball than u is integrated over. One cannot, in general, reduce the radii of the balls on the right hand side. To see this, consider the following example: let $\Omega = (0,1)\cup(2,3)\cup(4,5)$ and $u \equiv 0$ on $(0,1)$, $u \equiv 1$ on $(2,3)\cup(4,5)$, $g \equiv 0$ on $(0,1)\cup(2,3)$ and $g \equiv$ const. is very large on $(4,5)$. The details are left to the reader.

Hence in the Sobolev type inequalities like Theorem 5.1 or Theorem 6.1 we have to integrate g over a larger ball as well.

We show in this section that one can use balls of the same size provided the geometry of balls is sufficiently nice. This leads us to define John domains.

The first subsection is devoted to study of the geometry of John domains and in the second subsection we study Sobolev inequalities in John domains.

9.1 John domains

When dealing with Sobolev type inequalities in domains in $\mathbb{R}^n$ one usually assumes that the domain is "nice" in the sense that its boundary is locally a graph of a Lipschitz function. This notion of being "nice" is not appropriate for the setting of metric spaces and so one has to define a "nice" domain using only its interior properties. This leads to John domains.

Definition. A bounded open subset Ω of a metric space is called a *John domain* provided it satisfies the following "twisted cone" condition: There exist a distinguished point $x_0 \in \Omega$ and a constant $C > 0$ such that, for every $x \in \Omega$, there is a curve $\gamma : [0, l] \to \Omega$ parametrized by the arclength and such that $\gamma(0) = x$, $\gamma(l) = x_0$ and

$$\operatorname{dist}(\gamma(t), \Omega^c) \geq Ct. \tag{47}$$

(The length l depends on x.)

Notice that every rectifiable curve in a metric space can be parametrized by arclength, see Busemann [22], or Väisälä [246].

John domains in $\mathbb{R}^n$ were introduced by Martio and Sarvas, [192]. They are named after F. John who considered similar domains in [137].

The class of John domains in $\mathbb{R}^n$ is much larger than the class of domains with the interior cone condition. In general, the Hausdorff dimension of the boundary of a John domain can be strictly larger than $n - 1$.

The above definition of John domain is still not appropriate for many metric spaces, as points in an arc-wise connected metric space may not be joinable by rectifiable curves. For example, if $\Gamma \subset \mathbb{R}^2$ is the von Koch snowflake curve and $\Omega \subset \Gamma$ is a nontrivial subcurve, then Ω is not a John domain. However, we would like to include at least some of the metric spaces that lack rectifiable curves in the class of John domains; see Example 9.1. The following definition seems to give a proper generalization of John domains.

Definition. A bounded open subset Ω of a metric space (X, d) is called a *weak John domain* provided there is a distinguished point $x_0 \in \Omega$ and a constant $1 \geq C_J > 0$ such that for every $x \in \Omega$ there exists a curve $\gamma : [0, 1] \to \Omega$ such that $\gamma(0) = x$, $\gamma(1) = x_0$ and for every $t \in [0, 1]$

$$\operatorname{dist}(\gamma(t), \Omega^c) \geq C_J d(x, \gamma(t)). \tag{48}$$

We call such a curve a *weak John curve.* If $\Omega^c = \emptyset$, then we set $\operatorname{dist}(\gamma(t), \Omega^c) = +\infty$ and hence (48) is always satisfied.

Notice that this definition can be also used in the setting of quasimetric spaces (i.e., when the triangle inequality is replaced by $\rho(x,y) \le K(\rho(x,z) + \rho(z,y))$, $K \ge 1$). In general, it can happen that even a very nice metric space does not contain nontrivial rectifiable curves. With the metric $\rho(x,y) = |x-y|^{1/2}$ on the real axis, any interval is of infinite length.

Example 9.1 If $f : S^2 \to S^2$ is a quasiconformal mapping and $\gamma \subset S^2$ is any smooth Jordan curve, then any connected part of $\Gamma = f(\gamma)$ is a weak John domain. This includes the class of Jordan curves $\Gamma \subset S^2$ such that both components of $S^2 \setminus \Gamma$ are John domains, see Näkki and Väisälä, [205].

There are also many other fractal sets whose "nice" subsets are weak John domains, while they cannot be John domains because of the lack of rectifiable curves.

Example 9.2 Let X be a bounded arc-wise connected metric space. If we take $\Omega = X$, then Ω is weak John domain, since (48) is satisfied for *any* curve joining x and x_0.

The reader may find the preceding example somewhat artificial. Let us briefly clarify the issue. Our aim is to deduce a p-Poincaré inequality for Ω whenever such an inequality holds for all balls in Ω, that is, $B(x,r) \subset \Omega$. If we are given an underlying space X, then we consider the balls of the space X that are contained in Ω. Otherwise, the collection of the balls can be fairly large. For example, let Ω be a bounded domain in $\mathbb{R}^n$ so that Ω equipped with the restrictions of the Euclidean distance and volume is a doubling space. If we neglect $\mathbb{R}^n$, and consider Ω as our entire doubling space, we shall obtain a Poincaré inequality for Ω, provided we assume a Poincaré inequality for the pair u, g for all balls of Ω. These balls consist of the intersections of all Euclidean balls centered in Ω with Ω, see Corollary 9.9.

It is known that in the Euclidean case $X = \mathbb{R}^n$ the class of weak John domains coincides with the class of John domains, see Väisälä, [247, Theorem 2.18]. In a metric space this is no longer true. Clearly every John domain is a weak John domain, but the converse implication may fail. However, we generalize the result by proving that under a mild additional condition on the space X, the above two definitions are equivalent, see Proposition 9.6.

The crucial property of John domains for us is that they satisfy a chain condition that is essential in order to effectively use the Poincaré inequality on the balls contained in the domain.

Let us slightly modify the chain condition we employed in connection with the Trudinger inequality.

Definition. We say that Ω satisfies a $\mathcal{C}(\lambda, M)$ condition, where $\lambda,\ M \ge 1$, if there is a "central" ball $B_0 \subset \Omega$ such that to every $x \in \Omega$ and every $r > 0$ there is a sequence of balls $B_0, B_1, B_2, \dots, B_k$, ($B_0$ is fixed and k may depend on x, r) with the following properties

1. $\lambda B_i \subset \Omega$ for $i = 0, 1, 2, \ldots, k$ and $B_k \subset B(x, r)$,
2. $M^{-1} r_i \le \operatorname{dist}(x, \lambda B_i) \le M r_i$ for $i = 1, 2, \ldots$,
3. there is a ball $R_i \subset B_i \cap B_{i+1}$, such that $B_i \cup B_{i+1} \subset M R_i$ for $i = 0, 1, 2, \ldots$,
4. no point of Ω belongs to more than M balls λB_i.

A variant of the above chain condition was employed in Hajłasz and Koskela, [105], [106].

Theorem 9.3 *Assume that X is a metric space which is doubling on $\Omega \subset X$. If $\Omega \subset X$ is a weak John domain, then Ω satisfies the $\mathcal{C}(\lambda, M)$ condition for any $\lambda \ge 1$ with some M depending on λ, C_J and doubling constant only.*

Proof. Assume first that $\Omega \ne X$. Let $B_0 = B(x_0, \operatorname{dist}(x_0, \Omega^c)/(4\lambda))$. Assume that x is far away from B_0. Say $x \in \Omega \setminus 2B_0$. Let γ be a weak John curve. First we define a sequence of balls B_i' as follows. The centers x_i of all balls B_i' lie on γ. Let $B_0' = \frac{1}{2} B_0$. Assume that we have already defined B_i'. Then we trace along γ, starting from the center x_i towards x, until we leave B_i' for the last time. We let this point be the center x_{i+1} of B_{i+1}' and we define

$$r_{i+1}' = \frac{C_J}{4\lambda} d(x, x_{i+1}) . \tag{49}$$

Now we define $B_i = 2B_i'$. Properties 1. and 2. are evidently satisfied provided we choose k large enough. Property 3. follows from the fact that consecutive balls have comparable radii and x_{i+1} lies on the boundary of B_{i+1}' (ball R_i is centered at x_{i+1} and of radius equal to minimum of r_i', r_{i+1}'). For property 4. assume that $y \in \lambda B_{i_1} \cap \lambda B_{i_2} \cap \ldots \cap \lambda B_{i_l}$. It follows from the construction that the radii of the balls λB_{i_j}, $j = 1, 2, \ldots, l$ and the distances between centers of the balls are all comparable to $d(x, y)$. Indeed, the radii are comparable and the distance of the centers times 2λ are no less than the radii. Thus there exists a constant $t > 0$ such that $tB_{i_1}, tB_{i_2}, \ldots, tB_{i_l}$ are pairwise disjoint and all these balls are contained in a ball centered at y and having radius comparable to $d(x, y)$, so the doubling condition implies the upper bound for l.

The case when $x \in 2B_0$ is similar. If $x \in \Omega \setminus B_0$, we define $B_1 = \frac{1}{4} B_0$ and the rest of the argument goes as above. Otherwise, we consider the union of two curves: the weak John curve for x and the weak John curve for some point y with $d(y, x_0) = \frac{1}{2} \operatorname{dist}(x_0, \Omega^c)$. This curve, traced first from x to x_0 and then from x_0 to y, is easily seen to be a John curve with a new distinguished point y and a new John constant only depending on C_J. One can then define desired chain by first replacing the role of x_0 in the above argument by y and then adding a chain joining y to x_0. We leave the details to the reader.

Suppose finally that $\Omega = X$. Then X is bounded. We fix $x_0 \in \Omega$ and define $B_0 = B(x_0, \operatorname{diam}(X)/4)$. The rest of the argument is the same as in the case $X \ne \Omega$ except that in (49) we replace C_J by 1. The proof is complete.

Lemma 9.4 *Let (X, d) be an arcwise connected metric space such that bounded and closed sets in X are compact. Assume that the metric d has the property that for every two points $a, b \in X$ the distance $d(a, b)$ is equal to the infimum of the lengths of curves that join a and b. Then there exists a shortest curve γ from a to b with $d(a, b) = d(a, z) + d(z, b)$, for every $z \in \gamma$.*

Remark. Observe that the compactness of the sets that are bounded and closed is a stronger assumption than the space being locally compact. The two conditions are equivalent if the space is complete.

This lemma is due to Busemann, [22, page 25], (cf. [76, page 592]). The idea of the proof is the following. Let $\{\gamma_k\}_{k=1}^{\infty}$ be a sequence of rectifiable curves joining a with b and such that the length of γ_k converges to $d(a, b)$. Parametrize each γ_k by arclength. Scaling the arclength parametrizations we may assume that all curves are defined on the interval $[0, 1]$. Now it easily follows that the family $\{\gamma_k\}$ is equicontinuous (because of the good parametrization). By a standard diagonal method we can find a subsequence of $\{\gamma_k\}$ which converges on a dense subset of $[0, 1]$. The equicontinuity implies the uniform convergence on the whole interval. It is easy to prove that the length of the limiting curve is $d(a, b)$. For a detailed proof, see [22, page 25].

Corollary 9.5 *Let the metric space (X, d) satisfy the assumptions of the above lemma. Then each ball $B \subset X$ is a John domain with a universal constant C.*

This corollary shows that balls in a Carnot–Carathéodory metric (see Section 11) are John domains.

The chain condition is closely connected to the concept of a John domain as the following proposition shows. Analogs of this result can be found in Buckley, Koskela and Lu, [20], and in Garofalo and Nhieu, [92], where the authors employ a Boman chain condition that is different from ours.

Proposition 9.6 *Let X be a metric space which is doubling on $\Omega \subset X$. Assume that Ω has the following local connectivity property: there exists a constant $\delta \geq 1$ such that for every ball B with $\delta B \subset \Omega$, every two points $x, y \in B$ can be connected by a rectifiable curve contained in δB and of length less than or equal to $\delta d(x, y)$. Then the following three conditions are equivalent:*

1. *Ω is a John domain,*
2. *Ω is a weak John domain,*
3. *Ω is a $\mathcal{C}(\lambda, M)$-domain for each $\lambda \geq 2\delta$ and for some M.*

Proof. The implications $1 \Rightarrow 2 \Rightarrow 3$ hold without any local connectivity assumptions on Ω: the first implication is immediate from the definitions and the second one follows from Theorem 9.3. We prove the implication $3 \Rightarrow 1$. Fix $x \in \Omega$ and let $\{B_i\}_0^k$ be an associated chain, for $\lambda = 2\delta$ and $r = d(x, \Omega^c)/\lambda$. We define $B_{k+1} = B(x, d(x, \Omega^c)/2)$. If the radii of the balls B_i were to increase geometrically when i decreases we would obtain the John curve simply by joining the centers of the balls in our chain by curves obtained from the local connectivity condition. However, this need not be the case. This difficulty can be rectified as follows.

If the entire chain is contained in $B(x, Cd(x, \Omega^c))$, the bounded overlap condition for the chain and the doubling property imply that the length of the subchain (i.e., the number of balls) joining the boundary of B_{k+1} to x_0 does not exceed a uniform constant that depends on the doubling constant, the constants in the chain condition and on C. The above connect-the-dots argument applies in this case. Otherwise, we consider the subchain joining B_{k+1} to $\partial B(x, Cd(x, \Omega^c))$. Again, the length of this subchain is bounded by a uniform constant and the radii are bounded from below by a multiple of $d(x, \Omega^c)$. Pick a point y_1 with $d(y_1, x) = Cd(x, \Omega^c)$ that is contained in one of the balls of the subchain. If the constant C is sufficiently large, then $d(y_1, \Omega^c) \geq 2d(x, \Omega^c)$. Consider a chain joining y_1 to x_0. We now repeat the above argument for the subchain joining y_1 to $\partial B(x, C^2 d(x, \Omega^c))$. By continuing inductively we obtain a new chain with the appropriate geometric behavior. We leave it to the reader to provide the details.

9.2 Sobolev type inequalities

In the main theorem of this section (Theorem 9.7) we show how the claims of Theorem 5.1 and Theorem 6.1 extend to the setting of John domains.

The study of the Sobolev type inequalities in John domains in $\mathbb{R}^n$ originated in the papers of Boman, [13], Bojarski, [12], Goldshtein and Reshetnyak, [97], Hurri, [130], Iwaniec and Nolder, [132], Kohn, [158], and Martio, [191]. It was then generalized to the Carnot–Carathéodory spaces by Jerison, [133], and then to more general situations by Franchi, Gutiérrez and Wheeden, [76], Garofalo and Nhieu, [92], Hajłasz and Koskela, [105], Lu [175], [178]. Other related references include Buckley and Koskela, [16], [17], [18], Buckley, Koskela and Lu, [19], [20], Chen and Li, [37], Chua, [42], Hurri-Syrjänen, [131], Hajłasz and Koskela, [106], Franchi, [74], Lu, [178], Maheux and Saloff-Coste, [186], Saloff-Coste, [221], Smith and Stegenga, [231], [232].

Buckley and Koskela, [17], [18], showed that the class of John domains is nearly the largest one for which one can prove the Sobolev-Poincaré embedding theorem.

Theorem 9.7 *Let X be a metric space equipped with a measure which is doubling on a weak John domain $\Omega \subset X$. Assume that the measure μ satisfies the condition*

$$\mu(B(x,r)) \geq C_b \left(\frac{r}{\operatorname{diam} \Omega} \right)^s \mu(\Omega) \,,$$

whenever $x \in \Omega$ and $r \leq \operatorname{diam} \Omega$. If the pair u, g satisfies a p-Poincaré inequality (5), $p > 0$, in Ω, then all the claims of Theorem 5.1 hold with B and $5\sigma B$ replaced *by Ω. For example, we get that if $0 < p < s$ and the pair u, g has the truncation property, then*

$$\inf_{c\in\mathbb{R}} \left(\fint_\Omega |u-c|^{p^*}\, d\mu \right)^{1/p^*} \leq C(\operatorname{diam}\Omega) \left(\fint_\Omega g^p\, d\mu \right)^{1/p}, \tag{50}$$

where $p^* = sp/(s-p)$.

If in addition the space is connected and $p = s > 1$, then the Trudinger inequality holds in Ω, i.e.,

$$\fint_\Omega \exp\left(\frac{C_1|\Omega|^{1/s}|u-u_\Omega|}{\operatorname{diam}\Omega\, \|g\|_{L^s(\Omega)}} \right)^{s/(s-1)} d\mu \leq C_2\,.$$

The constants C, C_1, C_2 depend on p, s, σ, C_P, C_d, C_b and C_J only.

Remarks. 1) As follows from the proof, the above theorem actually holds for any open set that satisfies the $C(\sigma, M)$ condition. 2) We have stated explicitly only a generalization of one part of Theorem 5.1, inequality (50). It is left to the reader to formulate generalizations of the other cases. 3) Observe that $|u-u_\Omega|$ is replaced by $|u-c|$ and infimum over $c \in \mathbb{R}$ is taken. This is necessary if $p^* < 1$, as then we cannot apply inequality (9).

Proof. By Theorem 9.3 the domain Ω satisfies the chain condition for any given $\lambda = \sigma$. Thus we obtain inequality (43) with balls B_i as in the chain condition; in particular with $\sigma B_i \subset B$. The proofs of Theorem 5.3 and Theorem 6.1 give the claim.

Corollary 9.8 *Let X be a doubling space satisfying the assumptions of Lemma 9.4. Suppose that the measure μ satisfies condition (21). Then all the claims of Theorem 5.1 hold with the integrals of g taken over B instead of $5\sigma B$. If, in addition, the space is connected and $s > 1$, then the Trudinger inequality (42) holds with the integral of g taken over B.*

Remark. This corollary applies to the Carnot–Carathéodory spaces, see Proposition 11.5.

Proof. By Corollary 9.5 every ball is a John domain with a universal constant C and hence we may apply Theorem 9.7. The proof is complete.

We have already mentioned that any bounded arc-wise connected set $\Omega = X$ is a weak John domain. To illustrate this issue we state the following special case of the above results.

Corollary 9.9 *Let $\Omega \subset \mathbb{R}^n$ be an arbitrary bounded domain. Assume that $|B(x,r)\cap\Omega| \geq Cr^n$, whenever $x \in \Omega$ and $r \leq \operatorname{diam}\Omega$. Assume that $u \in W^{1,p}(\Omega)$, $1 \leq p < n$, satisfies*

$$\fint_{\Omega\cap B} |u-u_B|\, dx \leq Cr \fint_{2B\cap\Omega} |\nabla u|\, dx\,,$$

whenever $B = B(x, r)$, $x \in \Omega$ and $r \leq \operatorname{diam} \Omega$. Then the global Sobolev inequality

$$\left(\fint_{\Omega} |u - u_{\Omega}|^{p^*} \, dx \right)^{1/p^*} \leq C(\operatorname{diam} \Omega) \left(\fint_{\Omega} |\nabla u|^p \, dx \right)^{1/p}$$

holds, where $p^* = np/(n-p)$.

Proof. Take $X = \Omega$. The condition $|B(x, r) \cap \Omega| \geq Cr^n$ means that the space X equipped with the Lebesgue measure and the Euclidean metric is doubling. Since $X = \Omega$, we conclude that Ω is a weak John domain and hence the claim follows from Theorem 9.7. The proof is complete.

As the last application of the chain method we improve the so-called representation formula (33). The result below is a generalization and simplification of earlier results due to Capogna, Danielli and Garofalo, [30], Franchi, Lu and Wheeden, [80], [81], and Franchi and Wheeden, [86].

Theorem 9.10 *Assume that X is a metric space which is doubling on a weak John domain $\Omega \subset X$. If the pair u, g satisfies a p-Poincaré inequality (5), $p > 0$, in Ω, then for almost every $x \in \Omega$ we have the inequality*

$$|u(x) - u_{B_0}| \leq C \sum_{j=-\infty}^{\infty} \left(\int_{A_j(x) \cap \Omega} \frac{g(y)^p d(x, y)^p}{\mu(B(x, d(x, y)))} \, d\mu(y) \right)^{1/p},$$

where $A_j(x) = B(x, 2^j) \setminus B(x, 2^{j-1})$ and $B_0 = B(x_0, \operatorname{dist}(x_0, \Omega^c)/(4\sigma))$, $x_0 \in \Omega$, is a fixed ball.

In particular when $p = 1$ we get

$$|u(x) - u_{B_0}| \leq C \int_{\Omega} \frac{g(y) d(x, y)}{\mu(B(x, d(x, y)))} \, d\mu(y) \, .$$

Proof of Theorem 9.10. By Theorem 9.3 the domain satisfies the $\mathcal{C}(\lambda, M)$ condition with $\lambda = \sigma$. Then we have

$$\begin{aligned} |u(x) - u_{B_0}| &\leq 2 \sum_{i=0}^{k} |u_{B_i} - u_{B_{i+1}}| \\ &\leq C \sum_{i=0}^{k} r_i \left(\fint_{\sigma B_i} g^p \, d\mu \right)^{1/p} . \end{aligned}$$

Each ball σB_i is covered by a finite number, say no more than l, of the annuli $A_j(x)$. Hence if $\sigma B_i \cap A_j(x) \neq \emptyset$ we get

$$r_j \left(\fint_{\sigma B_i} g^p \, d\mu \right)^{1/p} \leq C \sum_{\nu=j-l}^{j+l} \left(\int_{A_\nu(x) \cap \Omega} \frac{g^p(y) d(x, y)^p}{\mu(B(x, d(x, y)))} \, d\mu(y) \right)^{1/p} .$$

Now observe that the doubling condition and the bounded overlapping of the balls σB_i implies that the number of balls σB_i with $\sigma B_i \cap A_j(x) \neq \emptyset$ is bounded by a constant not depending on j. This easily implies the claim.

There are several related results when $p = 1$. For the Euclidean case see Goldshtein and Reshetnyak [97], Martio [191] and Hajłasz and Koskela [106]; for the Carnot–Carathéodory case see Franchi, Lu and Wheeden [80], Capogna, Danielli and Garofalo [30], and for the case of doubling spaces see Franchi, Lu and Wheeden [81], Franchi and Wheeden [86].

10 Poincaré inequality: examples

The purpose of this section is to illustrate the results obtained up to now in the paper: we collect basic examples of pairs that satisfy p-Poincaré inequalities.

We will pay particular attention to the validity of the truncation property. Recall that this property is used to prove the Sobolev embedding in the borderline case with the sharp exponent.

Two classes of examples, Carnot–Carathéodory spaces and graphs, require a longer presentation, and so we discuss them in Sections 11 and 12.

10.1 Riemannian manifolds.

The pair u, $|\nabla u|$, where $u \in \mathrm{Lip}\,(\mathbb{R}^n)$, satisfies the 1-Poincaré inequality and hence all the p-Poincaré inequalities for $1 \leq p < \infty$. Obviously the pair u, $|\nabla u|$ also has the truncation property.

This result extends to those Riemannian manifolds whose Ricci curvature is bounded from below. Let M be a complete Riemannian manifold of dimension n, and let g denote the Riemannian metric tensor. Denote the canonical measure on M by μ. Assume that the Ricci curvature is bounded from below i.e. $Ric \geq -Kg$ for some $K \geq 0$. Then the Bishop–Gromov comparison theorem implies that

$$\mu(B(x,2r)) \leq 2^n \exp(\sqrt{(n-1)K}2r)\mu(B(x,r)),$$

see Cheeger, Gromov and Taylor [35]. Moreover Buser's inequality, [23], implies that

$$\int_B |u - u_B|\, d\mu \leq C(n) \exp(\sqrt{K}r) r \int_B |\nabla u|\, d\mu.$$

This shows that for any $R > 0$ both the doubling property and the 1-Poincaré inequality hold on all balls with radii less than R. If we assume that the Ricci curvature is nonnegative (i.e. $K = 0$), then we can take $R = \infty$. Obviously, the pair of a Lipschitz function and the length of its gradient has the truncation property in this setting as well.

Thus the results of our paper imply that in the above setting, the Sobolev-Poincaré inequality holds, see Maheux and Saloff-Coste [186] and also Hajłasz and Koskela [105].

An excellent introduction to the Buser inequality and the comparison theorems can be found in Chavel's book, [33].

For related and earlier works on Poincaré and Sobolev inequalities on manifolds with a bound on the Ricci curvature see Chen and Li [37], Gallot [88], Kusuoka and Stroock [163], Li and Schoen [170], Li and Yau [172], Saloff-Coste [220], [221].

10.2 Upper gradients.

Let (X, d, μ) be a metric space with a Borel measure, not necessarily doubling.

Definition. We say that a Borel function $g : \Omega \to [0, \infty]$ is *an upper gradient on* Ω of another Borel function $u : \Omega \to \mathbb{R}$, if for *every* 1-Lipschitz curve $\gamma : [a, b] \to \Omega$ we have

$$|u(\gamma(b)) - u(\gamma(a))| \leq \int_a^b g(\gamma(t))\, dt. \tag{51}$$

Note that $g \equiv \infty$ is an upper gradient of any Borel function u.

Definition. We say that the space *supports a p-Poincaré inequality inequality on* Ω, $1 \leq p < \infty$, if every pair u, g of a continuous function u and its upper gradient g on Ω satisfies the p-Poincaré inequality (5) in Ω (with some fixed constants $C_P > 0$, $\sigma \geq 1$).

If we say that the space supports a p-Poincaré inequality, then we mean that above $\Omega = X$.

Since every rectifiable curve admits an arc-length parametrization that makes the curve 1-Lipschitz, the class of 1-Lipschitz curves coincides with the class of rectifiable curves, modulo a parameter change.

It is necessary to assume that the function g is defined *everywhere*, as we require the condition (51) for *all* rectifiable curves. We refer the reader to Busemann, [22], or Väisälä, [246, Chapter 1], for more information on integration over rectifiable curves.

The notions of an upper gradient and a space supporting a p-Poincaré inequality were introduced by Heinonen and Koskela, [118], and then applied and further developed by Bourdon and Pajot, [15], Cheeger, [34], Franchi, Hajłasz and Koskela, [77], Heinonen and Koskela, [119], Kallunki and Shanmugalingam, [148], Koskela and MacManus, [161], Laakso, [164], Semmes, [226], Tyson, [245], and Shanmugalingam, [228].

Notice that above we required the p-Poincaré inequality for continuous functions and their upper gradients. If X is sufficiently nice, say quasiconvex and each closed ball in X is compact, then the p-Poincaré inequality follows for any measurable function and its upper gradient. In fact, it would in such a setting suffice to assume the p-Poincaré inequality for Lipschitz functions and their upper gradients. For this see [119].

Proposition 10.1 *If u is a Lipschitz function on a Riemannian manifold M, then any measurable function g such that $g \geq |\nabla u|$ everywhere is an upper gradient of u. On the other hand, if $g \in L^p(M)$ is an upper gradient of $u \in L^p(M)$, then $u \in W^{1,p}(M)$ and $g \geq |\nabla u|$ almost everywhere.*

Proof. The first part of the proposition is immediate; the second one follows from the ACL characterization of the Sobolev space, see for example Ziemer [263, Theorem 2.1.4].

Remark. It is not true, in general, that any upper gradient g of $u \in C^\infty(M)$ satisfies $g \geq |\nabla u|$ a.e., unless we assume that g is locally integrable. As an example take $u(x) \equiv x$ on $[0,1]$. Let $E \subset [0,1]$ be a Cantor set with positive length and set $g(x) = 0$ if $x \in E$, $g(x) = \infty$ if $x \notin E$. One can then improve this example and even obtain $g < \infty$ everywhere.

Proposition 10.2 *If u is a locally Lipschitz function defined on an open subset of a metric space X, then the function $|\nabla^+ u|(x) = \limsup_{y\to x} |u(x)-u(y)|/d(x,y)$ is an upper gradient of u.*

Remark. The proposition is no longer true if we only assume that u is continuous. Indeed, if u is the familiar non-decreasing continuous function $u : [0,1] \to [0,1]$ such that $u(0) = 0$, $u(1) = 1$ and u is constant on connected components of the complement of a Cantor set, then $|\nabla^+ u(x)| = 0$ a.e. in $[0,1]$.

Proof of the proposition. Let $\gamma : [a,b] \to \Omega$ be 1-Lipschitz. The function $u \circ \gamma$ is Lipschitz and hence differentiable a.e. It easily follows that $|(u\circ\gamma)'(t)| \leq |\nabla^+ u(\gamma(t))|$ whenever $u \circ \gamma$ is differentiable at t. Hence

$$|u(\gamma(b)) - u(\gamma(a))| \leq \int_a^b |(u\circ\gamma)'(t)|\, dt \leq \int_a^b |\nabla^+ u(\gamma(t))|\, dt.$$

The proof is complete.

Theorem 10.3 *Assume that the space X supports a p-Poincaré inequality on Ω. Then any pair u, g of a continuous function and its upper gradient on Ω has the truncation property.*

Proof. Let g be an upper gradient of a continuous function u. We have to prove a family of p-Poincaré inequalities for all the pairs $v_{t_1}^{t_2}$, $g\chi_{\{t_1 < v \leq t_2\}}$, where $v = \varepsilon(u-b)$ (see the definition of the truncation property). Since g is an upper gradient of each of the functions v, we can assume that $u = v$. Thus it remains to prove the inequality

$$\fint_B |u_{t_1}^{t_2} - u_{t_1\,B}^{t_2}|\, d\mu \leq Cr \left(\fint_{\sigma B} g^p \chi_{\{t_1 < u \leq t_2\}}\, d\mu \right)^{1/p}. \tag{52}$$

The following lemma is due to Semmes, [226, Lemma C.19]. For reader's convenience we recall the proof.

Lemma 10.4 *Let g be an upper gradient of a continuous function u. Let $0 < t_1 < t_2 < \infty$, and let V be an arbitrary open set such that $\{t_1 \leq u \leq t_2\} \subset V$. Then $g\chi_V$ is an upper gradient of $u_{t_1}^{t_2}$.*

Proof. Let γ be a curve as in the definition of the upper gradient. We have to prove the analog of (51) for $u_{t_1}^{t_2}$ and $g\chi_V$. If either γ is contained in V or γ is contained in $X \setminus \{t_1 \leq u \leq t_2\}$, the claim is very easy. In the general case the curve γ splits into a finite number of parts, each of which is contained in V or in $X \setminus \{t_1 \leq u \leq t_2\}$ and the lemma follows by applying the preceding special cases to those pieces of γ.

Now we can complete the proof of the theorem. Take $t_1 < s_1 < s_2 < t_2$. Then $\{s_1 \leq u \leq s_2\} \subset V$, where $V = \{t_1 < u < t_2\}$. Applying the p-Poincaré inequality to the pair $u_{s_1}^{s_2}$, $g\chi_V$ and passing to the limit as $s_1 \searrow t_1$, $s_2 \nearrow t_2$ we obtain the desired inequality. This completes the proof.

Theorem 10.3 is interesting provided we can find sufficiently many examples of non-smooth metric spaces that support a p-Poincaré inequality. The rest of Section 10 is devoted to the discussion of such examples.

10.3 Topological manifolds.

Definition. A metric space X is called *Q-regular*, $Q > 0$, if it is a complete metric space and there is a measure μ on X so that $C_1 r^Q \leq \mu(B(x,r)) \leq C_2 r^Q$ whenever $x \in X$ and $r \leq \operatorname{diam} X$.

It is well known that one can replace μ in the above definition by the Q-dimensional Hausdorff measure, see for example [226, Lemma C.3].

Semmes, [226], proved a p-Poincaré inequality on a large class of Q-regular metric spaces including some topological manifolds.

The following result is a direct consequence of a more general result of Semmes, [226].

Theorem 10.5 *Let X be a connected Q-regular metric space that is also an orientable topological Q-dimensional manifold, $Q \geq 2$, integer. Assume that X satisfies the local linear contractibility condition: there is $C \geq 1$ so that, for each $x \in X$ and $R \leq C^{-1} \operatorname{diam} X$, the ball $B(x,R)$ can be contracted to a point inside $B(x,CR)$. Then the space supports a 1-Poincaré inequality.*

For related inequalities also see, David and Semmes [62], and Semmes [227].

10.4 Gluing and related constructions.

Heinonen and Koskela, [118, Theorem 6.15], proved that gluing two spaces that support a p-Poincaré inequality along a sufficiently large common part results in a new space that also supports a p-Poincaré inequality. For example, one can glue two copies of the unit ball of $\mathbb{R}^2$ along the usual $\frac{1}{3}$-Cantor set and the resulting doubling space supports a p-Poincaré inequality for all $p > 2 - \frac{\log 2}{\log 3}$. This procedure allows one to build plethora of examples. Hanson and Heinonen, [111], used this type of a construction recently to build a space that supports the 1-Poincaré inequality but has no manifold points.

Laakso, [164], constructed recently for each $Q > 1$ a Q-regular metric space that supports the 1-Poincaré inequality. Notice that here Q need not be an integer. These spaces do not admit bi-Lipschitz imbeddings into any Euclidean space. They are obtained as quotients by finite to one maps of products of intervals with Cantor sets.

The first authors to find non-integer dimensional Q-regular spaces that support a Poincaré inequality were Bourdon and Pajot, [15]. Their examples are boundaries of certain hyperbolic buildings.

10.5 Further examples.

A huge class of examples of spaces that support p-Poincaré inequalities is contained in the class of so-called Carnot–Carathéodory spaces that are discussed in Section 11. This class includes the Carnot groups that have been mentioned above.

One can also investigate p-Poincaré inequalities on graphs, see Section 12. Here the situation is however different. Since the space is disconnected, the notion of an upper gradient is absurd. Moreover, the truncation property does not hold and it has to be modified.

There are also many other examples that will not be discussed in the paper. The main class of such examples is given by Poincaré inequalities on Dirichlet spaces. Roughly speaking we are given a pair u, g satisfying a p-Poincaré inequality on a doubling space and in addition g is related to u in terms of a Dirichlet form, see Biroli and Mosco [8], [9], [10], Jost [143], [144], [145], Sturm [236], [237], [238], [239]. Thus this example fits precisely into the setting of our paper. However, the presence of the Dirichlet form gives additional structure that may lead to results not under the scope of our more general setting.

The analysis of Dirichlet forms is closely related to the analysis on fractals and especially with the spectral theory of Laplace operators, see, e.g., Barlow and Bass [5], Jonsson [139], Kozlov [162], Kigami [151], [152], [153], Kigami and Lapidus [154], Lapidus [167], [168], Metz and Sturm [197], Mosco [201]. As it follows from recent works of Jonsson, [139], and Jonsson and Wallin, [141], the spectral theory of the Laplace operators on fractals is also related to the theory of function spaces on fractal subsets of $\mathbb{R}^n$ developed by Jonsson and Wallin, [138], [140], see also Triebel [244]. Some connections with the theory presented in this paper seem evident, but a better understanding of those connections is still lacking.

11 Carnot–Carathéodory spaces

In this section we give an introduction to the analysis of vector fields — one of the main areas where the theory of Sobolev spaces on metric spaces is applicable.

In the first subsection we define the so called Carnot–Carathéodory metric associated with a family of vector fields $X = (X_1, \dots, X_k)$. Then, in the second subsection, we prove that with respect to this metric the "gradient" $|Xu|$ associated with the given family of vector fields becomes the smallest upper gradient of u. We also deal with Poincaré inequalities and Sobolev spaces associated with the given system of vector fields.

The main novelty in our approach is that we develop the analysis on Carnot–Carathéodory spaces from the point of view of upper gradients. The prime results of the section are Theorem 11.7 and Theorem 11.12.

In the last three subsections we consider Carnot groups and vector fields satisfying Hörmander's condition — both are examples where pairs u, $|Xu|$ satisfy such a Poincaré inequality. We also discuss some other classes of vector fields that do not satisfy Hörmander's condition, but still support Poincaré inequalities.

11.1 Carnot–Carathéodory metric.

Let $\Omega \subset \mathbb{R}^n$ be an open and connected set and let X_1, X_2,…,X_k be vector fields defined in Ω, with real locally Lipschitz continuous coefficients. We identify the X_j's with the first order differential operators that act on $u \in \mathrm{Lip}\,(\Omega)$ by the formula

$$X_j u(x) = \langle X_j(x), \nabla u(x) \rangle, \quad j = 1, 2, \dots, k.$$

We set $Xu = (X_1 u, \dots, X_k u)$, and hence

$$|Xu(x)| = \left(\sum_{j=1}^{k} |X_j u(x)|^2 \right)^{1/2}.$$

With such a family of vector fields one can associate a suitable degenerate elliptic operator, like for example $L = -\sum_{j=1}^{k} X_j^* X_j$, where X_j^* is the formal adjoint of X_j in L^2 i.e., $\int X_j u\, v = \int u X_j^* v$ for all $u, v \in C_0^\infty$. Both the Poincaré and Sobolev inequalities for the pair u, $|Xu|$ are then crucial for the Harnack inequality for positive solutions to $Lu = 0$ via Moser's iteration. Since the Poincaré inequality implies the Sobolev inequality, one needs only check the validity of a Poincaré inequality. Of course this requires strong restrictions on the class of vector fields.

Even if one considers degenerate elliptic equations of the divergence form

$$Lu(x) = \operatorname{div}(A(x)\nabla u(x)) \tag{53}$$

with a symmetric, nonnegative semi-definite matrix A with smooth coefficients, it is, in general, necessary to deal with vector fields that have only Lipschitz coefficients as they arise in the factorization $L = -\sum_j X_j^* X_j$, see Oleinik and Radkevic [208].

For more applications to PDE and references, see Section 13.

How does one prove a Poincaré inequality for the pair u, $|Xu|$? The natural approach is to bound u by integrals of $|Xu|$ along curves and then average the resulting one-dimensional integrals to obtain the desired Poincaré inequality.

In order to have such bounds for u in terms of integrals of $|Xu|$, one would like to know that $|Xu|$ is an upper gradient of u. Unfortunately this is rarely the case.

For example, if we have only the single vector field $X_1 = \partial/\partial x_1$ and $\gamma(t) = (0,t)$, $u(x_1,x_2) = x_2$ in $\mathbb{R}^2$, then $|u(\gamma(1)) - u(\gamma(0))| = 1$, while $|Xu| \equiv 0$, and so $|Xu|$ is not an upper gradient of u. It is not an upper gradient even up to a constant factor. Roughly speaking, the problem is caused by the fact that $\dot\gamma$ is not spanned by the X_j's.

There is a brilliant idea that allows one to avoid this problem by introducing a new metric (that is described below) in Ω that makes $|Xu|$ an upper gradient of u on a new metric space. The metric is such that it restricts the class of 1-Lipschitz curves to those for which $\dot\gamma$ is a linear combination of the X_j's. To be more precise, it is not always a metric as it allows the distance to be infinite.

We say that an absolutely continuous curve $\gamma : [a,b] \to \Omega$ is *admissible* if there exist measurable functions $c_j(t)$, $a \le t \le b$, satisfying $\sum_{j=1}^k c_j(t)^2 \le 1$ and $\dot\gamma(t) = \sum_{j=1}^k c_j(t) X_j(\gamma(t))$.

Note that if the vector fields are not linearly independent at a point, then the coefficients c_j are not unique.

Then we define the distance $\rho(x,y)$ between $x,y \in \Omega$ as the infimum of those $T > 0$ such that there exists an admissible curve $\gamma : [0,T] \to \Omega$ with $\gamma(0) = x$ and $\gamma(T) = y$.

If there is no admissible curve that joins x and y, then we set $\rho(x,y) = \infty$.

Note that the space (Ω,ρ) splits into a (possibly infinite) family of metric spaces $\Omega = \bigcup_{i\in I} A_i$, where $x,y \in A_i$ if and only if x and y can be connected by an admissible curve in A_i. Obviously (A_i,ρ) is a metric space and the distance between distinct A_i's is infinite.

If we only have the single vector field $X_1 = \partial/\partial x_1$ in $\mathbb{R}^2$, then $\rho(x,y) = |x-y|$ if x and y lie on a line parallel to the x_1 axis; otherwise $\rho(x,y) = \infty$. On the other hand, if $X_j = \partial/\partial x_j$ for $j = 1,2,\ldots,n$ in $\mathbb{R}^n$, then ρ is the Euclidean metric.

The distance function ρ is given many names in the literature. We will use the name *Carnot–Carathéodory distance.* A space equipped with the Carnot–

Carathéodory distance is called a *Carnot–Carathéodory space.*

There are several other equivalent definitions for the Carnot–Carathéodory distance, see, e.g., Jerison and Sanchez-Calle [136] and Nagel, Stein and Wainger [204]. The Carnot–Carathéodory distance can also be defined for Dirichlet forms, see Sturm [238].

It has already been mentioned that Lipschitz vector fields arise in connection with degenerate elliptic equations of the divergence form (53). It seems that Fefferman and Phong, [69], where the first to realize that many important properties of the operator can be read off from the properties of the associated Carnot–Carathéodory metric. Roughly speaking, they proved that, locally, subellipticity of (53) is equivalent to the estimate $\rho(x,y)\leq C|x-y|^{\varepsilon}$ for some $\varepsilon>0$.

Other connections with degenerate elliptic equations will be discussed later on in Section 13. We want to emphasize that the scope of applications of the Carnot–Carathéodory geometry goes far beyond degenerate elliptic equations and it includes control theory, CR geometry, and more recently quasiconformal mappings. We refer the reader to the collection [240] of papers for a comprehensive introduction to the Carnot–Carathéodory geometry. Other important references include: Franchi, [74], Franchi and Lanconelli, [78], Garofalo and Nhieu, [92], [91], Gole and Karidi, [98], Karidi, [150], Liu and Sussman, [174], Nagel, Stein and Wainger, [204], Pansu, [210], Saloff-Coste, [221], [223], Strichartz, [235], Varopoulos, Saloff-Coste and Coulhon, [251], to name a few.

Lemma 11.1 *Let $B(x,R)\subset\subset\Omega$ and let $\sup_{B(x,R)}|X|=M$. If $\gamma:[0,T]\rightarrow\Omega$, $T<R/M$, is an admissible curve with $\gamma(0)=x$, then $\gamma([0,T])\subset B(x,R)$.*

Proof. Assume by contradiction that the image of η is not contained in the ball $B(x,R)$. Then there is the smallest $t_0\in(0,T]$ such that $|x-\gamma(t_0)|=R$. Note that by the Schwartz inequality, $|\dot\gamma(t)|\leq|X(\gamma(t))|$. Hence

$$R=|x-\gamma(t_0)|=\left|\int_0^{t_0}\dot\gamma(t)\,dt\right|\leq\int_0^{t_0}|X(\gamma(t))|\,dt\leq MT,$$

which contradicts the assumption $T<R/M$. The proof is complete.

As as a corollary we obtain the following well known result.

Proposition 11.2 *Let $G\subset\subset\Omega$. Then there is a constant $C>0$ such that*

$$\rho(x,y)\geq C|x-y|,$$

for all $x,y\in G$.

Proof. Let $x,y\in G$ and let $\gamma:[0,T]\rightarrow\Omega$, $\gamma(0)=x$, $\gamma(T)=y$, be any admissible curve. Fix $\varepsilon>0$ such that $G^{\varepsilon}=\{x\in\mathbb{R}^n:\,\mathrm{dist}\,(x,G)<\varepsilon\}\subset\subset\Omega$ and set $M=\sup_{G^{\varepsilon}}|X|$. Obviously $B(x,R)\subset G^{\varepsilon}$, when $R=\min\{|x-y|,\varepsilon\}$, and hence

Lemma 11.1 implies that $T \geq R/M \geq \min\{M^{-1}, \varepsilon(M\operatorname{diam} G)^{-1}\}|x-y|$. This completes the proof.

If $\rho(x,y) < \infty$ for all $x, y \in \Omega$, then ρ is a true metric (called the *Carnot–Carathéodory metric*). Proposition 11.2 implies that $\mathrm{id} : (\Omega, \rho) \to (\Omega, |\cdot|)$ is continuous. However, it need not be a homeomorphism as the simple example of the two vector fields ∂_x and $x_+\partial_y$ in $\mathbb{R}^2$ shows.

In order to avoid such pathological situations, it is often assumed in the literature that

$$\mathrm{id} : (\Omega, \rho) \to (\Omega, |\cdot|) \qquad \text{is a homeomorphism.} \tag{54}$$

Fortunately, (54) is true for a large class of vector fields satisfying the so-called Hörmander condition which includes Carnot groups (see the following subsections), and the case of Grushin type vector fields like those in Franchi, [74], Franchi, Gutiérrez and Wheeden, [76], and Franchi and Lanconelli, [78].

To keep the generality we *do not* assume (54) unless it is explicitly stated.

By a Lipschitz function on Ω we mean Lipschitz continuity with respect to the Euclidean metric in Ω, but when we say that a function is Lipschitz on (Ω, ρ) we mean Lipschitz with respect to the distance ρ. The same convention extends to functions with values in Ω or in (Ω, ρ). Functions that are Lipschitz with respect to ρ will be also called *metric Lipschitz*. Balls with respect to ρ will be called *metric balls* and denoted by $\widetilde{B}$.

Lemma 11.3 *Every admissible curve $\gamma : [0,T] \to \Omega$ is Lipschitz.*

Proof. Use the Schwartz inequality.

Proposition 11.4 *A mapping $\gamma : [0,T] \to (\Omega, \rho)$ is an admissible curve if and only if it is 1-Lipschitz i.e., $\rho(\gamma(b), \gamma(a)) \leq |b-a|$ for all a, b.*

Proof. $\Rightarrow$. This implication directly follows from the definition of ρ.

$\Leftarrow$. Let $\gamma : [0,T] \to (\Omega, \rho)$ be a 1-Lipschitz curve. By Lemma 11.3 it is Lipschitz with respect to the Euclidean metric on Ω and hence it is differentiable a.e. We have to prove that γ is admissible. Let $t_0 \in (0,T)$ be any point where γ is differentiable. Since $\rho(\gamma(t_0+\varepsilon), \gamma(t_0)) \leq \varepsilon$ for $\varepsilon > 0$, there exists an admissible curve $\eta : [0, \varepsilon+\delta] \to \Omega$, $\eta(0) = \gamma(t_0)$, $\eta(\varepsilon+\delta) = \gamma(t_0+\varepsilon)$ for any $\delta > 0$. We have

$$\int_0^{\varepsilon+\delta} \dot{\eta}(t)\, dt = \gamma(t_0+\varepsilon) - \gamma(t_0) = \dot{\gamma}(t_0)\varepsilon + o(\varepsilon).$$

By the definition of an admissible curve there are measurable functions $c_j(t)$ such that $\sum_j c_j(t)^2 \leq 1$ and

$$\dot{\eta}(t) \;=\; \sum_{j=1}^{k} c_j(t) X_j(\gamma(t_0)) + \sum_{j=1}^{k} c_j(t) \Big(X_j(\eta(t)) - X_j(\eta(0)) \Big)$$

$$= \sum_{j=1}^{k} c_j(t) X_j(\gamma(t_0)) + a(t).$$

Note that, by Proposition 11.2, $C|\eta(t) - \eta(0)| \le \rho(\eta(t), \eta(0)) \le t$, provided ε and δ are sufficiently small. Hence $|a(t)| \le |X(\eta(t)) - X(\eta(0))| \le Ct$, as the vector fields have locally Lipschitz coefficients. Thus we conclude that

$$\begin{aligned}\dot\gamma(t_0) &= \varepsilon^{-1} \int_0^{\varepsilon+\delta} \dot\eta(t)\, dt + \frac{o(\varepsilon)}{\varepsilon} \\ &= \frac{\varepsilon+\delta}{\varepsilon} \sum_{j=1}^{k} \left(\fint_0^{\varepsilon+\delta} c_j(t)\, dt \right) X_j(\gamma(t_0)) + \varepsilon^{-1} \int_0^{\varepsilon+\delta} a(t)\, dt + \frac{o(\varepsilon)}{\varepsilon}.\end{aligned}$$

Selecting suitable sequences $\varepsilon_l \to 0$ and $\delta_l \to 0$ we conclude that $\dot\gamma(t_0) = \sum_j b_j X_j(\gamma(t_0))$, $\sum_j b_j^2 \le 1$. This completes the proof.

It is well known that any rectifiable curve in a metric space admits an arc-length parametrization, see [22] or [246, Chapter 1]. This also holds for the Carnot–Carathéodory distance as a Carnot–Carathéodory space splits into a family of metric spaces such that each rectifiable curve is entirely contained in one of these metric spaces. Note also that the arc-length parametrization makes the curve 1-Lipschitz and hence admissible. This observation implies the following result.

Proposition 11.5 *The Carnot–Carathéodory distance between any two points equals the infimum of lengths (with respect to ρ) of curves that join those two points. If the points cannot be connected by a rectifiable curve, then their distance is infinite.*

11.2 Upper gradients and Sobolev spaces.

The following two results generalize Proposition 10.1.

Proposition 11.6 *$|Xu|$ is an upper gradient of $u \in C^\infty(\Omega)$ on the space (Ω, ρ).*

Proof. Let $\gamma : [a,b] \to (\Omega, \rho)$ be a 1-Lipschitz curve. By Lemma 11.3, $u \circ \gamma$ is Lipschitz and hence

$$|u(\gamma(b)) - u(\gamma(a))| = \left| \int_a^b \langle \nabla u(\gamma(t)), \dot\gamma(t) \rangle\, dt \right| \le \int_a^b |Xu(\gamma(t))|\, dt;$$

the inequality follows from the fact that γ is admissible by Lemma 11.4 and from the Schwartz inequality. The proof is complete.

Theorem 11.7 *Let $0 \le g \in L^1_{\mathrm{loc}}(\Omega)$ be an upper gradient on (Ω, ρ) of a function u which is continuous with respect to the Euclidean metric. Then the distributional derivatives $X_j u$, $j = 1, 2, \ldots, k$, are locally integrable and $|Xu| \le g$ a.e.*

Notice that when (54) holds, the continuity assumption above can as well be given with respect to the metric ρ.

The proof of the theorem is rather complicated and thus we first make some comments and give applications and postpone the proof until the end of the subsection.

The proof is particularly easy if $u \in C^\infty(\Omega)$ and the vector fields have C^1 smooth coefficients. We present it now as it may help to understand the proof for the general case.

Since u is smooth, we do not have to worry about distributional derivatives and we simply prove that $g \geq |Xu|$ a.e.

The set of the points where $|Xu(x)| > 0$ is open. Since the desired inequality holds trivially outside this set, we can assume that $|Xu| > 0$ everywhere in Ω. Let $a_j(x) = X_j u(x)/|Xu(x)|$ and let γ be any integral curve of the vector field $Y = \sum_j a_j X_j$ i.e., $\gamma : (-T, T) \to \Omega$, $\dot\gamma(t) = \sum_j a_j(\gamma(t))X_j(\gamma(t))$. Obviously γ is an admissible curve. Thus $\gamma : (-T, T) \to (\Omega, \rho)$ is 1-Lipschitz and hence

$$|u(\gamma(t_2)) - u(\gamma(t_1))| \leq \int_{t_1}^{t_2} g(\gamma(t))\, dt,$$

for any $-T < t_1 < t_2 < T$. On the other hand

$$|u(\gamma(t_2)) - u(\gamma(t_1))| = \left| \int_{t_1}^{t_2} \langle \nabla u(\gamma(t)), \dot\gamma(t)\rangle\, dt \right| = \int_{t_1}^{t_2} |Xu(\gamma(t))|\, dt.$$

This yields

$$\int_{t_1}^{t_2} |Xu(\gamma(t))|\, dt \leq \int_{t_1}^{t_2} g(\gamma(t))\, dt. \tag{55}$$

If the vector field Y were parallel to one of the coordinate axes, then (55) would imply that $g \geq |Xu|$ a.e. on almost every line parallel to that axis and hence $g \geq |Xu|$ a.e. in Ω. The general case can be reduced to the case of a vector field of parallel directions by the rectification theorem. This is obvious if the vector field is C^1-smooth as it is the usual requirement in the rectification theorem, see Arnold [3]. However the same argument can be also used in the general Lipschitz case. A construction of the Lipschitz rectification is provided in the proof of Theorem 11.7 (look for $\Phi : G \to \Omega$).

Now we give two applications of the theorem.

Note that if u is metric L-Lipschitz, then the constant function L is an upper gradient of u. However, the function u need not be continuous with respect to the Euclidean metric, even if the Carnot–Carathéodory distance is a metric. This is easily seen in the previously discussed example of ∂_x and $x_+\partial_y$ in $\mathbb{R}^2$. Thus, in order to apply Theorem 11.7 to a metric L-Lipschitz function, we need to assume either that the function is continuous with respect to the Euclidean metric or simply that (54) holds.

The following special case of Theorem 11.7 was proved by Franchi, Hajłasz and Koskela, [77], and in a slightly weaker form earlier by Chernikov and Vodop'yanov, [38], Franchi, Serapioni and Serra Cassano, [85], and Garofalo and Nhieu, [91].

Corollary 11.8 *Assume that (54) holds. If u is metric L-Lipschitz, then the distributional derivatives $X_j u$, $j = 1, 2, \ldots, k$, are represented by bounded functions and $|Xu| \le L$ a.e.*

The following version of Meyers–Serrin's theorem was discovered in its local form by Friedrichs, [87], (cf. [214, Lemma 11.27]), and later by Chernikov and Vodop'yanov, [38, Lemma 1.2], Franchi, Serapioni and Serra Cassano, [84, Proposition 1.2.2], [85, p. 90], and Garofalo and Nhieu, [92, Lemma 7.6].

Since later on we will need estimates from the proof, rather than the statement alone, we recall the proof following Friedrich's argument.

Theorem 11.9 *Let $X = (X_1, X_2, \ldots, X_k)$ be a system of vector fields with locally Lipschitz coefficients in $\Omega \subset \mathbb{R}^n$ and let $1 \le p < \infty$. If $u \in L^p(\Omega)$ and (the distributional derivative) $Xu \in L^p(\Omega)$, then there exists a sequence $u_k \in C^\infty(\Omega)$ such that $\|u_k - u\|_{L^p(\Omega)} + \|Xu_k - Xu\|_{L^p(\Omega)} \to 0$ as $k \to \infty$.*

Proof. We will prove that if u has compact support in Ω, then the standard mollifier approximation gives a desired approximating sequence. The general case follows by a partition of unity argument.

Let $Y(x) = \sum_{j=1}^n c_j(x)\partial/\partial x_j$, where the functions c_j are locally Lipschitz, denote one of X_j's. Let $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon)$, $0 \le \varphi \in C_0^\infty(B^n(0,1))$, $\int \varphi = 1$, be a standard mollifier kernel. For a locally integrable function u we have

$$
\begin{aligned}
Y(u * \varphi_\varepsilon)(x) &= \sum_{j=1}^n \int c_j(x-y)\frac{\partial u}{\partial x_j}(x-y)\varphi_\varepsilon(y)\,dy \\
&+ \sum_{j=1}^n \int (c_j(x) - c_j(x-y))\frac{\partial u}{\partial x_j}(x-y)\varphi_\varepsilon(y)\,dy \\
&= (Yu) * \varphi_\varepsilon(x) \\
&+ \sum_{j=1}^n \int (u(x-y) - u(x))\frac{\partial}{\partial y_j}\Big(\big(c_j(x) - c_j(x-y)\big)\varphi_\varepsilon(y)\Big)\,dy \\
&= (Yu) * \varphi_\varepsilon(x) + A_\varepsilon u(x), \qquad (56)
\end{aligned}
$$

where the integrals are understood in the sense of distributions. Note that

$$
\left|\frac{\partial}{\partial y_j}\Big(\big(c_j(x) - c_j(x-y)\big)\varphi_\varepsilon(y)\Big)\right| \le CL(x,\varepsilon)\varepsilon^{-n}\chi_{B^n(0,\varepsilon)}, \qquad \text{a.e.},
$$

where $L(x,\varepsilon)$ is the Lipschitz constant of all c_j's on $B^n(x,\varepsilon)$. Hence

$$
|A_\varepsilon u(x)| \le CL(x,\varepsilon) ⨍_{B^n(x,\varepsilon)} |u(y) - u(x)|\,dy. \qquad (57)
$$

If $u \in L^p(\Omega)$ has compact support in Ω, then it easily follows that $\|A_\varepsilon u\|_{L^p(\Omega)} \to 0$ as $\varepsilon \to 0$. Indeed, it is obvious if u is continuous and in the general case we can approximate u by compactly supported continuous functions in the L^p norm. If in addition $Yu \in L^p(\Omega)$, then $(Yu) * \varphi_\varepsilon \to Yu$ in $L^p(\Omega)$ and hence by (56), $Y(u * \varphi_\varepsilon) \to Yu$ in $L^p(\Omega)$. The proof is complete.

The following result is a corollary of the above proof.

Proposition 11.10 *Assume that (54) holds. Let u be metric L-Lipschitz in Ω. Then the standard mollifier approximation converges to u uniformly on compact subsets of Ω and*

$$|X(u * \varphi_\varepsilon)(x)| \leq L + |A_\varepsilon u(x)|,$$

where $|A_\varepsilon u| \to 0$ as $\varepsilon \to 0$ uniformly on compact subsets of Ω. The above inequality holds for all $x \in \Omega$ of distance at least ε to the boundary.

Proof. Condition (54) is used to guarantee that u is continuous with respect to the Euclidean metric, which in turn together with (57) implies that $|A_\varepsilon u(x)|$ converges to zero uniformly on compact sets. By Corollary 11.8, $|Yu| * \varphi_\varepsilon \leq L$, for any $Y = \sum_{j=1}^k c_j X_j$ with $\sum_j c_j^2 \leq 1$. The desired inequality then results using (56) with the following choice of the coefficients. Fix an arbitrary point $x_0 \in \Omega$. If $|X(u*\varphi_\varepsilon)(x_0)| = 0$, then we are done; otherwise we take $c_j = X_j(u*\varphi_\varepsilon)(x_0)/|X(u*\varphi_\varepsilon)(x_0)|$. The proof is complete.

The theorem below shows that, in a certain sense, the analysis of vector fields is determined by the associated Carnot–Carathéodory metric. The result is also an affirmative answer to a question posed by Bruno Franchi.

Theorem 11.11 *Let X and Y be two families of vector fields with locally Lipschitz coefficients in Ω and such that (54) holds for the induced Carnot–Carathéodory metrics ρ_X and ρ_Y. Then the following conditions are equivalent.*

1. *There exists a constant $C \geq 1$ such that $C^{-1}\rho_X \leq \rho_Y \leq C\rho_X$.*
2. *There exists a constant $C \geq 1$ such that $C^{-1}|Xu| \leq |Yu| \leq C|Xu|$ for all $u \in C^\infty(\Omega)$.*

Proof. 1. $\Rightarrow$ 2. Note that if g is an upper gradient of $u \in C^\infty(\Omega)$ on (Ω, ρ_X), then the equivalence of the metrics implies that Cg is an upper gradient of u on (Ω, ρ_Y). This fact, Proposition 11.6, and Theorem 11.7 imply that $|Yu| \leq C|Xu|$. The opposite inequality follows by the same argument.

2. $\Rightarrow$ 1. Fix $x, y \in \Omega$, and let $u(z) = \rho_X(x, z)$. Let $\gamma : [0, T] \to (\Omega, \rho_Y)$ be an arbitrary 1-Lipschitz curve such that $\gamma(0) = x$, $\gamma(T) = y$.

Let $u_\varepsilon = u * \varphi_\varepsilon$ be the standard mollifier approximation. By Proposition 11.6, $|Yu_\varepsilon|$ is an upper gradient of u_ε on (Ω, ρ_Y), and hence, invoking Proposition 11.10, we get

$$\begin{aligned}\rho_X(x,y) \overset{\varepsilon\to 0}{\longleftarrow} & \ |u_\varepsilon(\gamma(T)) - u_\varepsilon(\gamma(0))| \le \int_0^T |Yu_\varepsilon(\gamma(t))|\,dt \\ \le & \ C\int_0^T |Xu_\varepsilon(\gamma(t))|\,dt \le CT + C\int_0^T |A_\varepsilon u(\gamma(t))|\,dt \overset{\varepsilon\to 0}{\longrightarrow} CT.\end{aligned}$$

Now it follows from the definition of ρ_Y that $\rho_X \le C\rho_Y$. The opposite inequality follows by the same argument. The proof is complete.

Let us come back to the question posed at the beginning of the section. How does one prove a Poincaré inequality for the pair u, $|Xu|$? The natural approach is to bound the oscillation of u by integrals of $|Xu|$ over admissible curves — this can be done as $|Xu|$ is an upper gradient of u. Then the Poincaré inequality should follow by averaging the resulting line integrals. Unfortunately, in general, this program is very difficult to handle, and it turns out that many additional assumptions on the vector fields are needed. One such a proof of a Poincaré inequality will be presented later on (see Theorem 11.17). Anyhow, if one succeeds in proving a Poincaré inequality using the above idea, the resulting inequality holds on metric balls.

Thus the Poincaré inequality we should expect is

$$⨍_{\widetilde{B}} |u - u_{\widetilde{B}}|\,dx \le C_P r \left(⨍_{\sigma\widetilde{B}} |Xu|^p\,dx \right)^{1/p}, \tag{58}$$

whenever $\sigma\widetilde{B} \subset \Omega$ and $u \in C^\infty(\sigma\widetilde{B})$. Here $C_P > 0$, $\sigma \ge 1$, $1 \le p < \infty$, are fixed constants and, as usual, $\widetilde{B}$ denotes a ball with respect to the Carnot–Carathéodory metric ρ.

Even if proving inequalities like (58) requires many assumptions on X, there are sufficiently many important examples where (58) holds. Some of them will be discussed in the following subsections.

Theorem 11.12 *Assume that a system of locally Lipschitz vector fields is such that condition (54) is satisfied. Fix $\sigma \ge 1$, $C_P > 0$, and $1 \le p < \infty$. Then the space (Ω, ρ, H^n) supports a p-Poincaré inequality (with given σ and C_P) if and only if inequality (58) holds whenever $\sigma\widetilde{B} \subset \Omega$ and $u \in C^\infty(\sigma\widetilde{B})$.*

Proof. The left-to-right implication is easy to obtain as the function $|Xu|$ is an upper gradient of u, for $u \in C^\infty(\Omega)$. If $u \in C^\infty(\sigma\widetilde{B})$, then we can extend u from the ball $(1-\varepsilon)\sigma\widetilde{B}$ to a continuous function on Ω; next extending $|Xu|$ from the same ball to Ω by ∞ gives an upper gradient on Ω of the extension of u. Now applying the p-Poincaré inequality on $(1-\varepsilon)\widetilde{B}$ to the extended pair and passing to the limit as $\varepsilon \to 0$ yields (58).

For the right-to-left implication we have to prove that, whenever g is an upper gradient of a continuous function u, then

$$\fint_{\widetilde{B}} |u - u_{\widetilde{B}}|\, dx \leq C_P r \left(\fint_{\sigma\widetilde{B}} g^p\, dx \right)^{1/p}, \tag{59}$$

for all $\sigma\widetilde{B} \subset \Omega$. Fix a ball $\widetilde{B}$. We may assume that $g \in L^p(\sigma\widetilde{B})$; otherwise the inequality is obvious.

Since g is an upper gradient of u on $(\sigma\widetilde{B}, \rho)$, Theorem 11.7 implies that $|Xu| \leq g$ a.e. in $\sigma\widetilde{B}$. Then, by Theorem 11.9, there is a sequence of functions $u_k \in C^\infty(\sigma\widetilde{B})$ such that $\|u_k - u\|_{L^p(\sigma\widetilde{B})} + \|Xu_k - Xu\|_{L^p(\sigma\widetilde{B})} \to 0$. Thus, if we pass to the limit in the inequality (58) applied to u_k's, we obtain the p-Poincaré inequality for the pair u, $|Xu|$. This together with the estimate $|Xu| \leq g$ yields (59). The proof is complete.

There is an obvious way to define a Sobolev space associated with a system of vector fields. Namely, we define $W^{1,p}_X(\Omega)$, $1 \leq p \leq \infty$, as the set of those $u \in L^p(\Omega)$ such that $|Xu| \in L^p(\Omega)$, where Xu is defined in the sense of distributions, and we equip the space with the norm $\|u\|_{L^p(\Omega)} + \|Xu\|_{L^p(\Omega)}$ under which $W^{1,p}_X(\Omega)$ becomes a Banach space.

According to Theorem 11.9, when $1 \leq p < \infty$, one can equivalently define the space as the completion of $C^\infty(\Omega)$ in the above norm.

We will return to the construction of a Sobolev space associated with the system of vector fields in Section 13.

Proof of Theorem 11.7. Let $Y = \sum_{j=1}^k c_j X_j$, where c_j's are arbitrarily chosen constant coefficients with $\sum_{j=1}^k c_j^2 \leq 1$.

Let $\Phi(x,t)$ be the function uniquely defined by the conditions $\Phi(x,0) = x$ and $\frac{d}{dt}\Phi(x,t) = Y(\Phi(x,t))$. The properties of Φ are collected in the following lemma. For the proof, see Franchi, Serapioni and Serra Cassano, [85, p.101], or Hartman, [112], Hille, [122].

Lemma 11.13 *If $\Omega' \subset\subset \Omega$, then there exists $T > 0$ such that $\Phi : \Omega' \times (-2T, 2T) \to \Omega$. Moreover for every $t \in (-2T, 2T)$, the mapping $\Phi(\cdot, t) : \Omega' \to \Omega$ is bi-Lipschitz onto the image with the inverse $\Phi(\cdot, -t)$; the mapping $\Phi(\cdot, t)$ is differentiable a.e. and*

$$\frac{\partial \Phi^i}{\partial x_j}(x,t) = \delta_{ij} + a_{ij}(x,t),$$

where δ_{ij} is the Kronecker symbol and $|a_{ij}(x,t)| \leq C|t|$, with a constant C which does not depend neither on $x \in \Omega'$ nor on $t \in (-T, T)$. This implies that the Jacobian of Φ satisfies

$$J_\Phi(x,t) = 1 + \widetilde{J}_\Phi(x,t), \qquad |\widetilde{J}_\Phi(x,t)| \leq C|t|, \tag{60}$$

for the given range of x and t.

Let $\Omega' \subset\subset \Omega$. It suffices to show that $|Xu| \leq g$ a.e. in Ω'; the theorem will follow then by an exhaustion of the domain Ω.

Define the directional derivative of u in the direction of Y by the formula $\widetilde{Y}u(x) = \frac{d}{dt}|_{t=0} u(\Phi(x,t))$.

The plan of the proof of the theorem is the following. In the first step we prove that $\widetilde{Y}u$ exists a.e. and that $|\widetilde{Y}u| \leq g$ a.e. In the second step we prove that $\widetilde{Y}u$ is actually the distributional derivative and in the last step we show that, by an appropriate choice of the c_j's, we get $|\widetilde{Y}u| = |Xu|$.

Step 1. We show that $\widetilde{Y}u$ exists a.e. and $|\widetilde{Y}u| \leq g$ a.e.

If $Y(x) = 0$, then $\widetilde{Y}u(x) = 0$, and hence $|\widetilde{Y}u(x)| \leq g(x)$. Thus it remains to prove the inequality in the open set where $Y \neq 0$.

Observe that the curves $t \mapsto \Phi(x,t)$ are admissible, and hence, for $-2T < t_1 < t_2 < 2T$,

$$|u(\Phi(x,t_2)) - u(\Phi(x,t_1))| \leq \int_{t_1}^{t_2} g(\Phi(x,t))\, dt.$$

Thus, if for given x, the function $t \mapsto g(\Phi(x,t))$ is locally integrable, then the above inequality implies that the function $t \mapsto u(\Phi(x,t))$ is absolutely continuous and

$$|\widetilde{Y}u(\Phi(x,t))| = \left| \frac{d}{dt} u(\Phi(x,t)) \right| \leq g(\Phi(x,t)), \tag{61}$$

for almost all $t \in (-2T, 2T)$.

Fix $x_0 \in \Omega'$ with $Y(x_0) \neq 0$, and let $B^{n-1}(x_0,\delta)$ be a sufficiently small ball contained in the hyperplane perpendicular to $Y(x_0)$. For a moment restrict the domain of definition of Φ to $G = B^{n-1}(x_0,\delta) \times (-T,T)$. The uniqueness theorem for ODE implies that Φ is one-to-one on G. Moreover the properties of Φ collected in Lemma 11.13 imply that Φ is Lipschitz on G and the Jacobian of $\Phi : G \to \Omega$ satisfies $C_1 \geq |J_\Phi| \geq C_2 > 0$ on G provided δ and T are sufficiently small (note that this is the Jacobian of a different mapping than that in (60)). Hence $|J_\Phi(\Phi^{-1}(z))|^{-1}$ is bounded on $\Phi(G)$. Note that $|J_\Phi(\Phi^{-1}(z))|^{-1}$ is defined almost everywhere on $\Phi(G)$. This follows from the observation that if $E \subset \Phi(G)$, $|E| > 0$, then by the change of variables formula $0 < |E| = \int_{\Phi^{-1}(E)} |J_\Phi|$ and hence $|\Phi^{-1}(E)| > 0$. The last observation implies also that if we prove that some property holds for almost all $(x,t) \in B^{n-1}(x_0,\delta) \times (-T,T)$, then it is equivalent to say that the property holds for almost all $z \in \Phi(G)$.

The set $\Phi(G)$ is open and it contains x_0. Since we can cover the set where $Y \neq 0$ with such $\Phi(G)$'s it remains to prove that $|\widetilde{Y}u| \leq g$ a.e. in $\Phi(G)$.

In order to prove that for almost every $z \in \Phi(G)$ the directional derivative $\widetilde{Y}u(z)$ exists and satisfies $|\widetilde{Y}u(z)| \leq g(z)$, it suffices to prove that for almost every $x \in B^{n-1}(x_0,\delta)$ the function $t \mapsto g(\Phi(x,t))$, $t \in (-T,T)$ is integrable, and then the claim follows from (61). The integrability follows immediately from the estimate

$$\begin{aligned}\int_{B^{n-1}(x_0,\delta)}\int_{-T}^{T} g(\Phi(x,t))\,dt\,dx &= \int_{\Phi(G)} g(z)|J_\Phi(\Phi^{-1}(z))|^{-1}\,dz \\ &\le C\int_{\Phi(G)} g(z)\,dz<\infty\end{aligned}$$

and the Fubini theorem. Thus we proved that $|\widetilde{Y}u|\le g$ a.e. in $\Phi(G)$ and hence a.e. in Ω'.

Step 2. Now we prove that $\widetilde{Y}u=Yu$, where Yu is the distributional derivative defined by its evaluation on $\varphi\in C_0^\infty(\Omega')$ by the formula

$$\langle Yu,\varphi\rangle = -\int uY^*\varphi = -\int uY\varphi - \int u\varphi\,\mathrm{div}\,Y.$$

In the proof we will need a stronger result than just the inequality $|\widetilde{Y}u|\le g$. Let $u_\sigma(z)=(u(\Phi(z,\sigma))-u(z))/\sigma$. We claim that for every $\varphi\in C_0^\infty(\Omega')$

$$\int u_\sigma(z)\varphi(z)\,dz\longrightarrow\int \widetilde{Y}u(z)\varphi(z)\,dz. \tag{62}$$

Since $u_\sigma\to\widetilde{Y}u$ a.e. it suffices to prove that, locally, the family $\{u_\sigma\}_{0<\sigma<T}$ is uniformly integrable. Then the convergence (62) will follow from Proposition 14.9. According to the Vallée Poussin theorem (see Theorem 14.8), it suffices to prove that there exists a convex function $F:[0,\infty)\to[0,\infty)$ such that $F(0)=0$, $F(x)/x\to\infty$ as $x\to\infty$, and $\sup_\sigma\int_{\Phi(G)}F(|u_\sigma|)<\infty$, where G was defined in the first step.

Since $g\in L^1(\Phi(G))$, then again by Vallée Poussin's theorem there is a convex function F with growth properties as above and such that $\int_{\Phi(G)}F(g)<\infty$. Now

$$|u_\sigma(\Phi(x,t))| = \sigma^{-1}|u(\Phi(x,t+\sigma))-u(\Phi(x,t))|\le\sigma^{-1}\int_t^{t+\sigma} g(\Phi(x,s))\,ds.$$

Hence Jensen's inequality implies

$$F(|u_\sigma(\Phi(x,t))|)\le\sigma^{-1}\int_t^{t+\sigma}F\Big(g(\Phi(x,s))\Big)\,ds,$$

and thus denoting $B_\delta^{n-1}=B^{n-1}(x_0,\delta)$ we get

$$\begin{aligned}\int_{B_\delta^{n-1}}\int_{-T}^{T}F\Big(|u_\sigma(\Phi(x,t))|\Big)\,dt\,dx &\le \sigma^{-1}\int_{B_\delta^{n-1}}\int_{-T}^{T}\int_t^{t+\sigma}F\Big(g(\Phi(x,s))\Big)\,ds\,dt\,dx \\ &\le \int_{B_\delta^{n-1}}\int_{-T}^{T+\sigma}F\Big(g(\Phi(x,s))\Big)\,ds\,dx.\end{aligned}$$

Therefore

$$\begin{aligned}\sup_{0<\sigma<T}\int_{\Phi(G)} F(|u_\sigma(z)|)\,dz &= \sup_{0<\sigma<T}\int_{B_\delta^{n-1}}\int_{-T}^{T} F\Big(|u_\sigma(\Phi(x,t))|\Big)|J_\Phi(x,t)|\,dt\,dx\\ &\leq C\int_{B_\delta^{n-1}}\int_{-T}^{2T} F(g(\Phi(x,s))\,ds\,dx<\infty,\end{aligned}$$

which yields desired uniform integrability. This completes the proof of (62).

Now we proceed to prove that $\widetilde{Y}u = Yu$. Fix an arbitrary $x_0 \in \Omega'$. Note that $Yu = Y(u-u(x_0))$, and hence

$$|\langle \widetilde{Y}u - Yu,\varphi\rangle| \leq \left|\int \widetilde{Y}u\varphi + (u-u(x_0))Y\varphi\right| + \int |u-u(x_0)|\,|\varphi|\,|\mathrm{div}\,Y| = I_1 + I_2.$$

First we prove that $\sup|\langle \widetilde{Y}u - Yu,\varphi\rangle| = C(x_0,\varepsilon)<\infty$, where the supremum is taken over all $\varphi\in C_0^\infty(B(x_0,\varepsilon))$ with $\|\varphi\|_\infty\leq 1$. This inequality implies that $\widetilde{Y}u - Yu$ is a signed Radon measure with total variation on $B^n(x_0,\varepsilon)$ equal to $C(x_0,\varepsilon)$.

In what follows we assume that φ is compactly supported in $B^n(x_0,\varepsilon)$ with the supremum norm no more than 1. As Y has locally Lipschitz coefficients, $|\mathrm{div}\,Y|$ is locally bounded and hence

$$I_2 \leq C\varepsilon^n \sup_{B^n(x_0,\varepsilon)} |u-u(x_0)|.$$

The estimates for I_1 are more difficult to handle. In what follows we write u instead of $u-u(x_0)$ and simply assume that $u(x_0)=0$. We have

$$\int u(x)Y\varphi(x)\,dx = \lim_{t\to 0}\frac{1}{t}\left(\int u(x)\varphi(x)\,dx - \int u(x)\varphi(\Phi(x,-t))\,dx\right) = A.$$

The change of variables $\widetilde{x} = \Phi(x,-t)$ together with (60) yields

$$\int u(x)\varphi(\Phi(x,-t))\,dx = \int u(\Phi(\widetilde{x},t))\varphi(\widetilde{x})(1+\widetilde{J}_\Phi(\widetilde{x},t))\,d\widetilde{x},$$

and hence by (62)

$$\begin{aligned}A &= \lim_{t\to 0}\int\frac{u(x)-u(\Phi(x,t))}{t}\varphi(x)\,dx - \lim_{t\to 0}\frac{1}{t}\int u(\Phi(x,t))\varphi(x)\widetilde{J}_\Phi(x,t)\,dx\\ &= -\int \widetilde{Y}u(x)\varphi(x)\,dx - \lim_{t\to 0}\frac{1}{t}\int u(\Phi(x,t))\varphi(x)\widetilde{J}_\Phi(x,t)\,dx.\end{aligned}$$

Hence

$$\begin{aligned}I_1 &= \left|\lim_{t\to 0}\frac{1}{t}\int u(\Phi(x,t))\varphi(x)\widetilde{J}_\Phi(x,t)\,dx\right|\\ &\leq C\lim_{t\to 0}\int_{B^n(x_0,\varepsilon)}|u(\Phi(x,t))|\,dx \leq C\varepsilon^n\sup_{B^n(x_0,\varepsilon)}|u-u(x_0)|.\end{aligned}$$

This and the estimate for I_2 yields that $\widetilde{Y}u - Yu$ is a signed Radon measure whose total variation on $B^n(x_0,\varepsilon)$ is estimated from above by

$$|\widetilde{Y}u - Yu|(B^n(x_0,\varepsilon)) \leq C\varepsilon^n \sup_{B(x_0,\varepsilon)} |u - u(x_0)|.$$

This in turn implies that the measure $|\widetilde{Y}u - Yu|$ is absolutely continuous with respect to the Lebesgue measure, so $\widetilde{Y}u - Yu \in L^1_{\rm loc}$, and then by the Lebesgue differentiation theorem

$$|\widetilde{Y}u(x_0) - Yu(x_0)| = \lim_{\varepsilon\to 0} ⨍_{B^n(x_0,\varepsilon)} |\widetilde{Y}u - Yu| \leq C \lim_{\varepsilon\to 0} \sup_{B^n(x_0,\varepsilon)} |u - u(x_0)| = 0$$

for almost all x_0. Thus $\widetilde{Y}u = Yu$, and hence by Step 1, $|Yu| \leq g$ a.e.

Step 3. Repeating the above arguments for all the rational coefficients c_j, we conclude that there is a subset of Ω' of full measure such that for all rational c_j's with $\sum_j c_j^2 \leq 1$ there is $|\sum_j c_j X_j u| \leq g$ at all points of the set. If $|Xu| = 0$ at a given point, then $|Xu| \leq g$ at the point. If $|Xu| \neq 0$, then approximating coefficients $\widetilde{c}_j = X_j u/|Xu|$ by rational coefficients and passing to 0 with the accuracy of the approximation yields $|Xu| = |\sum_j \widetilde{c}_j X_j u| \leq g$. The proof of the theorem is complete.

11.3 Carnot groups.

The aim of this subsection is to give a background on the so-called Carnot groups which are prime examples of spaces that support the p-Poincaré inequality for any $1 \leq p < \infty$. Carnot groups are special cases of Carnot–Carathéodory spaces associated with a system of vector fields satisfying Hörmander's condition that will be described in the next subsection. For a more complete introduction to Carnot groups, see Folland and Stein [73, Chapter 1] and also Heinonen [114].

Before we give the definition we need to collect some preliminary notions and results.

Let $\mathfrak{g}$ be a finite dimensional real Lie algebra. We say that $\mathfrak{g}$ is *nilpotent of step* m if for some positive integer m, $\mathfrak{g}_{(m)} \neq \{0\}$, $\mathfrak{g}_{(m+1)} = \{0\}$, where $\mathfrak{g}_{(1)} = \mathfrak{g}$ and $\mathfrak{g}_{(j+1)} = [\mathfrak{g}, \mathfrak{g}_{(j)}]$. A Lie algebra is called *nilpotent* if it is nilpotent of some step m. A Lie group G is called *nilpotent (of step m)* if its Lie algebra is nilpotent (of step m).

Let V be the underlying vector space of the nilpotent Lie algebra $\mathfrak{g}$. Define the polynomial mapping $\circ : V \times V \to V$ by the Campbell–Hausdorff formula

$$X \circ Y = \sum_{p=1}^{\infty} \frac{(-1)^{p+1}}{p} \sum_{\substack{n_i+m_i\geq 1\\ i=1,2,\ldots,p}} \frac{(n_1+m_1+\ldots+n_p+m_p)^{-1}}{n_1!m_1!\cdots n_p!m_p!}$$
$$\times (\operatorname{ad} X)^{n_1}(\operatorname{ad} Y)^{m_1}\ldots(\operatorname{ad} X)^{n_p}(\operatorname{ad} Y)^{m_p-1}Y,$$

where $(\mathrm{ad}\, A)B = [A, B]$. We adopt here the convention that if $m_p = 0$, then the term in the sum ends with $\dots(\mathrm{ad}X)^{n_p-1}X$. Note also that if $m_p > 1$, then the term in the sum is zero.

The formal series on the right hand side of the Campbell–Hausdorff formula is in fact a polynomial, because the Lie algebra is nilpotent. One can check that the mapping defines a group structure on V with the Lie algebra $\mathfrak{g}$. Since connected and simply connected Lie groups with isomorphic Lie algebras are isomorphic, we obtained a full description of simply connected nilpotent Lie groups.

In what follows the group identity will be denoted by 0; however for the group law we use multiplicative notation: xy.

One can write the Campbell–Hausdorff formula in the form

$$X \circ Y = X + Y + \frac{1}{2}[X, Y] + \dots$$

where the dots indicate terms of order greater than or equal to 3. Note that the map $t \mapsto tX$ is a one parameter subgroup of V. Hence the exponential map $\exp : \mathfrak{g} \to V$ is identity. Then one can find a basis in V so that in the induced coordinate system the Jacobi matrix of the left multiplication by $a \in V$ is a lower triangular matrix with ones on the diagonal. Thus the Jacobi determinant equals one. Hence the Lebesgue measure is the left invariant Haar measure. The same argument applies to the right multiplication, and so the Lebesgue measure is the bi-invariant Haar measure.

A *Carnot group* is a connected and simply connected Lie group G whose Lie algebra $\mathfrak{g}$ admits a *stratification* $\mathfrak{g} = V_1 \oplus \cdots \oplus V_m$, $[V_1, V_i] = V_{i+1}$, $V_i = \{0\}$ for $i > m$. Obviously a Carnot group is nilpotent. Moreover a Carnot group is nilpotent of step m if $V_m \neq 0$. Note that the basis of V_1 generates the whole Lie algebra $\mathfrak{g}$. Carnot groups are also known as *stratified groups*.

Being nilpotent, Carnot group is diffeomorphic to $\mathbb{R}^n$ for some n.

Let $X_1, X_2, \dots, X_k$ form a basis of V_1. We identify X_1, X_2,…,X_k with the left invariant vector fields.

The following result is due to Chow, [40], and Rashevsky, [213]. For modern proofs see Bellaïche [7], Gromov [101], Herman [121], Nagel, Stein and Wainger [204], Strichartz [235], Varopoulos, Saloff-Coste and Coulhon [251].

Proposition 11.14 *The Carnot–Carathéodory distance associated with the basis X_1, X_2,…,X_k of V_1 is a metric i.e., every two points of the Carnot group can be connected by an admissible curve.*

The aim of this subsection is to prove that a Carnot group with the above Carnot–Carathéodory metric supports the p-Poincaré inequalities for all $1 \leq p < \infty$ (see Theorem 11.17). This is a special case of Jerison's result (see Theorem 11.20) that will be described in the next subsection.

By G we will denote a Carnot group of step m and ρ will be the Carnot–Carathéodory metric associated with the basis $X_1, \ldots, X_k$ of V_1.

As the Carnot–Carathéodory metric is not given in an explicit form, it is quite difficult to handle. Therefore it is convenient to introduce new distances that can be defined explicitly and that are equivalent to the Carnot–Carathéodory metric.

A Carnot group admits a one-parameter family of dilations that we next describe.

For $X \in V_i$ and $r > 0$ we set $\delta_r X = r^i X$. This extends to a linear map that is an automorphism of the Lie algebra $\mathfrak{g}$. This in turn induces an automorphism of the Lie group via the exponential map.

Observe that the metric ρ has the two important properties of being left invariant and commutative with δ_r in the sense that $\rho(\delta_r x, \delta_r y) = r\rho(x, y)$.

A *continuous homogeneous norm* on G is a continuous function $|\cdot| : G \to [0, \infty)$ that satisfies 1) $|x^{-1}| = |x|$, 2) $|\delta_r x| = r|x|$ for all $r > 0$ and 3) $|x| = 0$ if and only if $x = 0$.

One such homogeneous norm is given by $|x| = \rho(0, x)$.

Proposition 11.15 *Let $|\cdot|$ be a continuous homogeneous norm. Then the following results hold.*

1. *There exist constants $C_1, C_2 > 0$ such that $C_1\|x\| \leq |x| \leq C_2\|x\|^{1/m}$, for $|x| \leq 1$. Here $\|\cdot\|$ denotes a fixed Euclidean norm.*
2. *The distance $\varrho(x, y) = |x^{-1}y|$ is a quasimetric i.e., it has all the properties of metric but the triangle inequality that is replaced by a weaker condition: there is a constant $C > 0$ such that for all $x, y, z \in G$*
$$\varrho(x, y) \leq C(\varrho(x, z) + \varrho(z, y)). \tag{63}$$
3. *Balls $B(x, r) = \{y : \varrho(x, y) < r\}$ are the left translates of $B(0, r)$ by x, and $B(0, r) = \delta_r B(0, 1)$.*
4. *The number $Q = \sum_{j=1}^m j \dim V_j$ will be called the homogeneous dimension. It satisfies $|\delta_r(E)| = r^Q|E|$ and hence $|B(x, r)| = Cr^Q$ for all $x \in G$ and all $r > 0$, where $|E|$ denotes the Lebesgue measure of the set E.*
5. *Any two continuous homogeneous norms are equivalent in the sense that if $|\cdot|'$ is another continuous homogeneous norm on G, then there exist $C_1, C_2 > 0$ such that $C_1|x|' \leq |x| \leq C_2|x|'$ for all $x \in G$.*

For a proof, see Folland and Stein [73, Chapter 1]. Anyway the proof is easy and it could be regarded as a very good exercise.

In the literature the concept of a *homogeneous norm* is defined as above but with the additional property of being C^∞-smooth on $G \setminus \{0\}$. This property is irrelevant to us. Thus we delete it and add the adjective "continuous" to indicate that we do not assume smoothness.

To give an explicit example of a continuous homogeneous norm note that any element $x \in V$ can be represented as $x = \sum_{j=1}^m x_j$, where $x_j \in V_j$. Fix an Euclidean norm $\| \cdot \|$ in V. Then

$$|x| = \sum_{j=1}^m \|x_j\|^{1/j},$$

is a continuous homogeneous norm on G (after identification of G with V).

The continuous homogeneous norm induced by the Carnot–Carathéodory metric $x \mapsto \rho(0,x)$ satisfies (63) with $C = 1$. For general continuous norms we only have $C \geq 1$. See also Hebisch and Sikora, [113], for a construction of a homogeneous norm (i.e. smooth on $G \setminus \{0\}$) with $C = 1$.

Mitchell, [199], proved that the Hausdorff dimension of a Carnot group is equal to its homogeneous dimension, see also [101, p. 102]. This dimension, in general, is larger than the Euclidean dimension of the underlying Euclidean space. This shows that the Carnot–Carathéodory geometry is pretty wild and the metric is not equivalent to any Riemannian metric.

It is an exercise to show that inequality 1. of the above proposition implies that for every bounded domain $\Omega \subset G$, there are constants $C_1, C_2 > 0$ such that

$$C_1 \|x - y\| \leq \rho(x,y) \leq C_2 \|x - y\|^{1/m}, \tag{64}$$

whenever $x, y \in \Omega$. Note that inequality (64) along with Lemma 9.4 imply that every two points can be connected by a geodesic — the shortest admissible curve.

So far we have not given any examples of the Carnot group. Let us fill the gap right now.

Example 11.16 The most simple nontrivial example of a Carnot group is the Heisenberg group $\mathbb{H}_1 = \mathbb{C} \times \mathbb{R}$ with the group law

$$(z,t) \circ (z',t') = (z + z', t + t' + 2\mathrm{Im}\, z\overline{z}').$$

The basis consisting of the left invariant vector fields X, Y, Z, such that $X(0) = \partial/\partial x$, $Y(0) = \partial/\partial y$, $T(0) = \partial/\partial t$, is given by

$$X = \frac{\partial}{\partial x} + 2y\frac{\partial}{\partial t}, \qquad Y = \frac{\partial}{\partial y} - 2x\frac{\partial}{\partial t}, \qquad T = \frac{\partial}{\partial t}.$$

Note that $[X, Y] = -4T$ and all the other commutators are trivial. Thus the Lie algebra is stratified, $\mathfrak{h} = V_1 \oplus V_2$ with $V_1 = \mathrm{span}\{X, Y\}$ and $V_2 = \mathrm{span}\{T\}$. The Carnot–Carathéodory metric is defined with respect to the vector fields X, Y. The group $\mathbb{H}_1$ is a nilpotent group of step 2 and its homogeneous dimension is 4. The family of dilations is given by $\delta_r(z,t) = (rz, r^2 t)$ for $r > 0$. Moreover the function $|(z,t)| = (t^2 + |z|^4)^{1/4}$ is a homogeneous norm.

The following theorem states that a Carnot group supports a 1-Poincaré inequality. This is a corollary of a much more general theorem of Jerison (see Theorem 11.20). For completeness we provide a clever proof due to Varopoulos, [249] (see also [223, page 461]).

Proposition 11.17 *Any Carnot group equipped with the Lebesgue measure and the Carnot–Carathéodory metric supports a* 1*-Poincaré inequality.*

Proof. Let G be a Carnot group with the Carnot–Carathéodory metric that we denote by ρ. Let u, g be a pair of a continuous function and its upper gradient. It suffices to prove that

$$\int_B |u(x) - u_B|\, dx \leq Cr \int_{2B} g(x)\, dx \tag{65}$$

on every ball of radius r. Obviously we can assume that the ball B is centered at 0. Set $|z| = \rho(0, z)$ and let $\gamma_z : [0, |z|] \to G$ be a geodesic path that joins 0 with z. Observe that $s \mapsto x\gamma_z(s)$ is the shortest path that joins x with xz. Hence

$$|u(x) - u(xz)| \leq \int_0^{|z|} g(x\gamma_z(s))\, ds.$$

This and the left invariance of the Lebesgue measure yields

$$\begin{aligned}
\int_B |u(x) - u_B|\, dx &\leq \frac{1}{|B|} \int_B \int_B |u(x) - u(y)|\, dy\, dx \\
&= \frac{1}{|B|} \int_G \int_G \chi_B(x)\chi_B(xz)|u(x) - u(xz)|\, dz\, dx \\
&\leq \frac{1}{|B|} \int_G \int_G \int_0^{|z|} \chi_B(x)\chi_B(xz) g(x\gamma_z(s))\, ds\, dx\, dz.
\end{aligned}$$

Invoking the right invariance of the Lebesgue measure we obtain

$$\begin{aligned}
\int_G \chi_B(x)\chi_B(xz) g(x\gamma_z(s))\, dx &= \int_G \chi_{B\gamma_z(s)}(\xi)\chi_{Bz^{-1}\gamma_z(s)}(\xi) g(\xi)\, d\xi \\
&\leq \chi_{2B}(z) \int_{2B} g(\xi)\, d\xi, \qquad (66)
\end{aligned}$$

Here we denote by Bh the right translation of B by h. The above inequality requires some explanations. If the expression under the sign of the middle interval has a nonzero value, then $\xi = x\gamma_z(s) = yz^{-1}\gamma_z(s)$ for some $x, y \in B$. Hence $z = x^{-1}y \in 2B$. Thus $\xi = x\gamma_{x^{-1}y}(s)$ lies on a geodesic that joins x with y and so $\rho(x, \xi) + \rho(y, \xi) = \rho(x, y)$, which together with the triangle inequality implies $\xi \in 2B$ and hence (66) follows. Now

$$\int_B |u(x) - u_B|\, dx \leq \frac{1}{|B|} \int_G \int_0^{|z|} \chi_{2B}(z) \int_{2B} g(\xi)\, d\xi\, ds\, dz$$

$$= \frac{1}{|B|}\int_{2B}\int_{2B}|z|g(\xi)\,d\xi\,dz$$
$$\leq Cr\int_{2B}g(\xi)\,d\xi.$$

The proof is complete.

Remarks. 1) The above proof easily generalizes to more general unimodular groups, see [249], [223, page 461].

2) Applying Theorem 9.8 to inequality (65) we conclude that the ball $2B$ on the right hand side can be replaced by B, and, moreover, the exponent on the left hand side can be replaced by $Q/(Q-1)$, where Q is the homogeneous dimension of the group. This inequality in turn implies the isoperimetric inequality. Such an isoperimetric inequality was proved first in the case of the Heisenberg group by Pansu, [209], and in the general case of the Carnot groups by Varopoulos, [251].

For a more complete treatment of Sobolev inequalities on Lie groups with the Carnot–Carathéodory metric, see Gromov [101], and Varopoulos, Saloff-Coste and Coulhon [251] and also Folland [71], [72], Nhieu [206], [207].

11.4 Hörmander condition.

Definition. Let $\Omega \subset \mathbb{R}^n$ be an open, connected set, and let X_1, X_2,…,X_k be vector fields defined in a neighborhood of $\overline{\Omega}$, real valued, and C^∞ smooth. We say that these vector fields satisfy Hörmander's condition, provided there is an integer p such that the family of commutators of $X_1, X_2, \ldots, X_k$ up to the length p i.e., the family of vector fields X_1,…,X_k, $[X_{i_1}, X_{i_2}]$,…, $[X_{i_1}, [X_{i_2}, [\ldots, X_{i_p}]]\ldots]$, $i_j = 1, 2, \ldots, k$, span the tangent space $\mathbb{R}^n$ at every point of Ω.

The definition easily extends to smooth manifolds, but for simplicity we will consider the Euclidean space only.

As an example take the vector fields $X_1 = \partial/\partial x_1$, $X_2 = x_1^k\partial/\partial x_2$, where k is a positive integer. These two vector fields do not span $\mathbb{R}^2$ along the line $x_1 = 0$. However X_1, X_2 and commutators of the length $k+1$ do.

Another example is given by vector fields on a Carnot group. Namely, if G is a Carnot group (see the previous subsection) with the stratification $\mathfrak{g} = V_1 \oplus \ldots \oplus V_m$ of its Lie algebra, then the left invariant vector fields associated with a basis of V_1 satisfy Hörmander's condition.

The above condition was used by Hörmander [128], in his celebrated work on hypoelliptic operators, see also Bony [14], Chemin and Xu [36], Fefferman and Sánchez-Calle [70], Hörmander and Melin [129], Jerison [133], Morbidelli [200], Nagel, Stein, and Wainger [204], Rothschild and Stein [218], Sánchez-Calle [224], Varopoulos, Saloff-Coste and Coulhon [251]. Related references will also be given in Section 13.

As usual, the Carnot–Carathéodory distance associated with a family of vector fields satisfying Hörmander's condition will be denoted by ρ.

The following result provides the full version of the theorem of Chow and Raschevsky, whose special case was discussed earlier (see Proposition 11.14). For the proof, see the references given there. In some more simple settings the theorem was proved earlier by Carathéodory, [32].

Theorem 11.18 *Let an open and connected set $\Omega \subset \mathbb{R}^n$ and a system of vector fields satisfying Hörmander's condition in Ω be given. Then any two points in Ω can be connected by a piecewise smooth admissible curve, and hence the Carnot–Carathéodory distance is a metric.*

Nagel, Stein and Wainger, [204], studied the geometry of Carnot–Carathéodory spaces in detail and, in particular, they gave a more quantitative version of Chow–Raschevsky's theorem. Let us quote some of their results.

In what follows $\tilde{B}(x,r)$ will denote a ball with respect to the metric ρ.

Theorem 11.19 *Let $X_1, \dots, X_k$ be a system of vector fields satisfying Hörmander's condition as above, and let ρ be the associated Carnot–Carathéodory metric. Then for every compact set $K \subset \Omega$ there exist constants C_1 and C_2 such that*

$$C_1|x-y| \leq \rho(x,y) \leq C_2|x-y|^{1/p} \tag{67}$$

for every $x, y \in K$. Moreover there are $r_0 > 0$ and $C \geq 1$ such that

$$|\tilde{B}(x,2r)| \leq C|\tilde{B}(x,r)| \tag{68}$$

whenever $x \in K$ and $r \leq r_0$.

Here, as usual, $|\tilde{B}|$ denotes the Lebesgue measure. In the previous subsection we proved the theorem in the special case of a Carnot group. The general case is however much more difficult, see also Gromov [101], and Varopoulos, Saloff-Coste and Coulhon [251, Section IV.5]. Estimate (67) has been obtained independently by Lanconelli [165].

Assume for a moment that $\Omega = \mathbb{R}^n$. If $\Omega' \subset \mathbb{R}^n$ is bounded with respect to the Euclidean metric, then by (67) it is also bounded with respect to ρ. However, if Ω' is bounded with respect to ρ, then it need not be bounded with respect to the Euclidean metric. Indeed, if one of the vector fields is $x_1^2 \partial/\partial x_1$, then the Carnot–Carathéodory distance to infinity is finite because of the rapid growth of the coefficient. Hence, in general, (68) holds only for $r < r_0$ for some sufficiently small r_0 and r_0 cannot be replaced with $5 \operatorname{diam}_\rho(\Omega')$, even if $\operatorname{diam}_\rho(\Omega') < \infty$, as was required for the measure in the definition of doubling in Ω'.

Proposition 11.17 is a special case of the following Poincaré inequality of Jerison [133], see also Jerison and Sanchez–Calle [136], and Lanconelli and Morbidelli [166].

Theorem 11.20 *Let $X_1, \dots, X_k$ be a system of vector fields satisfying Hörmander's condition in Ω. Then for every compact set $K \subset \Omega$ there are con-*

stants $C > 0$ and $r_0 > 0$ such that for $u \in \mathrm{Lip}\,(\tilde{B})$

$$\int_{\tilde{B}} |u - u_{\tilde{B}}|\, dx \le Cr \int_{2\tilde{B}} |Xu|\, dx, \tag{69}$$

whenever $\tilde{B}$ is a ball centered at K with radius $r < r_0$.

In fact, Jerison proved the inequality with the L^2 norms on both sides, but the same argument works with the L^1 norm. Then Jerison proved that one can replace the ball $2\tilde{B}$ on the right hand side of (69) with $\tilde{B}$. As we have already seen this can be done in a much more general setting, see Section 9.

11.5 Further generalizations

The results of the previous two subsections concern Poincaré inequalities for smooth vector fields satisfying Hörmander's condition. It is a difficult problem to find a large class of vector fields with Lipschitz coefficients such that the Poincaré type inequalities hold on the associated Carnot–Carathéodory spaces. The lack of smoothness does not permit one to use a Hörmander type condition. There are few results of that type, see Franchi [74], Franchi, Gutiérrez and Wheeden [76], Franchi and Serapioni [83], Franchi and Lanconelli [78], Jerison and Sanchez-Calle [136]. It seems that Franchi and Lanconelli, [78], were the first to prove a Poincaré type inequality for a Carnot–Carathéodory space. They probably also were the first to prove estimates of the type as in Theorem 11.19.

12 Graphs

Let $G = (V, E)$ be a graph, where V is the vertex set and E the set of edges. We say that $x, y \in V$ are neighbors if they are joined by an edge; we denote this by $x \sim y$. Assume that the graph is connected in the sense that any two vertices can be connected by a sequence of neighbors. We let the distance between two neighbors to be 1. This induces a geodesic metric on V that we denote by ϱ. The graph is endowed with the counting measure: the measure of a set $E \subset V$ is simply the number $V(E)$ of elements of E. For a ball $B = B(x, r)$ we use also the notation $V(B) = V(x, r)$. We say that G is *locally uniformly finite* if $d = \sup_{x \in V} d(x) < \infty$, where $d(x)$ is the number of neighbors of x. The length of the gradient of a function u on V at a point x is

$$|\nabla_G u|(x) = \sum_{y \sim x} |u(y) - u(x)|.$$

Many graphs have the following two properties:

1. The counting measure is doubling i.e.,

$$V(x, 2r) \le C_d V(x, r),$$

for every $x \in V$ and $r > 0$.

2. The p-Poincaré inequality holds i.e., there are constants $C > 0$ and $\sigma \geq 1$ such that

$$\frac{1}{V(B)} \sum_{x \in B} |u(x) - u_B| \leq C_P r \left(\frac{1}{V(\sigma B)} \sum_{x \in \sigma B} |\nabla_G u|^p(x) \right)^{1/p}, \tag{70}$$

for any ball B and any function $u : V \to \mathbb{R}$.

Observe that the doubling condition implies that the graph is locally uniformly finite.

The Euclidean, or more generally, the upper gradients have the truncation property. Unfortunately the truncation property is no longer valid for the length of the gradient on a graph. This is because, in general, $|\nabla_G u_{t_1}^{t_2}|$ is not supported on the set $\{t_1 < u \leq t_2\}$. However, intuition suggests that $|\nabla_G u|$ should still have properties similar to those of a gradient with the truncation property.

If $v \in \mathrm{Lip}\,(\mathbb{R}^n)$ and $p > 0$, then

$$\sum_{k=-\infty}^{\infty} |\nabla v_{2^{k-1}}^{2^k}|^p = \sum_{k=-\infty}^{\infty} |\nabla v|^p \chi_{\{2^{k-1} < v \leq 2^k\}} \leq |\nabla v|^p \tag{71}$$

almost everywhere. It turns out that a version of inequality (71) is satisfied also by the length of the gradient on a graph. More precisely we have the following estimate.

Lemma 12.1 *Let G be locally uniformly finite i.e., $d = \sup_{x \in V} d(x) < \infty$. If $v : V \to \mathbb{R}$ and $p > 0$, then*

$$\sum_{k=-\infty}^{\infty} \left| \nabla_G v_{2^{k-1}}^{2^k} \right|^p (x) \leq C(p,d) |\nabla_G v|^p(x)$$

for each $x \in V$.

Proof. Fix $x \in V$ and let

$$v_M(x) = \max\{v(w) \, : \, \varrho(w,x) \leq 1\},$$

$$v_m(x) = \min\{v(w) \, : \, \varrho(w,x) \leq 1\}.$$

Note that $|\nabla_G v|(x) \geq |v_M(x) - v_m(x)|$. Assume for simplicity that $v_m(x) > 0$ (the case $v_m(x) \leq 0$ follows by the same argument). Let $j \in \mathbb{Z}$ be the least integer and $i \in \mathbb{Z}$ the largest integer such that

$$2^j \geq v_M(x) \geq v_m(x) \geq 2^i.$$

We have

$$|v_M(x) - v_m(x)| = v_M(x) - 2^{j-1} + \sum_{k=i+1}^{j-2} (2^{k+1} - 2^k) + 2^{i+1} - v_m(x).$$

Hence

$$\begin{aligned}
|\nabla_G v|^p(x) &\geq C\left(|v_M(x) - 2^{j-1}|^p + \left(\sum_{k=i+1}^{j-2} 2^k\right)^p + |2^{i+1} - v_m(x)|^p\right) \\
&\geq C\left(|v_M(x) - 2^{j-1}|^p + (1-2^{-p})\sum_{k=i+1}^{j-2} 2^{kp} + |2^{i+1} - v_m(x)|^p\right) \\
&\geq C\left(\frac{1}{d^p}|\nabla_G v_{2^{j-1}}^{2^j}|^p(x) + \frac{1-2^{-p}}{d^p}\sum_{k=i+1}^{j-2} |\nabla_G v_{2^k}^{2^{k+1}}|^p(x)\right. \\
&\qquad\qquad \left. + \frac{1}{d^p}|\nabla_G v_{2^i}^{2^{i+1}}|^p(x)\right) \\
&\geq \frac{C(p)}{d^p}\sum_{k=i}^{j-1} |\nabla_G v_{2^k}^{2^{k+1}}|^p(x) \\
&= \frac{C(p)}{d^p}\sum_{k=-\infty}^{\infty} |\nabla_G v_{2^k}^{2^{k+1}}|^p(x).
\end{aligned}$$

The proof is complete.

The inequality of the lemma is a good substitute for the truncation property; it allows one to mimic the proofs of Theorems 2.1 and 2.3. We will generalize Corollary 9.8. This result deals with sharp inequalities with integrals on the different sides of the inequality taken over the same domain. As pointed out in Section 9, a Poincaré inequality does not, in general, guarantee that one could use balls of the same size on the different sides of the inequality. We described a sufficient condition in terms of the geometry of balls that, in particular, holds for the Carnot-Caratheodory metrics. As ρ is a geodesic metric, it should come as no surprise that we can reduce the size of σ in (70) down to 1.

The following theorem is related to some results in Bakry, Coulhon, Ledoux and Saloff-Coste [4], Coulhon [54].

Theorem 12.2 *Assume that the counting measure is doubling, and that for some constants $C_b > 0, s \geq 1$*

$$\frac{V(x,r)}{V(x,r_0)} \geq C_b\left(\frac{r}{r_0}\right)^s$$

whenever $B(x,r) \subset B(x_0,r_0)$. Suppose that each function $u : V \to \mathbb{R}$ satisfies the p-Poincaré inequality (70) with a fixed $p > 0$.

1. *If $0<p<s$, then there is a constant $C>0$ such that*

$$\left(\frac{1}{V(B)}\sum_{x\in B}|u(x)-u_B|^{p^*}\right)^{1/p^*}\leq Cr\left(\frac{1}{V(B)}\sum_{x\in B}|\nabla_G u|^p(x)\right)^{1/p} \tag{72}$$

for any ball B of radius r and any function $u:V\to\mathbb{R}$, where $p^*=sp/(s-p)$.

2. *If $p=s>1$, then there are constants $C_1,C_2>0$ such that*

$$\frac{1}{V(B)}\sum_{x\in B}\exp\left(\frac{C_1V(B)^{1/s}|u(x)-u_B|}{(\operatorname{diam}B)\|\nabla_G u\|_{L^s(B)}}\right)^{s/(s-1)}\leq C_2$$

for any ball B of radius r and any function $u:V\to\mathbb{R}$.

3. *If $p>s$, then there is $C>0$ such that*

$$|u(x)-u(y)|\leq C\varrho(x,y)^{1-s/p}r^{s/p}V(B)^{-1/p}\|\nabla_G u\|_{L^p(B)}\,,$$

for all $x,y\in B$, where B is an arbitrary ball of radius r, and for any function $u:V\to\mathbb{R}$.

The constants C, C_1, C_2 depend on p, s, σ, C_d, C_b and C_P only.

Remark. If $sp/(s-p)<1$, then we have to replace u_B by u_{B_0} in (72), where $B_0=(2\sigma)^{-1}B$.

Proof. The proof involves arguments similar to those used in the previous sections, and so we only sketch the main ideas, leaving the details to the reader.

First of all, under the local finiteness of the graph, one may assume that $r>10\sigma$ (as for $r\leq 10\sigma$ we only have a finite collection of non-isometric balls). We follow the line of ideas from Section 9. Given a ball $B(x_0,r)$ and a point $x\in B(x_0,r)$, we join x to x_0 by a chain $x=x_1,...,x_m$ of length less than r of vertices. If we trace along the chain for l steps with l the least integer larger or equal to 4σ, then $B(x_l,2\sigma)\subset B(x_0,r)$. Following the chain towards x we may construct a chain $B_0=B(x_0,r/(2\sigma)),B_1,...,B_k=B(x_l,2)$ of balls as in the $C(\sigma,M)$ condition of Section 9. Next,

$$\begin{aligned}|u(x)-u_{B_0}| &\leq \sum_{i=1}^{l-1}|u(x_{i+1})-u(x_i)|+|u_{B(x_l,2)}-u(x_l)|+\sum_{i=0}^{k-1}|u_{B_{i+1}}-u_{B_i}|\\ &\leq \sum_{i=1}^{l-1}|\nabla_G u|(x_i)+\sum_{y\in B(x_l,2)}|\nabla_G u|(y)\\ &+ C\sum_{i=0}^{k}r_i\left(\frac{1}{V(\sigma B_i)}\sum_{y\in\sigma B_i}|\nabla_G u|^p(y)\right)^{1/p}. \end{aligned}\tag{73}$$

We employed here the observation that $|u(x_{i+1}) - u(x_i)| \leq |\nabla_G u|(x_i)$. Inequality (73) is a good substitute for (33), (43) as $|\nabla_G u|(y)$ equals to the product of the radius and the L^p-average of $|\nabla_G u|$ over the ball $\sigma B(y, \sigma^{-1})$.

Now assume that $p < s$. Write $sp/(s-p) = p^*$. Using a version of Theorem 5.1 as in Section 9 we conclude the weak type inequality

$$\sup_{t \geq 0} \frac{V(\{x \in B : |u(x) - u_{B_0}| > t\}) t^{p^*}}{V(B)} \leq C r^{p^*} \left(\frac{1}{V(B)} \sum_{x \in B} |\nabla_G u|^p(x) \right)^{p^*/p}.$$

To obtain the desired strong type inequality one reasons as follows.

Define $v_\pm$ as in the proof of Theorem 2.1 (with Ω replaced by B). It suffices to prove suitable L^{p^*} estimates for v_+ and v_-. In what follows v denotes either v_+ or v_-. We have

$$\sup_{t \geq 0} \frac{V\{x \in B : v_{t_1}^{t_2} > t\} t^{p^*}}{V(B)} \leq C r^{p^*} \left(\frac{1}{V(B)} \sum_{x \in B} |\nabla_G v_{t_1}^{t_2}|^p \right)^{p^*/p},$$

and hence

$$\begin{aligned}
\frac{1}{V(B)} \sum_{x \in B} v^{p^*}(x) &\leq \sum_{k=-\infty}^{\infty} \frac{2^{kp^*} V(\{x \in B : v_{2^{k-2}}^{2^{k-1}} \geq 2^{k-2}\})}{V(B)} \\
&\leq C r^{p^*} \left(\frac{1}{V(B)} \sum_{x \in B} \sum_{k=-\infty}^{\infty} |\nabla_G v_{2^{k-2}}^{2^{k-1}}|^p(x) \right)^{p^*/p} \\
&\leq C r^{p^*} \left(\frac{1}{V(B)} \sum_{x \in B} |\nabla_G v|^p(x) \right)^{p^*/p} \\
&\leq C r^{p^*} \left(\frac{1}{V(B)} \sum_{x \in B} |\nabla_G u|^p(x) \right)^{p^*/p}.
\end{aligned}$$

If $p = s$, then the method described above provides us with a chains that are sufficiently good to mimic the proof of Theorem 6.1.

Once we have good chains also the Hölder continuity with the same balls on both sides follows when $p > s$.

The proof is complete.

Remarks. 1) The doubling property 1. and the Poincaré inequality 2. are very important in the potential theory on graphs. Indeed, independently Delmotte, [64], [65], Holopainen and Soardi, [127], and Rigoli, Salvatori and Vignati, [216], proved that 1. and 2. imply the so-called Harnack inequality for p-harmonic functions. As a consequence, they concluded a Liouville type theorem stating that every bounded p-harmonic function on G is constant. Recall that $u : V \to \mathbb{R}$ is 2-*harmonic* if it satisfies the *mean value property*, i.e., $u(x) = d(x)^{-1}\sum_{y\sim x} u(y)$ for all $x \in V$. For a definition of p-harmonic functions with $p \neq 2$, see, e.g., [127]. The proof of the Harnack inequality employs the Sobolev–Poincaré inequality as it relies on the Moser iteration. As we have already seen properties 1. and 2. imply the Sobolev–Poincaré inequality. Papers related to the Harnack inequalities on graphs include Auscher and Coulhon [2], Chung [43], Chung and Yau [44], Lawler [169], Merkov [196], Rigoli, Salvatori and Vignati [217], Schinzel [225], Zhou [262].

The Moser iteration was originally employed in the setting of elliptic and parabolic equations; see the next section.

2) There are many examples of graphs for which both properties 1. and 2. are satisfied. A very nice example is given by a Cayley graph associated with a finitely generated group. We say that the group G *is finitely generated* if there is a finite set $\{\gamma_i\}_{i=1}^k$ such that every element $g \in G$ can be presented as a product $g = \gamma_{i_1}^{\varepsilon_1}\cdots\gamma_{i_l}^{\varepsilon_l}$, $\varepsilon_i = \pm 1$. Then the vertex set of the *Cayley graph* is the set of all elements of G and two elements $g_1, g_2 \in G$ are connected by an edge if $g_1 = g_2\gamma_i^{\pm 1}$ for some generator γ_i. Thus we may visualize finitely generated groups as geometric objects. This point of view has been intensively used after Milnor's paper, [198].

We say that the group is of polynomial growth if $V(r) \leq Cr^C$ for all $r > 0$ and some $C \geq 1$. One of the most beautiful results in the area is due to Gromov [100]. He proved that a group is of polynomial growth if and only if it is virtually nilpotent, and hence by the theorem of Bass, [6], $V(r) \approx r^d$ for some positive integer d.

Thus if the group is of polynomial growth, then it satisfies the doubling property 1. It is also known that it satisfies the 1-Poincaré inequality.

Proposition 12.3 *If G is a finitely generated group of polynomial growth $V(r) \approx r^d$, d positive integer, then there is a constant $C > 0$ such that*

$$\frac{1}{V(B)}\sum_{x\in B}|u(x) - u_B| \leq Cr\frac{1}{V(2B)}\sum_{x\in 2B}|\nabla_G u(x)|$$

for every ball $B \subset X$.

The reader may prove the proposition as an easy exercise mimicing the proof of Theorem 11.17.

Other examples of graphs with properties 1. and 2. can be found in Holopainen and Soardi [127], Coulhon [54], Coulhon and Saloff-Coste [58], [59], Saloff-Coste [222].

3) The analysis on graphs is also important in the study of open Riemannian manifolds because, roughly speaking, one can associate with given manifold a graph with similar global properties. This method of discretization of manifolds has been in active use after the papers of Kanai, [149], and Mostow, [203]. Related references include Coulhon [52] [53], Coulhon and Ledoux [57], Coulhon and Saloff-Coste [56], [58], [59], Chavel [33], Delmotte [64], [65], Holopainen [123], Holopainen and

Soardi [126], [127], Soardi [230], Varopoulos [248], and Varopoulos, Saloff-Coste and Coulhon [251].

13 Applications to P.D.E and nonlinear potential theory

The results presented in the paper directly apply to the regularity theory of degenerate elliptic equations associated with vector fields. Below we describe some of the applications.

13.1 Admissible weights

Let $A = (A_1, \dots, A_m) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ be a Carathéodory function satisfying the growth conditions

$$|A(x,\xi)| \le C_1 \omega(x) |\xi|^{p-1}, \qquad A(x,\xi)\cdot\xi \ge C_2\omega(x)|\xi|^p,$$

where $1 < p < \infty$, $C_1, C_2 > 0$ are fixed constants and $0 < \omega \in L^1_{\rm loc}(\mathbb{R}^n)$. We will denote by $d\mu = \omega\, dx$ the measure with the density ω.

Given A, we consider the equation

$$\sum_{j=1}^m X_j^*\, A_j(x, X_1 u, \dots, X_m u) = 0, \tag{74}$$

where $X = (X_1, \dots, X_m)$ is a family of vector fields with locally Lipschitz coefficients in $\mathbb{R}^n$. Recall that X_j^* denotes the formal adjoint of X_j, that is, $\int X_j uv = \int u X_j^* v$ for all $u, v \in C_0^\infty$.

The theory of nonlinear equations of the type (74), especially when X is a system of vector fields satisfying Hörmander's condition, is an area of intensive research; see, e.g., Buckley, Koskela and Lu [19], Capogna [25], Capogna, Danielli and Garofalo [27], [31], Chernikov and Vodop'yanov [38], Citti [45], Citti and Di Fazio [46], Citti, Garofalo and Lanconelli [47], Danielli, Garofalo and Nhieu [61], Franchi, Gutiérrez and Wheeden [76], Franchi and Lanconelli [78], Franchi and Serapioni [83], Garofalo and Lanconelli [90], Garofalo and Nhieu [92], Hajłasz and Strzelecki [108], Jerison [133], Jerison and Lee [134], [135], Jost and Xu [146], Lu [177], [179], [180], [181], Marchi [189], Vodop'yanov and Markina [254], Xu [258], [259], Xu and Zuily [260]. The above papers mostly deal with the nonlinear theory. References to the broad literature on the linear theory can be found in these papers.

Equation (74) is a generalization of the classical weighted p-harmonic equation. Indeed, if $X = \nabla$ and $A(\xi) = \omega(x)|\xi|^{p-2}\xi$ we get the equation

$$\operatorname{div}\left(\omega(x)|\nabla u|^{p-2}\nabla u\right) = 0.$$

In what follows we assume that the condition (54) is satisfied i.e. we assume that the Carnot–Carathéodory distance ρ is a metric such that the identity map is a homeomorphism between the Euclidean metric and ρ.

We call u a *weak solution* of (74) if

$$\int_{\mathbb{R}^n} \sum_{j=1}^{m} A_j(x, Xu)\, X_j\varphi(x)\, dx = 0,$$

for any $\varphi \in C_0^\infty(\mathbb{R}^n)$. We assume that the weak solution belongs to the weighted Sobolev space $W_X^{1,p}(\mathbb{R}^n, \mu)$ defined as the closure of C^∞ functions in the norm

$$\|u\|_{1,p,X,\omega} = \left(\int_{\mathbb{R}^n} |u(x)|^p \omega(x)\, dx\right)^{1/p} + \left(\int_{\mathbb{R}^n} |Xu(x)|^p \omega(x)\, dx\right)^{1/p}.$$

Already in the "classical" case i.e. when $Xu = \nabla u$ one has to put many additional conditions on the weight ω in order to have a reasonable theory.

The first condition concerns the definition of the Sobolev space. One needs the so-called *uniqueness* condition which guarantees that the "gradient" X is well defined in the Sobolev space associated to X. Later we will clarify this condition.

The regularity results for solutions to (74), like Harnack inequality and Hölder continuity, are usually obtained via the Moser iteration technique. For that the essential assumptions are a doubling condition on μ, (with respect to the Carnot–Carathéodory metric), the Poincaré inequality

$$\left(\fint_{\widetilde{B}} |u - u_{\widetilde{B}}|^p\, d\mu\right)^{1/p} \le Cr \left(\fint_{\widetilde{B}} |Xu|^p\, d\mu\right)^{1/p}, \tag{75}$$

for all smooth functions u in a metric ball $\widetilde{B}$, and a Sobolev inequality

$$\left(\fint_{\widetilde{B}} |u|^q\, d\mu\right)^{1/q} \le Cr \left(\fint_{\widetilde{B}} |Xu|^p\, d\mu\right)^{1/p}, \tag{76}$$

with some $q > p$ for all smooth functions u with compact support in a metric ball $\tilde{B}$.

Given the above assumptions one can mimic the standard Moser iteration technique replacing Euclidean balls by metric balls. This leads to the Harnack inequality which states that if u is a positive solution to (74) on $2\widetilde{B}$, then

$$\sup_{\widetilde{B}} u \le C \inf_{\widetilde{B}} u\,,$$

where the constant C does not depend on $\widetilde{B}$. Then the iteration of the Harnack inequality implies that each weak solution to (74) is locally Hölder continuous with respect to ρ and hence — if condition (67) is satisfied — locally Hölder continuous with respect to the Euclidean metric.

The fact that the above conditions are essential for the Moser iteration was observed first by Fabes, Kenig and Serapioni, [67]. They considered the "classical" linear setting, $X = \nabla$, $p = 2$.

It seems that Franchi and Lanconelli, [78], were the first to apply the Moser technique for the Carnot–Carathéodory metric as above. Then the idea was extended by many authors to more difficult situations; see, e.g., Capogna, Danielli and Garofalo [27], Chernikov and Vodop'yanov [38], Franchi [74], Franchi and Lanconelli [79], Franchi, Lu and Wheeden [81], Franchi and Serapioni [83], Jerison [133], Lu [180]. There are moreover many other related papers.

Saloff-Coste, [221], and Grigor'yan, [99], independently realized that in certain settings, a Poincaré inequality implies a Sobolev inequality and hence one can delete assumption (76) as it follows from (75). This result was extended then to more general situations by several authors: Biroli and Mosco, [9], Maheux and Saloff-Coste, [186], Hajłasz and Koskela, [105], Sturm, [239], Garofalo and Nhieu, [92].

The result presented below (Theorem 13.1) is in the same spirit. This is a generalization of a result of Hajłasz and Koskela, [105].

The following definition is due to Heinonen, Kilpeläinen and Martio, [120], when $X = \nabla$ and due to Chernikov and Vodop'yanov, [38], in the case of general vector fields.

We say that $\omega \in L^1_{\rm loc}(\mathbb{R}^n)$, $\omega > 0$ a.e., is *p-admissible*, $1 < p < \infty$, if the measure defined by $d\mu = \omega(x)\,dx$ satisfies the following four conditions:

1. (Doubling condition) $\mu(2\widetilde{B}) \leq C_d \mu(\widetilde{B})$ for all metric balls $\widetilde{B} \subset \mathbb{R}^n$.
2. (Uniqueness condition) If Ω is an open subset of $\mathbb{R}^n$ and $\varphi_i \in C^\infty(\Omega)$ is a sequence such that $\int_\Omega |\varphi_i|^p\,d\mu \to 0$ and $\int_\Omega |X\varphi_i - v|^p\,d\mu \to 0$, where $v \in L^p(\mu)$, then $v \equiv 0$.
3. (Sobolev inequality) There exists a constant $k > 1$ such that for all metric balls $\widetilde{B} \subset \mathbb{R}^n$ and all $\varphi \in C_0^\infty(\widetilde{B})$
$$\left(\fint_{\widetilde{B}} |\varphi|^{kp}\,d\mu \right)^{1/kp} \leq C_2 r \left(\fint_{\widetilde{B}} |X\varphi|^p\,d\mu \right)^{1/p}.$$
4. (Poincaré inequality) If $\widetilde{B} \subset \mathbb{R}^n$ is a metric ball and $\varphi \in C^\infty(\widetilde{B})$, then
$$\int_{\widetilde{B}} |\varphi - \varphi_{\widetilde{B}}|^p\,d\mu \leq C_3 r^p \int_{\widetilde{B}} |X\varphi|^p\,d\mu.$$

One can easily modify the above definition and consider vector fields defined in an open subset Ω of $\mathbb{R}^n$ with the estimates in the above conditions depending on compact subsets of Ω. However for clarity we assume the global estimates. We do not care to present the results in their most general form. We aim to present the method. Various generalizations are then obvious.

The uniqueness condition guarantees that any function $u \in L^p(\mathbb{R}^n,\mu)$ that belongs to $W^{1,p}_X(\mathbb{R}^n,\mu)$ has a *uniquely* defined gradient Xu as the limit of gradients Xu_k of smooth functions u_k which converge to u in the Sobolev norm. If the uniqueness condition were not true, then we would find $u_k \in C^\infty$ such that $u_k \to 0$ in $L^p(\mu)$ and $Xu \to v \not\equiv 0$ in $L^p(\mu)$. Then the zero function would have at least two gradients 0 and v i.e. $(0,0)$ and $(0,v)$ would be two distinct elements in $W^{1,p}_X(\mathbb{R}^n,\omega)$.

Theorem 13.1 *Let $0<\omega\in L^1_{\rm loc}(\mathbb{R}^n)$ and let X be a system of vector fields in $\mathbb{R}^n$ satisfying condition (54). Then the weight ω is p-admissible, $1<p<\infty$, if and only if the measure μ associated with ω is doubling with respect to the metric ρ (i.e. $\mu(2\widetilde{B})\le C_d\mu(\widetilde{B})$ for all metric balls $\widetilde{B}\subset\mathbb{R}^n$) and there exists $\sigma\ge 1$ such that*

$$\fint_{\widetilde{B}} |u-u_{\widetilde{B}}|\,d\mu \le Cr\left(\fint_{\sigma\widetilde{B}}|Xu|^p\,d\mu\right)^{1/p},$$

whenever $\widetilde{B}\subset\mathbb{R}^n$ is a metric ball of radius r and $u\in C^\infty(\sigma\widetilde{B})$.

Proof. The necessity is obvious. Now we prove the sufficiency. First note that the uniqueness of the gradient 2. was recently proved by Franchi, Hajłasz and Koskela, [77, Corollary 13].

Next, by Corollary 9.8, we conclude the Sobolev–Poincaré inequality

$$\left(\fint_{\widetilde{B}}|\varphi-\varphi_{\widetilde{B}}|^{p^*}\,d\mu\right)^{1/p^*} \le Cr\left(\fint_{\widetilde{B}}|X\varphi|^p\,d\mu\right)^{1/p},$$

for all $\varphi\in C^\infty(\widetilde{B})$ with some $p^*>p$ (remember that the doubling condition implies (21) with $s=\log_2 C_d$). For our purpose the exact value of p^* is irrelevant. It is only important that $p^*>p$. This and the Hölder inequality imply the Poincaré inequality 4.

Now we are left with the Sobolev inequality 3. Since $p^*>p$ we have $p^*=kp$ for some $k>1$. For $\varphi\in C_0^\infty(\widetilde{B})$ we have

$$\left(\fint_{\widetilde{B}}|\varphi|^{kp}\,d\mu\right)^{1/kp} \le \left(\fint_{\widetilde{B}}|\varphi-\varphi_{\widetilde{B}}|^{kp}\,d\mu\right)^{1/kp} + |\varphi_{\widetilde{B}}|.$$

The Sobolev–Poincaré inequality provides us with the desired estimate for the first summand on the right hand side. Thus it suffices to estimate $|\varphi_{\widetilde{B}}|$. The Poincaré inequality applied to the ball $\widetilde{B}$ gives

$$\left(\int_{\widetilde{B}}|\varphi-\varphi_{\widetilde{B}}|^p\,d\mu\right)^{1/p} \le Cr\left(\int_{\widetilde{B}}|X\varphi|^p\,d\mu\right)^{1/p}, \tag{77}$$

and when applied to the ball $2\widetilde{B}$ gives

$$\left(\int_{\widetilde{B}}|\varphi-\varphi_{2\widetilde{B}}|^p\,d\mu\right)^{1/p} \quad\le\quad \left(\int_{2\widetilde{B}}|\varphi-\varphi_{2\widetilde{B}}|^p\,d\mu\right)^{1/p}$$

$$
\begin{aligned}
&\leq C2r\left(\int_{2\widetilde{B}} |X\varphi|^p\, d\mu\right)^{1/p} \\
&= 2Cr\left(\int_{\widetilde{B}} |X\varphi|^p\, d\mu\right)^{1/p}. \qquad (78)
\end{aligned}
$$

Thus

$$
\begin{aligned}
\left(1-\frac{\mu(\widetilde{B})}{\mu(2\widetilde{B})}\right)\left(\int_{\widetilde{B}} |\varphi_{\widetilde{B}}|^p\, d\mu\right)^{1/p} &= \left(\int_{\widetilde{B}} |\varphi_{\widetilde{B}}-\varphi_{2\widetilde{B}}|^p\, d\mu\right)^{1/p} \\
&\leq 3Cr\left(\int_{\widetilde{B}} |X\varphi|^p\, d\mu\right)^{1/p}.
\end{aligned}
$$

In the proof of the equality we employ the fact that φ is supported in $\widetilde{B}$ and the inequality follows from the triangle inequality and inequalities (77) and (78). It follows from the doubling property and the geometry of metric balls in $\mathbb{R}^n$ that $1-\mu(\widetilde{B})/\mu(2\widetilde{B}) > C > 0$ and hence

$$
|\varphi_{\widetilde{B}}| \leq C'r\left(\fint_{\widetilde{B}} |X\varphi|^p\, d\mu\right)^{1/p}.
$$

The proof is complete.

13.2 Sobolev embedding for $0 < p < 1$

The classical Sobolev–Poincaré inequality

$$
\left(\fint_B |u-u_B|^{p^*}\, dx\right)^{1/p^*} \leq C\left(\fint_B |\nabla u|^p\right)^{1/p},
$$

holds when $1 \leq p < n$. It is easy to see that it fails when $0 < p < 1$, and even a weaker version of the Poincaré inequality fails for the range $0 < p < 1$. For an explicit example, see Buckley and Koskela [16].

However Buckley and Koskela, [16], and in a more general setting Buckley, Koskela and Lu, [19], proved that if u is a solution to the equation div $A(x, Xu) = 0$ in a John domain with respect to the Carnot–Carathéodory metric, then u satisfies a Sobolev–Poincaré inequality for any $0 < p < s$, where s is given by condition (21).

As we will see, one of the results of the paper, Theorem 9.7, which states that for any $0 < p < s$, a p-Poincaré inequality implies a Sobolev–Poincaré inequality, can be regarded as an abstract version of the above result. In particular this gives a new proof of the result of Buckley, Koskela and Lu.

More precisely, assume that $X = (X_1, \ldots, X_m)$ are locally Lipschitz vector fields in $\mathbb{R}^n$. Assume that the associated Carnot–Carathéodory metric satisfies condition (54), the Lebesgue measure is doubling with respect to the Carnot–Carathéodory distance i.e. $|2\widetilde{B}| \leq C_d|\widetilde{B}|$ for all metric balls $\widetilde{B} \subset \mathbb{R}^n$, and that condition (21) is satisfied.

In addition we assume that the 1-Poincaré inequality is satisfied i.e. there is $C > 0$ and $\sigma \geq 1$ such that

$$\fint_{\widetilde{B}} |u - u_{\widetilde{B}}|\, dx \leq C_P r \fint_{\sigma\widetilde{B}} |Xu|\, dx\,,$$

for all metric balls $\widetilde{B} \subset \mathbb{R}^n$ and all $u \in C^\infty(\sigma\widetilde{B})$.

Let $A : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ be a Carathéodory function such that

$$|A(x,\xi)| \leq C_1|\xi|^{q-1}, \qquad A(x,\xi)\cdot\xi \geq C_2|\xi|^q\,,$$

where $1 < q < \infty$ is given. (Observe that in contrast with the previous section we do not allow a weight ω.)

The following result is a variant of the result of Buckley, Koskela and Lu, [19].

Theorem 13.2 *Let $\Omega \subset \mathbb{R}^n$ be a John domain with respect to the Carnot–Carathéodory metric. Then for any $0 < p < s$ there is a constant $C > 0$ such that if u is a solution to equation* $\operatorname{div} A(x, Xu) = 0$, *in Ω, then*

$$\inf_{c\in\mathbb{R}} \left(\fint_\Omega |u-c|^{p^*}\, dx\right)^{1/p^*} \leq C \operatorname{diam}\Omega \left(\fint_\Omega |Xu|^p\right)^{1/p}.$$

The constant C depends on n, p, s, C_b, C_d, C_1, C_2, C_P, and C_J only.

Proof. Let u be a solution to $\operatorname{div} A(x, Xu) = 0$ in Ω. The first fact we need is that the gradient $|Xu|$ of the solution u satisfies a weak reverse Hölder inequality. This is well known. However, for the sake of completeness, we provide a proof.

Given a metric ball $\widetilde{B}$, let η_R be a cut-off function such that $0 \leq \eta_R \leq 1$, $\eta_R|_{\widetilde{B}} \equiv 1$, $\eta_R \equiv 0$ outside $2\widetilde{B}$ and $|X\eta_R| \leq 1/R$. Using the distance function with respect to ρ we easily construct a cut-off function with the metric Lipschitz constant $1/R$. Then the estimate $|X\eta_R| \leq 1/R$ follows from Corollary 11.8.

Now, using the test function $(u - u_{2\widetilde{B}})\eta_R$, where $2\widetilde{B} \subset \Omega$ is any metric ball and η_R is the associated cut-off function, we conclude from a standard computation the Caccioppoli estimate

$$\int_{\widetilde{B}} |Xu|^q \leq \frac{C}{R^q}\int_{2\widetilde{B}} |u - u_{2\widetilde{B}}|^q\,.$$

Then we estimate the right hand side by the Sobolev–Poincaré inequality and conclude that there is $p < q$ such that for all metric balls $\widetilde{B}$ with $2\widetilde{B} \subset\subset \Omega$

$$\left(\fint_B |Xu|^q\, dx\right)^{1/q} \leq C\left(\fint_{2\widetilde{B}} |Xu|^p\, dx\right)^{1/p}. \tag{79}$$

This inequality is known under the name *weak reverse Hölder inequality*.

It is well known that the weak reverse Hölder inequality has the self-improving property: if inequality (79) holds for *some* $0 < p < q$ and all $\widetilde{B}$ with $2\widetilde{B} \subset \Omega$, then for *any* $0 < p < q$ there is a new constant C such that (79) holds for any $\widetilde{B}$ with $2\widetilde{B} \subset \Omega$, see [19, Lemma 1.4]. This together with the 1-Poincaré inequality shows that the pair u, g satisfies a p-Poincaré inequality in Ω for any $p > 0$. Hence the claim follows from Theorem 9.7. The proof is complete.

14 Appendix

Here we collect the results in the measure theory that are needed in the paper. All the material is standard. Since we could not find a single reference that would cover the material we need, we have made all the statements precise and sometimes we have even given proofs. Good references are Federer [68], Mattila [193], and Simon [229].

In the appendix we *do not* assume that the measure μ is doubling.

14.1 Measures.

Throughout the paper by a measure we mean an outer measure, and by a Borel measure, an outer, Borel-regular measure i.e., such a measure μ on a metric space (X, d) that all Borel sets are μ-measurable and for every set A there exists a Borel set B such that $A \subset B$ and $\mu(A) = \mu(B)$. In the case of a Borel measure we also assume that the measure of each ball is strictly positive and $X = \bigcup_{j=1}^{\infty} U_j$, where U_j are open sets with $\mu(U_j) < \infty$.

Note that if the space X is locally compact, separable and $\mu(K) < \infty$ for every compact set K, then X can be written as a union of a countable family of open sets with finite measure.

Theorem 14.1 *Suppose that μ is a Borel measure on (X, d). Then*

$$\mu(A) = \inf_{\substack{U \supset A \\ U-open}} \mu(U)$$

for all subsets $A \subset X$, and

$$\mu(A) = \sup_{\substack{C \subset A \\ C-closed}} \mu(C)$$

for all measurable sets $A \subset X$.

For the proof, see [68, Theorem 2.2.2. and Section 2.2.3], [193, Theorem 1.10] or [229, Theorem 1.3].

If the space is locally compact and separable, the supremum over closed sets in the above theorem equals to the supremum over compact sets.

As a corollary to the above theorem we obtain the following well known result.

Theorem 14.2 *If μ is a Borel measure on a metric space (X,d), then for every $1 \le p < \infty$, continuous functions are dense in $L^p(X,\mu)$.*

Proof. Simple functions are dense in $L^p(X,\mu)$, see [219, Theorem 3.13], so it suffices to prove that characteristic functions can be approximated by continuous functions. Fix $\varepsilon > 0$. If $A \subset X$ is measurable, $\mu(A) < \infty$, then there exists a closed set C and an open set U such that $C \subset A \subset U$, $\mu(U \setminus C) < \varepsilon$. Now by Urysohn's lemma, there exists a continuous function φ^ε on X such that $0 \le \varphi^\varepsilon \le 1$, $\varphi^\varepsilon|_C = 1$ and $\varphi^\varepsilon|_{X\setminus U} = 0$. Then obviously $\|\chi_A - \varphi^\varepsilon\|_p \to 0$ as $\varepsilon \to 0$. This completes the proof.

In order to have a variety of Borel measures one usually assumes that the space be locally compact. In the definition of the doubling measure one does not assume anything about the metric space. However, as we will see, the existence of a doubling measure is such a strong condition that the space is "almost" locally compact.

We say that a subset A of a metric space (X,d) is an ε-net if for every $x \in X$ there is $y \in A$ with $d(x,y) < \varepsilon$. A metric space (X,d) is called *totally bounded* if for each $\varepsilon > 0$ there exists a finite ε-net.

The following two lemmas are well known.

Lemma 14.3 *A metric space (X,d) is compact if and only if it is complete and totally bounded.*

Lemma 14.4 *Every metric space is isometric to a dense subset of a complete metric space.*

The first lemma follows from a direct generalization of the proof that every bounded sequence of real numbers contains a convergent subsequence, while the second lemma follows by adding the "abstract limits" of Cauchy sequences to the space.

Theorem 14.5 *If a metric space (X,d) admits a Borel measure μ which is locally uniformly positive in the sense that for every bounded set $A \subset X$ and every $\varepsilon > 0$*

$$\inf_{x\in A} \mu(B(x,\varepsilon)) > 0, \tag{80}$$

then (X,d) is isometric to a dense subset of a locally compact separable metric space.

Proof. The fact that X is a union of countably many open sets of finite measure and (80) imply that X can be covered by balls $X = \bigcup_{j=1}^\infty B_j$ with $\mu(2B_j) < \infty$.

According to Lemma 14.3 and Lemma 14.4 it suffices to prove that for every $j = 1, 2, \ldots$ and every $\varepsilon > 0$ there is a finite ε-net in B_j. This, however, easily follows from (80) and the condition $\mu(2B_j) < \infty$. The proof is complete.

It is of fundamental importance to note that the doubling condition implies local uniform positivity of the measure, as follows from the following result.

Lemma 14.6 *Let μ be a Borel measure on a metric space X. Assume that μ is doubling, in the following sense, on a bounded subset $Y \subset X$: there is a constant $C_d \geq 1$ such that*

$$\mu(B(x,2r)) \leq C_d \mu(B(x,r)),$$

whenever $x \in Y$, and $r \leq \operatorname{diam} Y$. Then

$$\mu(B(x,r)) \geq (2 \operatorname{diam} Y)^{-s} \mu(Y) r^s,$$

for $s = \log_2 C_d$, $x \in Y$ and $r \leq \operatorname{diam} Y$.

The above lemma together with Theorem 14.5 shows that doubling spaces are isometric to dense subsets of locally compact separable metric spaces. The analogous result holds also when the measure is doubling on some open set only. Note that a doubling measure is finite on bounded sets.

The above remark together with the following result shows that a doubling measure can be extended to a doubling measure on the larger locally compact space.

Proposition 14.7 *Let $Y \subset X$ be a dense subset of a metric space (X, d). Let μ be a Borel measure on (Y, d), finite on bounded sets. Then there exists a unique Borel measure $\bar{\mu}$ on (X, d) such that*

$$\bar{\mu}(U) = \mu(U \cap Y)$$

for every open set $U \subset X$. Moreover, if μ is doubling on (Y, d), then $\bar{\mu}$ is doubling on (X, d) with the same doubling constant.

Proof. Set $\bar{\mu}(A) = \inf_{B \supset A,\ B-Borel} \mu(B \cap Y)$ for an arbitrary set $A \subset X$. One easily verifies that $\bar{\mu}$ is a Borel measure on (X, d). This proves the existence of the measure. The uniqueness follows form Theorem 14.1.

Assume now that μ is doubling. Then obviously $\bar{\mu}$ is doubling with the same doubling constant on all balls centered at Y. Since any ball in X can be "approximated" by balls centered at Y, the result follows.

Remark. If we removed the assumption that μ be finite on bounded sets, Y would still have the property $Y = \bigcup_{j=1}^{\infty} U_j$, where the sets U_j are open with $\mu(U_j) < \infty$. However then this property would not necessarily be true for $\bar{\mu}$. For example, let Y be the complement of a Cantor set in $[0,1]$, and $X = [0,1]$. Then Y consists of countable many intervals. Equip Y with a measure so that the measure of each of the intervals is 1. Then X cannot be decomposed into a countable number of open sets with finite $\bar{\mu}$-measure.

14.2 Uniform integrability.

In this section μ is an arbitrary measure on a set X.

Assume that $\mu(X) < \infty$. We say that a family $\{u_\alpha\}_{\alpha \in I}$ of μ-measurable functions on X is *uniformly integrable* if $\sup_{\alpha\in I} \int_X |u_\alpha|\, d\mu < \infty$ and

$$\lim_{\mu(A)\to 0} \sup_{\alpha \in I} \int_A |u_\alpha|\, d\mu = 0.$$

The following theorem is due to Vallée Poussin. For a proof, see Dellacherie and Meyer [63], or Rao and Ren [212].

Theorem 14.8 *Let μ be a measure on a set X with $\mu(X) < \infty$ and let $\{u_\alpha\}_{\alpha\in I}$ be a family of μ-measurable functions. Then the following two conditions are equivalent.*

1. *The family $\{u_\alpha\}_{\alpha\in I}$ is uniformly integrable.*
2. *There exists a convex smooth function $F : [0,\infty) \to [0,\infty)$ such that $F(0) = 0$, $F(x)/x \to \infty$ as $x \to \infty$ and*

$$\sup_{\alpha\in I} \int_X F(|u_\alpha|)\, d\mu < \infty.$$

The following well known result is a very useful criteria for convergence in L^1.

Proposition 14.9 *Let $\mu(X) < \infty$. If u_n are uniformly integrable on X and $u_n \to u$ a.e., then $\int_X |u_n - u|\, d\mu \to 0$.*

Proof. It follows directly from Egorov's theorem and the definition of uniform integrability that the sequence u_n is a Cauchy sequence in the L^1 norm, and hence u_n converges to u in L^1. The proof is complete.

14.3 L^p and L^p_w spaces.

In the following two theorems μ is an arbitrary σ-finite measure on X. The first result is known as Cavalieri's principle.

Theorem 14.10 *If $p > 0$ and u is measurable, then*

$$\int_X |u|^p\, d\mu = p \int_0^\infty t^{p-1} \mu(|u| > t)\, dt.$$

The claim follows from Fubini's theorem applied to $X \times [0,\infty)$.

We say that a measurable function u belongs to the Marcinkiewicz space $L^p_w(X)$ if there is $m > 0$ such that

$$\mu(|u| > t) \le m t^{-p} \quad \text{for all } t > 0. \tag{81}$$

If $u \in L^p(X)$, then (81) with $m = \int_X |u|^p\, d\mu$ is known as Chebyschev's inequality, so $L^p(X) \subset L^p_w(X)$. The converse inclusion does not hold. However, the following, well known result holds.

Theorem 14.11 *If $\mu(X) < \infty$ then $L^p_w(X) \subset L^q(X)$ for all $0 < q < p$. Moreover, if u satisfies (81), then*

$$\|u\|_{L^q(X)} \leq 2^{1/q} \left(\frac{qm}{p-q}\right)^{1/p} \mu(X)^{1/q-1/p}. \tag{82}$$

Proof. Fix $t_0 > 0$. Using Theorem 14.10 and the estimates $\mu(|u| > t) \leq \mu(X)$ for $t \leq t_0$ and $\mu(|u| > t) \leq mt^{-p}$ for $t > t_0$ we get

$$\int_X |u|^q \, d\mu \leq q \left(\int_0^{t_0} t^{q-1} \mu(X)\, dt + m \int_{t_0}^{\infty} t^{q-p-1}\, dt \right) = t_0^q \mu(X) + \frac{qm}{p-q} t_0^{q-p}.$$

Then inequality (82) follows by choosing $t_0 = (qm/(p-q))^{1/p} \mu(X)^{-1/p}$.

14.4 Covering lemma.

Theorem 14.12 (5r-covering lemma.) *Let $\mathcal{B}$ be a family of balls in a metric space (X, d) with $\sup\{\operatorname{diam} B : B \in \mathcal{B}\} < \infty$. Then there is a pairwise disjoint subcollection $\mathcal{B}' \subset \mathcal{B}$ such that*

$$\bigcup_{B \in \mathcal{B}} B \subset \bigcup_{B \in \mathcal{B}'} 5B$$

If (X, d) is separable, then $\mathcal{B}'$ is countable and we can represent $\mathcal{B}'$ as a sequence $\mathcal{B}' = \{B_i\}_{i=1}^{\infty}$, and so

$$\bigcup_{B \in \mathcal{B}} B \subset \bigcup_{i=1}^{\infty} 5B_i.$$

See Federer [68, 2.8.4-6], Simon [229, Theorem 3.3], or Ziemer [263, Theorem 1.3.1] for a clever proof.

14.5 Maximal function.

Assume that the measure μ is doubling on an open set $\Omega \subset X$. The following theorem is a version of the well known maximal theorem of Hardy, Littlewood and Wiener.

Theorem 14.13 (Maximal theorem.) *If X, Ω and μ are as above, and the maximal function $M_\Omega u$ is defined as in the introduction, then*

1. *$\mu(\{x \in \Omega : \ M_\Omega u(x) > t\}) \leq Ct^{-1} \int_\Omega |u|\, d\mu$ for $t > 0$ and*
2. *$\|M_\Omega u\|_{L^p(\Omega)} \leq C\|u\|_{L^p(\Omega)}$ for $1 < p \leq \infty$.*

In the first inequality the constant C depends on C_d only, while in the second one it depends on C_d and p.

For a proof in the case of Lebesgue measure, see Stein, [233, Chapter 1]. We assume that the reader is familiar with that proof and we show how to modify the argument in order to cover our setting. It sufices to prove 1.; one then proceeds as in [233]. Inequality 1. would follow from this inequality for the restricted maximal function $M_{\Omega,R}u$ provided we prove it with a constant C that does not depend on R. To this end, note first that the doubling condition implies that Ω is separable and hence the second part of Theorem 14.12 applies. Then the argument from the case of the Lebesgue measure works without any changes. We had to work with the restricted maximal function in order to obtain a suitable covering consisting of balls with radii less than R (if we did not have the upper bound for the radii, we could not apply Theorem 14.12).

We will also need a more general result. For $c \geq 1$ and $x \in \Omega$ define $\mathcal{F}_c(x)$ as the family of all measurable sets $E \subset \Omega$ such that $E \subset B(x,r)$ and $\mu(B(x,r)) \leq c\mu(E)$ for some $r > 0$. Then we define a new maximal function by

$$\mathcal{M}_\Omega^c u(x) = \sup_{E \in \mathcal{F}_c(x)} \fint_E |u|\, d\mu.$$

Obviously $\mathcal{M}_\Omega^c u \leq c M_\Omega u$, and thus we obtain as a corollary to Theorem 14.13 the following result.

Corollary 14.14 *Theorem 14.13 holds with M_Ω replaced by $\mathcal{M}_\Omega^c$. The only difference is that now the constants C in Theorem 14.13 depend also on c.*

14.6 Lebesgue differentiation theorem.

We say that a sequence of nonempty sets $\{E_i\}_{i=1}^\infty$ converges to x if there exists a sequence of radii $r_i > 0$ such that $E_i \subset B(x,r_i)$ and $r_i \to 0$ as $i \to \infty$.

Theorem 14.15 *Let μ be doubling on $\Omega \subset X$ and $u \in L^1_{\rm loc}(\Omega,\mu)$. Then for μ-a.e. $x \in \Omega$ we have*

$$\lim_{r\to 0} \fint_{B(x,r)} u(y)\, d\mu(y) = u(x). \tag{83}$$

Moreover, if we fix $c \geq 1$, then for μ-a.e. $x \in \Omega$ and every sequence of sets $E_i \in \mathcal{F}_c(x)$, $i = 1,2,\dots$ that converges to x we have

$$\lim_{i\to\infty} \fint_{E_i} u(y)\, d\mu(y) = u(x). \tag{84}$$

See [233, Chapter 1] for a proof in the case of the Lebesgue measure in $\mathbb{R}^n$. The same argument works also in our setting as it only relies on two facts: the weak type inequality for the maximal function (see Theorem 14.13 and Corollary 14.14) and the density of continuous functions in L^1 (see Theorem 14.2).

Let μ be doubling on $\Omega \subset X$. Given $u \in L^1_{\mathrm{loc}}(\Omega, \mu)$ it is often convenient to identify u with the representative given everywhere by the formula

$$u(x) := \limsup_{r\to 0} \fint_{B(x,r)} u(y)\, d\mu(y)\,. \tag{85}$$

Theorem 14.15 shows that taking the limit above only modifies u on a set of measure zero. We say that $x \in \Omega$ is a *Lebesgue point of* u if

$$\lim_{r\to 0} \fint_{B(x,r)} |u(y) - u(x)|\, d\mu(y) = 0,$$

where $u(x)$ is given by (85). It follows from Theorem 14.15 that almost all points of Ω are Lebesgue points of u. Observe that if $x \in \Omega$ is a Lebesgue point of u, then both (83) and (84) are true.

References

[1] Adams, D. A., Hedberg, L. I.: *Function spaces and potential theory.* Grundlehren der Mathematischen Wissenschaften, 314, Springer-Verlag, Berlin, 1996.

[2] Auscher P., Coulhon T.: Gaussian lower bounds for random walks from elliptic regularity. *Ann. Inst. H. Poincaré Probab. Stat.*, 35 (1999), 605–630.

[3] Arnol'd, V. I.: *Ordinary Differential Equations.* Springer, 1992.

[4] Bakry, D., Coulhon, T., Ledoux, M., Saloff-Coste, L.: Sobolev inequalities in disguise. *Indiana Univ. Math. J.* 44 (1995), 1033–1074.

[5] Barlow, M. T., Bass, R. F.: Coupling and Harnack inequalities for Sierpinski carpets. *Bull. Amer. Math. Soc. (N.S.)* 29 (1993), 208–212.

[6] Bass, H.: The degree of polynomial growth of finitely generated groups. *Proc. London Math. Soc.* 25 (1972), 603–614.

[7] Bellaïche, A.: The tangent space in sub-Riemannian geometry. In: [240], pp. 1–78.

[8] Biroli, M., Mosco, U.: Formed de Dirichlet et estimationes structurelles dans les mileux discontinues, *C. R. Acad. Sci. Paris*, 313 (1991), 593–598.

[9] Biroli, M. Mosco, U.: Sobolev inequalities on homogeneous spaces. *Pot. Anal.* 4 (1995), 311–324.

[10] Biroli, M., Mosco, U.: A Saint-Venant type principle for Dirichlet forms on discontinuous media. *Ann. Mat. Pura Appl.* 169 (1995), 125–181.

[11] Bobkov, S. G., Houdré, C.: Some connections between Sobolev–type inequalities and isoperimetry. *Mem. Amer. Math. Soc.* 129 (1997), no. 616, 1–111.

[12] Bojarski, B.: Remarks on Sobolev imbedding inequalities, *in: Proc. of the Conference on Complex Analysis (Joensu 1987)*, pp. 52–68, Lecture Notes in Math. 1351 (Springer 1988).

[13] Boman, J.: L^p–estimates for very strongly elliptic systems. Department of Mathematics, University of Stockholm, Sweden (1982), Reports no 29.

[14] Bony, J. M.: Principe du maximum, inégalité de Harnack et du probleme de Cauchy pour les opérateurs elliptiques dégénérés, *Ann. Inst. Fourier (Grenoble)* 19 (1969), 277–304.

[15] Bourdon, M., Pajot, H.: Poincaré inequalities and quasiconformal structure on the boundary of some hyperbolic buildings, *Proc. Amer. Math. Soc.*, 127 (1999), 2315–2324.

[16] Buckley, S. M., Koskela, P.: Sobolev–Poincaré inequalities for $p < 1$. *Indiana Univ. Math. J.* 43 (1994), 221–240.

[17] Buckley, S. M., Koskela, P.: Sobolev–Poincaré implies John. *Math. Res. Lett.* 2 (1995), 577–593.

[18] Buckley, S. M., Koskela, P.: Geometric criteria for inequalities of Sobolev-Poincaré type. *Internat. Math. Res. Notices* 2 (1995), 577–593.

[19] Buckley, S. M., Koskela, P., Lu, G.: Subelliptic Poincaré inequalities: the case $p < 1$. *Publ. Mat.* 39 (1995), 313–334.

[20] Buckley, S. M., Koskela, P., Lu, G.: Boman equals John. *in: Proceedings of the XVI Rolf Nevanlinna Colloquium (Joensuu 1995)*, pp. 91–99, de Gruyter, Berlin, 1996.

[21] Buckley, S., O'Shea, J.: Weighted Trudinger-type inequalities, *Indiana Univ. Math. J.*, to appear.

[22] Busemann, H.: *The geometry of geodesics*, Academic Press, New York 1955.

[23] Buser, P.: A note on the isoperimetric constant. *Ann. Sc. Ecole Norm. Sup.* 15 (1982), 213–230.

[24] Calderón, A. P., Scott, R.: Sobolev type inequalities for $p > 0$. *Studia Math.* 62 (1978), 75–92.

[25] Capogna, L.: Regularity of quasilinear equations in the Heisenberg group. *Comm. Pure Appl. Math.* 50 (1997), 867–889.

[26] Capogna, L.: Regularity for quasilinear equations and 1-quasiconformal maps in Carnot groups. *Math. Ann.* 313 (1999), 263–295.

[27] Capogna, L., Danielli, D., Garofalo, N.: An imbedding theorem and the Harnack inequality for nonlinear subelliptic equations. *Comm. Partial Differential Equations* 18 (1993), 1765–1794.

[28] Capogna, L., Danielli, D., Garofalo, N.: Subelliptic mollifiers and characterization of Rellich and Poincaré domains. *Rend. Sem. Mat. Univ. Polit. Torino* 54 (1993), 361–386.

[29] Capogna, L., Danielli, D., Garofalo, N.: The geometric Sobolev embedding for vector fields and the isoperimetric inequality. *Comm. Anal. Geom.* 2 (1994), 203–215.

[30] Capogna, L., Danielli, D., Garofalo, N.: Subelliptic mollifiers and a basic pointwise estimate of Poincaré type. *Math. Z.* 226 (1997), 147–154.

[31] Capogna, L., Danielli, D., Garofalo, N.: Capacitary estimates and the local behavior of solutions to nonlinear subelliptic equations. *Amer. J. Math.* 118 (1996), 1153–1196.

[32] Carathéodory, C.: Untersuchungen über die Grundlagen der Thermodynamik. *Math. Ann.* 67 (1909), 355–386.

[33] Chavel, I.: *Riemannian Geometry — A modern introduction.* Cambridge Tracts in Mathematics, vol. 108, Cambridge Univ. Press, 1993.

[34] Cheeger, J.: Differentiability of Lipschitz functions on metric measure spaces. *Geom. Funct. Anal.* 9 (1999) 428–517.

[35] Cheeger, J., Gromov, M., Taylor, M.: Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifolds. *J. Differential Geom.* 17 (1982), 15–53.

[36] Chemin, J. Y., Xu, C. J.: Inclusions de Sobolev en calcul de Weyl–Hörmander et champs de vecteurs sous-elliptiques. *Ann. Scient. Éc. Norm. Sup.* 30 (1997), 719–751.

[37] Chen, R. Li, P.: On Poincaré type inequalities. *Trans. Amer. Math. Soc.* 349 (1997), 1561–1585.

[38] Chernikov, V. M., Vodop'yanov, S. K.: Sobolev Spaces and hypoelliptic equations I,II. *Siberian Advances in Mathematics.* 6 (1996) no. 3, 27–67, no. 4, 64–96. Translation from: Trudy In-ta matematiki RAN. Sib. otd-nie. 29 (1995), 7–62.

[39] Chiarenza, F., Frasca, M.: A note on weighted Sobolev inequality. *Proc. Amer. Math. Soc.* 93 (1985), 703–704.

[40] Chow, W. L.: Über Systeme non linearen partiellen Differentialgleichungen erster Ordung. *Math. Ann.* 117 (1939), 98–105.

[41] Christ, M.: *Lectures on Singular Integral Operators*, Regional Conference Series in Math. 77, 1989.

[42] Chua, S. K.: Weighted Sobolev inequalities on domains satisfying the chain condition. *Proc. Amer. Math. Soc.* 117 (1993), 449–457.

[43] Chung, F. R. K.: *Spectral graph theory.* CBMS Regional Conference Series in Mathematics, 92; American Mathematical Society, 1997

[44] Chung, F. R. K., Yau, S. T.: A Harnack inequality for homogneous graphs and subgraphs. *Comm. Anal. Geom.* 2 (1994), 627–640.

[45] Citti, G.: Semilinear Dirichlet problem involving critical exponents for the Kohn Laplacian. *Ann. Mat. Pura Appl.* 169 (1995), 375–392.

[46] Citti, G., Di Fazio, G.: Hölder continuity of the solutions for operators which are sum of squares of vector fields plus a potential. *Proc. Amer. Math. Soc.* 122 (1994), 741–750.

[47] Citti, G., Garofalo, N., Lanconelli, E.: Harnack's inequality for sum of squares plus potential. *Amer. J. Math.* 115 (1993), 699–734.

[48] Coifman, R., Weiss, G.: *Analyse harmonique sur certains espaces homogenes*, Lecture Notes in Math. 242, Springer 1971.

[49] Coifman, R., Weiss, G.: Extensions of Hardy spaces and their use in analysis. *Bull. Amer. Math. Soc.* 83 (1977), 569–645.

[50] Colding, T. H., Minicozzi, W. P. II: Harmonic functions on manifolds. *Ann. of Math.* 146 (1997), 725–747.

[51] Colding, T. H., Minicozzi, W. P. II: Weyl type bounds for harmonic functions. *Invent. Math.* 131 (1988), 257–298.

[52] Coulhon T.: Analysis on graphs with regular volume growth, to appear in *Proceedings of the Cortona conference on Random walks and discrete potential theory*, Cambridge U.P.

[53] Coulhon, T.: Dimension at infinity for Riemannian manifolds. *Potential Anal.* 4 (1995), 335–344.

[54] Coulhon, T.: Espaces de Lipschitz et inégalités de Poincaré. *J. Funct. Anal.* 136 (1996), 81–113.

[55] Coulhon T.: Heat kernels on non-compact Riemannian manifolds: a partial survey. *Séminarie de Théorie Spectrale et Géométrie*, 15 (1996-1997), Institut Fourier, 1998, 167–187.

[56] Coulhon T., Grigor'yan A.: Random walks on graphs with regular volume. *Geom. Funct. Anal.* 8 (1998), 656–701.

[57] Coulhon, T., Ledoux, M.: Isopérimétrie, décroissance du noyau de la chaleur et transformations de Riesz: un contre-exemple. *Arkiv Mat.* 32 (1994), 63–77.

[58] Coulhon, T., Saloff-Coste, L.: Isopérimétrie pour les groupes et les variétés. *Rev. Mat. Iberoamericana* 9 (1993), 293–314.

[59] Coulhon, T., Saloff-Coste, L.: Variétés riemanniennes isométriques à l'infini. *Rev. Mat. Iberoamericana.* 11 (1995), 687–726.

[60] Danielli, D.: A compact embedding theorem for a class of degenerate Sobolev spaces. *Rend. Sem. Mat. Univ. Pol. Torino* 49 (1991), 399–420.

[61] Danielli, D., Garofalo, N., Nhieu, D.-M.: Trace inequalities for Carnot–Carathéodory spaces and applications. *Ann. Scuola Norm. Sup. Pisa* 27 (1998), 195–252.

[62] David, G., Semmes, S.: Strong A_∞ weights, Sobolev inequalities, and quasiconformal mappings, *in: Analysis and Partial Differential Equations*, pp. 101–111, Lecture Notes on Pure and Applied Math. 122, Marcel Dekker, 1990.

[63] Dellacherie, C., Meyer, P. A.: *Probabilities and potential.* North-Holland Mathematics Studies, 29. North-Holland Publishing Co., 1978.

[64] Delmotte, T.: Inégalité de Harnack elliptique sur les graphes, *Colloq. Math.* 72 (1997), 19–37.

[65] Delmotte T.: Parabolic Harnack inequality and estimates of Markov chains on graphs. *Rev. Mat. Iberoamericana* 15 (1999), 181–232.

[66] Evans, L. C., Gariepy, R. F.: *Measure Theory and Fine Properties of Functions.* Studies in Advanced Mathematics, 1992.

[67] Fabes, E. B., Kenig, C. E., Serapioni, R. P.: The local regularity of solutions of degenerate elliptic equations. *Comm. Partial Differential Equations* 7 (1982), 77–116.

[68] Federer, H.: *Geometric measure theory* (Springer 1969).

[69] Fefferman, C., Phong, D. H.: Subelliptic eigenvalue problems. In *Proceedings of the conference in harmonic analysis in honor of Antoni Zygmund*, Wadsworth Math. Ser., Wadsworth, Belmont, California, 1981, 590–606.

[70] Fefferman, C., Sanchez-Calle, A.: Fundamental solutions for second order subelliptic operators. *Ann. Math.* 124 (1986), 247–272.

[71] Folland, G. B.: A fundamental solution for a subelliptic operator. *Bull. Amer. Math. Soc.* 79 (1973), 373–376.

[72] Folland, G. B.: Subelliptic estimates and function spaces on nilpotent Lie groups. *Ark. Mat.* 13 (1975), 161–207.

[73] Folland, G. B., Stein, E. M.: *Hardy spaces on homogeneous groups.* Mathematical Notes vol. 28, Princeton Univ. Press, Princeton, New Jersey, 1982.

[74] Franchi, B.: Weighted Sobolev–Poincaré inequalities and pointwise inequalities for a class of degenerate elliptic equations. *Trans. Amer. Math. Soc.* 327 (1991), 125–158.

[75] Franchi, B., Gallot, S., Wheeden, R. L.: Sobolev and isoperimetric inequalities for degenerate metrics. *Math. Ann.* 300 (1994), 557–571.

[76] Franchi, B., Gutiérrez , C. E., Wheeden, R. L.: Weighted Sobolev–Poincaré inequalities for Grushin type operators. *Comm. Partial Differential Equations* 19 (1994), 523–604.

[77] Franchi, B., Hajłasz, P., Koskela, P.: Definitions of Sobolev classes on metric spaces. *Ann. Inst. Fourier (Grenoble)*, to appear.

[78] Franchi, B., Lanconelli, E.: Hölder regularity theorem for a class of non uniformly elliptic operators with measurable coefficients. *Ann. Scuola Norm. Sup. Pisa* 10 (1983), 523–541.

[79] Franchi, B., Lanconelli, E.: An imbedding theorem for Sobolev spaces related to non smooth vector fields and Harnack inequality. *Comm. Partial Differental Equations* 9 (1984), 1237–1264.

[80] Franchi, B., Lu, G., Wheeden, R.: Representation formulas and weighted Poincaré inequalities for Hörmander vector fields. *Ann. Inst. Fourier (Grenoble)* 45 (1995), 577-604.

[81] Franchi, B., Lu, G., Wheeden, R. L.: A relationship between Poincaré type inequalities and representation formulas in spaces of homogeneous type. *Internat. Math. Res. Notices* (1996), No. 1, 1–14.

[82] Franchi, B., Pérez, C., Wheeden R.L.: Self–improving properties of John–Nirenberg and Poincaré inequalities on spaces of homogeneous type, *J. Funct. Anal.* 153 (1998), 108–146.

[83] Franchi, B., Serapioni, R.: Pointwise estimates for a class of strongly degenerate elliptic operators: a geometric approach. *Ann. Scuola Norm. Sup. Pisa* 14 (1987), 527–568.

[84] Franchi, B., Serapioni R., Serra Cassano, F: Meyers–Serrin type theorems and relaxation of variational integrals depending on vector fields *Houston Math. J.* 22 (1996), 859–889.

[85] Franchi, B., Serapioni R., Serra Cassano, F: Approximation and imbedding theorems for weighted Sobolev spaces associated with Lipschitz continuous vector fields *Boll. Un. Mat. Ital.* (7) 11–B (1997), 83–117.

[86] Franchi, B., Wheeden, R.: Some remarks about Poincaré type inequalities and representation formulas in metric spaces of homogeneous type. *J. Inequalities and Appl.*, to appear.

[87] Friedrichs, K. O.: The identity of weak and strong extensions of differential operators. *Trans. Amer. Math. Soc.* 55 (1944), 132–151.

[88] Gallot, S.: Inégalités isopérimetriques, courbure de Ricci et invariants géométriques. I, II. *C. R. Acad. Sci. Paris Ser. I Math.* 296 (1983), 333–336, 365–368.

[89] Garattini, R.: Harnack inequality on homogeneous spaces, preprint.

[90] Garofalo, N., Lanconelli, E.: Existence and nonexistence results for semilinear equations on the Heisenberg group. *Indiana Univ. Math. J.* 41 (1992), 71–98.

[91] Garofalo, N., Nhieu, D. M.: Lipschitz continuity, global smooth approximation and extension theorems for Sobolev functions in Carnot–Carathéodory spaces. *J. Analyse Math.* 74 (1998), 67–97.

[92] Garofalo, N., Nhieu, D. M.: Isoperimetric and Sobolev inequalities for Carnot-Carathéodory spaces and the existence of minimal surfaces, *Comm. Pure Appl. Math.* 49 (1996), 1081–1144.

[93] Gatto, A. E., Vági, S.: On the exponential integrability of fractional integrals on spaces of homogeneous type. *Colloq. Math.* 64 (1993), 121–127.

[94] Gatto, A. E., Vági, S.: On functions arising as potentials on spaces of homogeneous type. *Proc. Amer. Math. Soc.* 125 (1997), 1149–1152.

[95] Genebahsvili, I., Gogatishvili, A., Kokilashvili, V., Krbec, M.: *Weighted theory for integral transforms on spaces of homogeneous type.* Pitman Monographs and Surveys in Pure and Applied Math. 92. Addison Wesley Longman Limited 1988.

[96] Gilbarg, D., Trudinger, N.: *Elliptic Partial Differential Equations of Second Order*, Springer–Verlag 1983.

[97] Goldshtein, V. M., Reshetnyak, Yu. G.: *Quasiconformal mappings and Sobolev spaces.* Mathematics and its Applications (Soviet Series), 54. Kluwer Academic Publishers Group, Dordrecht, 1990. (Translated and revised from the 1983 Russian original.)

[98] Gole, C., Karidi, R.: A note on Carnot geodesics in nilpotent Lie groups. *J. Dynam. Control Systems* 1 (1995), 535–549.

[99] Grigor'yan, A.: The heat equation on noncompact Riemannian manifolds. *Mat. Sb.* 182 (1991), 55–87 (in Russian).

[100] Gromov, M.: Groups of polynomial growth and expanding maps. *Publ. Math. I.H.E.S.* 53 (1981), 53–78.

[101] Gromov, M.: Carnot–Carathéodory spaces seen from within. In: [240], pp. 79–318.

[102] Hajłasz, P.: Sobolev spaces on an arbitrary metric space. *Potential Anal.* 5 (1996), 403–415.

[103] Hajłasz, P.: Geometric approach to Sobolev spaces and badly degenerated elliptic equations. *The Proceedings of Banach Center Minisemester: Nonlinear Analysis and Applications.* (N.Kenmochi, M.Niezgódka, P.Strzelecki, eds.) GAKUTO International Series; Mathematical Sciences and Applications, vol. 7 (1995), 141–168.

[104] Hajłasz, P., Kinnunen, J.: Hölder quasicontinuity of Sobolev functions on metric spaces. *Rev. Mat. Iberoamericana.* 14 (1998), 601–622.

[105] Hajłasz, P., Koskela, P.: Sobolev meets Poincaré. *C. R. Acad. Sci. Paris Ser. I Math.* 320 (1995), 1211–1215.

[106] Hajłasz, P., Koskela, P.: Isoperimetric inequalities and imbedding theorems in irregular domains. *J. London Math. Soc.* 58 (1998), 425–450.

[107] Hajłasz, P., Martio, O.: Traces of Sobolev functions on fractal type sets and characterization of extension domains. *J. Funct. Anal.* 143 (1997), 221–246.

[108] Hajłasz, P., Strzelecki, P.: Subelliptic p-harmonic maps into spheres and the ghost of Hardy spaces, *Math. Ann.* 312 (1998), 341–362.

[109] Han Y. S.: Triebel–Lizorkin spaces on spaces of homogeneous type, *Studia Math.* 108 (1994), 247–273.

[110] Han, Y. S. and Sawyer, E. T.: Littlewood–Paley Theory on Spaces of Homogeneous Type and Classical Function Spaces. *Mem. Amer. Math. Soc.* 110 (1994), no. 530, 1–126.

[111] Hanson, B. and Heinonen, J.: An n-dimensional space that admits a Poincaré inequality but has no manifold points. *Proc. Amer. Math. Soc.*, to appear.

[112] Hartman, P.: *Ordinary Differential Equations.* Birkhauser, 1982.

[113] Hebisch, W., Sikora, A.: A smooth subadditive homogeneous norm on a homogeneous group. *Studia Math.* 96 (1990), 321–236.

[114] Heinonen, J.: Calculus on Carnot groups. *Ber. Univ. Jyväskylä Math. Inst.* 68 (1995), 1–31.

[115] Heinonen, J.: Lectures on analysis on metric spaces, preprint.

[116] Heinonen, J., Holopainen, I.: Quasiregular maps on Carnot groups. *J. Geom. Anal.* 7 (1997), 109–148.

[117] Heinonen, J., Koskela, P.: Definitions of quasiconformality. *Inventiones Math.* 120 (1995), 61–79.

[118] Heinonen, J., Koskela, P.: Quasiconformal maps on metric spaces with controlled geometry. *Acta Math.* 181 (1998), 1–61.

[119] Heinonen, J., Koskela, P.: A note on Lipschitz functions, upper gradients and the Poincaré inequality. *New Zealand J. Math.* 28 (1999), 37–42.

[120] Heinonen, J., Kilpeläinen, T., Martio, O.: *Nonlinear Potential Theory of Degenerate Elliptic Equations*, Oxford Univ. Press, 1993.

[121] Herman, R.: *Differential Geometry and the Calculus of Variations*, Academic Press, 1968.

[122] Hille, E.: *Lectures on Ordinary Differential Equations.* Addison-Wesley Pub. Co. 1968.

[123] Holopainen, I.: Rough isometries and p-harmonic functions with finite Dirichlet integral. *Rev. Mat. Iberoamericana* 10 (1994), 143–176.

[124] Holopainen, I.: Volume growth, Green's functions, and parabolicity of ends. *Duke Math. J.* 97 (1999), 319–346.

[125] Holopainen, I., Rickman, S.: Ricci curvature, Harnack functions, and Picard type theorems for quasiregular mappings. *Analysis and Topology*, pp. 315–326. World Sci. Publishing, River Edge, NJ, 1998.

[126] Holopainen, I., Soardi, P.: p-harmonic functions on graphs and manifolds. *Manuscripta Math.* 94 (1997), 95–110.

[127] Holopainen, I., Soardi, P. M.: A strong Liouville theorem for p-harmonic functions on graphs. *Ann. Acad. Sci. Fenn. Math.* 22 (1997), 205–226.

[128] Hörmander, L.: Hypoelliptic second order differential equations, Acta Math. 119 (1967), 147–171.

[129] Hörmander, L., Melin, A.: Free systems of vector fields. *Ark. Mat.* 16 (1978), 83–88.

[130] Hurri, R.: Poincare domains in $\mathbb{R}^n$. *Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes* (1988), No. 71, 42 pp.

[131] Hurri-Syrjänen, R.: A weighted Poincaré inequality with a doubling weight. *Proc. Amer. Math. Soc.* 126 (1998), 545–552.

[132] Iwaniec, T., Nolder, C.: Hardy–Littlewood inequality for quasiregular mappings in certain domains in $\mathbb{R}^n$. *Ann. Acad. Sci. Fenn. Ser. A I Math.* 10 (1985), 267–282.

[133] Jerison, D.: The Poincaré inequality for vector fields satisfying Hörmander's condition. *Duke Math. J.* 53 (1986), 503–523.

[134] Jerison, D., Lee, J. M.: Intrinsic CR coordinates and the CR Yamabe problem. *J. Diff. Geom.* 29 (1989), 303–343.

[135] Jerison, D., Lee, J. M.: Extremals for the Sobolev inequality on the Heisenberg group and the CR Yambe problem. *J. Amer. Math. Soc.* 1 (1988), 1–13.

[136] Jerison, D., Sanchez-Calle, A.: Subelliptic, second order differential operators. In:*Complex analysis, III (College Park, Md., 1985–86).* pp. 46–77, Lecture Notes in Math., 1277, Springer, 1987.

[137] John, F.: Rotation and strain. *Comm. Pure Appl. Math.* 4 (1961), 391–414.

[138] Jonsson, A.: Besov spaces on closed subsets of $\mathbb{R}^n$. *Trans. Amer. Math. Soc.* 341 (1994), 355–370.

[139] Jonsson, A.: Brownian motion on fractals and function spaces. *Math. Z.* 222 (1996), 495–504.

[140] Jonsson, A., Wallin, H.: *Function Spaces on Subsets of* $\mathbb{R}^n$, Harwood Acad. Publ., 1984.

[141] Jonsson, A., Wallin, H.: Boundary value problems and Brownian motion on fractals. *Chaos Solitons Fractals* 8 (1997), 191–205.

[142] Jost, J.: Equilibrium maps between metric spaces. *Calc. Var.* 2 (1994), 173–205.

[143] Jost, J.: Generalized harmonic maps between metric spaces, in: *Geometric analysis and the calculus of variations* (J. Jost, ed.), International Press, 1966, pp. 143–174.

[144] Jost, J.: Generalized Dirichlet forms and harmonic maps. *Calc. Var.* 5 (1997), 1–19.

[145] Jost, J.: Nonlinear Dirichlet forms. In: *New directions in Dirichlet forms.* pp. 1–47. AMS/PI Stud. Adv. Math. 8, Amer. Math. Soc., Providence, RI, 1998.

[146] Jost, Xu, C. J.: Subelliptic harmonic maps, *Trans. Amer. Math. Soc.* 350 (1998), 4633–4649.

[147] Kałamajska, A.: On compactness of embedding for Sobolev spaces defined on metric spaces. *Ann. Acad. Sci. Ser. A. I Math.* 24 (1999), 123–132.

[148] Kallunki, S., Shanmugalingam, N.: Modulus and continuous capacity, preprint.

[149] Kanai, M.: Rough isometries and the parabolicity of Riemannian manifolds. *J. Math. Soc. Japan* 38 (1986), 227–238.

[150] Karidi, R.: Geometry of balls in nilpotent Lie groups. *Duke Math. J.* 74 (1994), 301–317.

[151] Kigami, J.: Harmonic calculus on p.c.f. self-similar sets. *Trans. Amer. Math. Soc.* 335 (1993), 721–755.

[152] Kigami, J.: Hausdorff dimension of self-similar sets and shortest path metrics. *J. Math. Soc. Japan* 47 (1995), 381–404.

[153] Kigami, J.: Harmonic metric and Dirichlet form on the Sierpinski gasket. In: *Asymptotic problems in probability theory: stochastic models and diffusions on fractals (Sanda/Kyoto, 1990)*, pp. 201–218, Pitman Res. Notes Math. Ser., 283, Longman Sci. Tech., Harlow, 1993.

[154] Kigami, J., Lapidus, M. L.: Weyl's problem for the spectral distribution of Laplacian on p.c.f. self-similar fractals. *Comm. Math. Phys.* 158 (1993), 93–125.

[155] Kilpeläinen, T.: Smooth approximation in weighted Sobolev spaces. *Comment. Math. Univ. Carolinae* 38 (1995), 1–8.

[156] Kilpeläinen, T., Kinnunen, J., Martio.: Sobolev spaces with zero boundary values on metric spaces. *Potential Anal.*, to appear.

[157] Kinnunen, J., Martio, O.: The Sobolev capacity on metric spaces. *Ann. Acad. Sci. Fenn. Math.* 21 (1996), 367–382.

[158] Kohn, R. V.: New integral estimates for deformations in terms of their nonlinear strains. *Arch. Rational Mech. Anal.* 78 (1982), 131–172.

[159] Korányi, A, Reimann, H. M.: Foundations of the theory of quasiconformal mappings on the Heisenberg group. *Adv. Math.* 111 (1995), 1–87.

[160] Koskela, P.: Removable sets for Sobolev spaces *Ark. Mat.* 37 (1999), 291–304.

[161] Koskela, P. MacManus, P.: Sobolev classes and quasisymmetric maps. *Studia Math.* 131 (1998), 1–17.

[162] Kozlov, S. M.: Harmonization and homogenization on fractals. *Comm. Math. Phys.* 153 (1993), 339–357.

[163] Kusuoka, S., Stroock, D. W.: Applications of the Malliavin calculus. III. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* 34 (1987), 391–442.

[164] Laakso, T.: Ahlfors Q-regular spaces with arbitrary Q admitting weak Poincaré inequality. *Geom. Funct. Anal.*, to appear.

[165] Lanconelli, E.: Stime sub-ellittiche e metriche riemanniane singolari. *Seminario di Analysi Matematica.* Istituto Matematico dell'Universita di Bologna. A.A. 1982-83.

[166] Lanconelli, E., Morbidelli, D.: On the Poincaré inequality for vector fields, *Ark. Mat.*, to appear.

[167] Lapidus, M. L.: Analysis on fractals, Laplacians on self-similar sets, noncommutative geometry and spectral dimensions. *Topol. Methods Nonlinear Anal.* 4 (1994), 137–195.

[168] Lapidus, M. L.: Fractals and vibrations: can you hear the shape of a fractal drum? In: *Symposium in Honor of Benoit Mandelbrot (Curacao, 1995). Fractals* 3 (1995), 725–736.

[169] Lawler, G. F.: Estimates for differences and Harnack inequality for difference operators coming from random walks with symmetric, spatially inhomogeneous, increments. *Proc. London Math. Soc.* 63 (1991), 552–568.

[170] Li, P., Schoen, R.: L^p and mean value properties of subharmonic functions on Riemannian manifolds. *Acta Math.* 153 (1984), 279–301.

[171] Li, P., Wang, J.: Convex hull properties of harmonic maps. *J. Diff. Geom.* 48 (1988), 487–530.

[172] Li, P., Yau, S. T.: Estimates of eigenvalues of a compact Riemannian manifold. In: *Geometry of the Laplace operator (Proc. Sympos. Pure Math., Univ. Hawaii, Honolulu, Hawaii, 1979)*, pp. 205–239, Proc. Sympos. Pure Math., 36, Amer. Math. Soc., Providence, R.I., 1980.

[173] Long, R. L., Nie, F. S.: Weighted Sobolev inequality and eigenvalue estimates of Schrödinger operators. *In: Harmonic Analysis.* (Tianjin, 1988). Lecture Notes in Math. 1494, Springer, 1991.

[174] Liu, S., Sussman, H.: Shortest paths for sub-Riemannian metrics on rank-two distributions. *Mem. Amer. Math. Soc.* 118 (1995), no. 564, 1–104.

[175] Lu, G.: Weighted Poincaré and Sobolev inequalities for vector fields satisfying Hörmander's condition and applications. *Rev. Mat. Iberoamericana* 8 (1992), 367–439.

[176] Lu, G.: Existence and size estimates for the Green's functions of differential operators constructed from degenerate vector fields. *Comm. Partial Differential Equations* 17 (1992), 1213–1251.

[177] Lu, G.: On Harnack's inequality for a class of strongly degenerate Schrödinger operators formed by vector fields. *J. Diff. Int. Equations* 7 (1994), 73–100.

[178] Lu, G.: The sharp Poincaré inequality for free vector fields: An endpoint result. *Rev. Mat. Iberoamericana* 10 (1994), 453–466.

[179] Lu, G.: Embedding theorems on Campanato-Morrey spaces for vector fields and applications. *C. R. Acad. Sci. Paris Ser. I Math.* 320 (1995), 429–434.

[180] Lu, G.: Embedding theorems into Lipschitz and BMO spaces and applications to quasilinear subelliptic differential equations. *Publ. Mat.* 40 (1996), 301–329.

[181] Lu, G.: A note on a Poincaré type inequality for solutions to subelliptic equations. *Comm. Partial Differential Equations* 21 (1996), 235–254.

[182] Luukkainen, J., Saksman, E.: Every complete doubling metric space carries a doubling measure. *Proc. Amer. Math. Soc.* 126 (1998), 531–534.

[183] Macias, R. A., Segovia, C.: Lipschitz functions on spaces of homogeneous type. *Adv. Math.* 33 (1979), 257–270.

[184] MacManus, P., Pérez, C.: Generalized Poincaré inequalities: sharp self-improving properties. *Internat. Math. Res. Notices* 1998, No. 2, 101–116.

[185] MacManus, P., Pérez, C.: Trudinger inequalities without derivatives. *Trans. Amer. Math. Soc.*, to appear.

[186] Maheux, P., Saloff-Coste, L.: Analyse sur les boules d'un opérateur sous-elliptique. *Math. Ann.* 303 (1995), 713–740.

[187] Malý, J., Ziemer, W.: *Fine Regularity Properties of Solutions of Elliptic Partial Differential Equations.* Mathematical Surveys and Monographs, 51. Americal Mathematical Society, 1997.

[188] Manfredini, M.: Compact embedding theorems for generalized Sobolev spaces. *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl.* 4 (1993), 251–263.

[189] Marchi, S.: Hölder continuity and Harnack inequality for De Giorgi classes related to Hörmander vector fields. *Ann. Mat. Pura Appl.* 168 (1995), 171–188.

[190] Margulis, G. A., Mostow, G. D.: The differential of a quasiconformal mapping of a Carnot–Carathéodory space. *Geom. Funct. Anal.* 5 (1995), 402–433.

[191] Martio, O.: John domains, bilipschitz balls and Poincaré inequality. *Rev. Roumaine Math. Pures Appl.* 33 (1988), 107–112.

[192] Martio, O., Sarvas, J.: Injectivity theorems in plane and space. *Ann. Acad. Sci. Fenn., Ser. A I Math.* 4 (1978/1979) 383–401.

[193] Mattila, P.: *Geometry of Sets and Measures in Euclidean Spaces.* Cambridge Studies in Advanced Mathematics, 44. Cambridge Univ. Press, 1995.

[194] Maz'ya, V. G.: A theorem on the multidimensional Schrödinger operator. *Izv. Akad. Nauk.* 28 (1964), 1145–1172 (in Russian).

[195] Maz'ya, V. G.: *Sobolev spaces* (Springer 1985).

[196] Merkov, A. B.: Second order elliptic equations on graphs. *Math. USSR-Sb.* 55 (1986), 493–509.

[197] Metz, V., Sturm, K. T.: Gaussian and non-Gaussian estimates for heat kernels on the Sierpinski gasket. In: *Dirichlet forms and stochastic processes (Beijing, 1993)*, pp. 283–289, de Gruyter, Berlin, 1995.

[198] Milnor, J.: A note on curvature and fundamental group. *J. Diff. Geom.* 2 (1968), 1–7.

[199] Mitchell, J.: On Carnot–Carathéodory metrics. *J. Diff. Geom.* 21 (1985), 35–45.

[200] Morbidelli, D.: Fractional Sobolev norms and structure of Carnot–Carathéodory balls for Hörmander vector fields, *Studia Math.* to appear.

[201] Mosco, U.: Variational metrics on self-similar fractals. *C. R. Acad. Sci. Paris Ser. I Math.* 321 (1995), 715–720.

[202] Moser, J.: On Harnack's theorem for elliptic differential equations. *Comm. Pure Appl. Math.* 14 (1961), 577–591.

[203] Mostow, G.: Quasiconformal mappings in n-space and the rigidity of hyperbolic space forms. *Publ. Math. I.H.E.S.* 34 (1968), 53–104.

[204] Nagel, A., Stein, E. M. and Wainger, S.: Balls and metrics defined by vector fields I: Basic properties, *Acta Math.* 55 (1985), 103–147.

[205] Näkki, R., Väisälä, J.: John disks, *Exposition. Math.* 9 (1991), 3–43.

[206] Nhieu, D. M.: Extensions of Sobolev spaces on the Heisenberg group. *C. R. Acad. Sci. Paris Ser. I Math.* 321 (1995), 1559–1564.

[207] Nhieu, D. M.: An extension theorem for the Folland–Stein Sobolev spaces on stratified nilpotent Lie groups, preprint.

[208] Oleinik, O. A., Radkevic, E. V.: *Second order equations with nonnegative characteristic form.* Plenum Press, New York-London, 1973.

[209] Pansu, P.: Une inégalité isopérimétrique sur le groupe d'Heisenberg. *C. R. Acad. Sci. Paris Ser. I Math.* 295 (1982), 127–131.

[210] Pansu, P.: Métriques de Carnot–Carathéodory et quasiisométries des espaces symétriques de rans un. *Ann. of Math.* 129 (1989), 1–60.

[211] Peetre, J.: A remark on Sobolev spaces. The case $0 < p < 1$. *J. Approx. Theory* 13 (1975), 218–228.

[212] Rao, M. M., Ren, Z. D.: *Theory of Orlicz Spaces.* Pure and Applied Math. 146. Marcel Dekker, Inc. 1991.

[213] Rashevsky, P. K.: Any two points of a totally nonholonomic space may be connected by an admissible line. *Uch. Zap. Ped. Inst. im. Liebknechta, Ser. Phys. Math.* 2 (1938), 83–94 (in Russian).

[214] Renardy, M., Rogers, R.: *An Introduction To Partial Differential Equations.* Texts in Applied Mathematics, 13. Springer, 1993.

[215] Rigoli, M., Salvatori, M., Vignati, M.: A note on p-subharmonic functions on complete manifolds. *Manuscripta Math.* 92 (1997), 339–359.

[216] Rigoli, M., Salvatori, M., Vignati, M.: Liouville properties on graphs. *Mathematika* 44 (1997), 133–148.

[217] Rigoli, M., Salvatori, M., Vignati, M.: Subharmonic functions on graphs. *Israel J. Math.* 99 (1997), 1–27.

[218] Rothschild, L. P., Stein, E. M.: Hypoelliptic differential operators and nilpotent groups. *Acta Math.* 137 (1976), 247–320.

[219] Rudin, W.: *Real and Complex Analysis*, Third edition, Mc Graw–Hill, New York 1987.

[220] Saloff-Coste, L.: Uniformly elliptic operators on Riemannian manifolds. *J. Diff. Geom.* 36 (1992), 417–450.

[221] Saloff-Coste, L.: A note on Poincaré, Sobolev, and Harnack inequalities. *Internat. Math. Res. Notices* (1992) no. 2, pp. 27–38.

[222] Saloff-Coste, L.: On global Sobolev inequalities. *Forum Math.* 6 (1994), 271–286.

[223] Saloff-Coste, L.: Parabolic Harnack inequality for divergence form second order differential operators. *Potential Anal.* 4 (1995), 429–467.

[224] Sánchez-Calle, A.: Fundamental solutions and geometry of sums of squares of vector fields. *Invent. Math.* 78 (1984), 143–160.

[225] Schinzel, A.: An analogue of Harnack's inequality for discrete superharmonic functions. *Demonstratio Math.* 11 (1978), 47–60.

[226] Semmes, S.: Finding curves on general spaces through quantitative topology, with applications to Sobolev and Poincaré inequalities. *Selecta Math. (N.S.)* 2 (1996), 155–295.

[227] Semmes, S.: Some remarks about metric spaces, spherical mappings and their derivatives. *Publ. Mat.* 40 (1996), 411–430.

[228] Shanmugalingam, N.: Newtonian spaces; a generalization of Sobolev spaces. *Rev. Mat. Iberoamericana*, to appear.

[229] Simon, L.: *Lectures on Geometric Measure Theory*, Proc. of Centre for Math. Anal. Austral. Nat. Univ. Vol. 3, 1983.

[230] Soardi, P. M.: Rough isometries and Dirichlet finite harmonic functions on graphs. *Proc. Amer. Math. Soc.* 119 (1993), 1239–1248.

[231] Smith, W., Stegenga, D. A.: Hölder domains and Poincaré domains. *Trans. Amer. Math. Soc.* 319 (1990), 67–100.

[232] Smith, W., Stegenga, D. A.: Sobolev imbeddings and integrability of harmonic functions in Hölder domains. *in: Potential theory* (de Gruyter, Berlin 1992).

[233] Stein, E.: *Singular integrals and differentiability properties of functions* (Princeton Univ. Press 1970).

[234] Stein, E.: *Harmonic Analysis: Real-Variable Methods, Orthogonality and Oscillatory Integrals* (Princeton Univ. Press 1993).

[235] Strichartz, R. S.: Sub-Riemannian geometry, *J. Diff. Geom.* 24 (1986), 221–263. Corrections: *J. Diff. Geom.* 30 (1989), 595–596.

[236] Sturm, T. K.: Analysis on local Dirichlet spaces. I. Recurrence, conservativeness and L^p-Liouville properties. *J. Reine Angew. Math.* 456 (1994), 173–196.

[237] Sturm, T. K.: Analysis on local Dirichlet spaces. II. Upper Gaussian estimates for the fundamental solutions of parabolic equations. *Osaka J. Math.* 32 (1995), 275–312.

[238] Sturm, K, T.: On the geometry defined by Dirichlet forms. In: *Seminar on Stochastic Analysis, Random Fields and Applications (Ascona, 1993)*, pp. 231–242 Progr. Probab., 36, Birkhauser, Basel, 1995.

[239] Sturm, K. T.: Analysis on local Dirichlet spaces III. The parabolic Harnack inequality. *J. Math. Pures Appl.* 75 (1996), 273–297.

[240] *Sub-Riemannian geometry.* Progress in Mathematics, 144. Edited by Andre Bellaiche and Jean-Jacques Risler. Birkhauser Verlag, Basel, 1996.

[241] Trudinger, N.: On imbedding into Orlicz spaces and some applications. *J. Math. Mech.* 17 (1967), 473–483.

[242] Tam, L. F.: Liouville properties of harmonic maps. *Math. Res. Lett.* 2 (1995), 719–735.

[243] Tartar, L.: Imbedding theorems in Lorentz spaces and applications. *Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8)* 1 (1998), 479–500.

[244] Triebel, H.: *Fractals and spectra.* Monographs in Mathematics 91, Birkhauser 1997.

[245] Tyson, J.: Quasiconformality and quasisymmetry in metric measure spaces. *Ann. Acad. Sci. Fenn. Math.* 23 (1998), 525–548.

[246] Väisälä, J.: *Lectures on n-dimensional quasiconformal mappings*, Lecture Notes in Math. 299, Springer 1971.

[247] Väisälä, J.: Uniform domains. *Tohoku Math. J.* 40 (1988), 101–118.

[248] Varopoulos, T.: Brownian motion and random walks on manifolds. *Ann. Inst. Fourier (Grenoble)* 34 (1984), 243–269.

[249] Varopoulos, T.: Fonctions harmoniques sur les groupes Lie. *C. R. Acad. Sci. Paris Ser. I Math.* 309 (1987), 519–521.

[250] Varopoulos, T.: Distances associees aux operateurs sous-elliptiques du second ordre. *C. R. Acad. Sci. Paris Ser. I Math.* 309 (1989), 663–667.

[251] Varopoulos, T., Saloff-Coste, L., Coulhon, T.: *Analysis and geometry on groups* (Cambridge Univ. Press 1992).

[252] Vodop'yanov, S. K.: Monotone functions on Carnot groups and their applications. *Sib. Math. Zh.* 37 (1996), 1269–1295 (in Russian); English transl. *Sib. Math. J.* 37 (1996), 1113–1136.

[253] Vodop'yanov, S. K., Greshnov, A. V.: Analytic properties of quasiconformal mappings on Carnot groups. *Sib. Math. Zh.* 36 (1995), 1317–1327 (in Russian); English transl. *Siberian Math. J.* 36 (1995), 1142–1151.

[254] Vodop'yanov, S. K., Markina, I. G.: Exceptional sets for solutions of subelliptic equations. *Sibirsk. Mat. Zh.* 36 (1995), 805–818 (in Russian); English trasl. *Siberian Math. J.* 36 (1995), 694–706.

[255] Volberg, A. L., Konyagin, S. V.: A homogeneous measure exists on any compactum in $\mathbb{R}^n$. *Dokl. Akad. Nauk SSSR* 278 (1984), 783–785 (in Russian); English transl. Soviet Math. Dokl. 30 (1984), 453–456.

[256] Volberg, A. L., Konyagin, S. V.: On measures with the doubling condition. *Izv. Acad. Nauk SSSR* 51 (1987), 666–675 (in Russian); English transl. Math. USSR Izv. 30 (1988), 629–638.

[257] Wu, J. M.: Hausdorff dimension and doubling measures on metric spaces. *Proc. Amer. Math. Soc.* 126 (1998), 1453–1459.

[258] Xu, C. J.: Subelliptic variational problems. *Bull. Soc. Math. France* 118 (1990), 147–159.

[259] Xu, C. J.: Regularity for quasilinear second-order subelliptic equations. *Comm. Pure Appl. Math.* 45 (1992), 77–96.

[260] Xu, C. J., Zuily, C.: Higher interior regularity for quasilinear subelliptic systems. *Calc. Var.* 5 (1997), 323–343.

[261] Yau, S. T.: Harmonic functions on complete Riemannian manifolds. *Comm. Pure Appl. Math* 28 (1975), 201–228.

[262] Zhou, X. Y.: Green function estimates and their applications to the intersections of symmetric random walks. *Stochastic Process. Appl.* 48 (1993), 31–60.

[263] Ziemer, W.: *Weakly differentiable functions*, Graduate Texts in Mathematics 120 (Springer 1989).

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