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ON APPROXIMATE DIFFERENTIABILITY OF THE MAXIMAL FUNCTION

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ABSTRACT. We prove that if $f \in L^1(\mathbb{R}^n)$ is approximately differentiable a.e., then the Hardy-Littlewood maximal function $\mathcal{M}f$ is also approximately differentiable a.e. Moreover, if we only assume that $f \in L^1(\mathbb{R}^n)$, then any open set of $\mathbb{R}^n$ contains a subset of positive measure such that $\mathcal{M}f$ is approximately differentiable on that set. On the other hand we present an example of $f \in L^1(\mathbb{R})$ such that $\mathcal{M}f$ is not approximately differentiable a.e.

1. Introduction

Juha Kinnunen [10] proved that the Hardy-Littlewood maximal function

$$\mathcal{M}f(x) = \sup_{r>0} |B(x,r)|^{-1} \int_{B(x,r)} |f(y)|\, dy$$

is a bounded operator in the Sobolev space $W^{1,p}(\mathbb{R}^n)$, $1 < p < \infty$. Recall that $W^{1,p}(\mathbb{R}^n)$ is the space of all functions $f \in L^p(\mathbb{R}^n)$ such that weak (distributional) partial derivatives $\partial f/\partial x_i$ also belong to $L^p(\mathbb{R}^n)$, and similarly for $W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$. Since the maximal function is not bounded in L^1, there is no apparent reason to expect any kind of boundedness of the maximal function in $W^{1,1}(\mathbb{R}^n)$. However, Tanaka [25] proved that in the one dimensional case the noncentered maximal function of $f \in W^{1,1}(\mathbb{R})$ belongs locally to $W^{1,1}(\mathbb{R})$. Since that time it has been an open problem to extend Tanaka's result to the case of the Hardy-Littlewood maximal function and to find analogous results in the higher dimensional case; cf. [8, Question 1]. To the best of our knowledge there are no known higher dimensional results in the case $p = 1$, and even in the one dimensional case it is still not known whether the Hardy-Littlewood maximal function (i.e. the centered one) of $f \in W^{1,1}(\mathbb{R})$ belongs locally to $W^{1,1}(\mathbb{R})$; see, however, [2], [3]. The results proved in the paper are clearly motivated by this challenging problem.

Theorem 1. *If $f \in L^1(\mathbb{R}^n)$ is approximately differentiable a.e., then the maximal function $\mathcal{M}f$ is approximately differentiable a.e.*

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Since every function $f \in W^{1,1}(\mathbb{R}^n)$ is approximately differentiable a.e., the result implies a.e. approximate differentiability of $\mathcal{M}f$. This, in particular, implies (see Lemma 5) that $\mathcal{M}f$ coincides with a C^1 function off an open set of arbitrarily small measure. However, a.e. approximate differentiability of $\mathcal{M}f$ is much less than weak differentiability of $\mathcal{M}f$, which is still an open problem. On the other hand, the assumption about f in the theorem is much weaker than $f \in W^{1,1}$. In addition to this result, Theorem 2 provides a formula for the approximate derivative of $\mathcal{M}f$ when $f \in W^{1,1}$.

Let $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$. It is easy to see (cf. [20]) that for a.e. $x \in \mathbb{R}^n$, either

$$\mathcal{M}f(x) = \fint_{B(x,r_x)} |f(y)|\, dy \quad \text{for some } r_x > 0 \tag{1}$$

or

$$\mathcal{M}f(x) = |f(x)|\,. \tag{2}$$

Denote by E and P the sets of points in $\mathbb{R}^n$ for which (1) and respectively (2) is satisfied. The following result is due to Luiro [20] when $p > 1$ and is new when $p = 1$. Our proof is new and simpler even in the case $p > 1$.

Theorem 2. *Let $f \in W^{1,p}(\mathbb{R}^n)$, $1 \leq p < \infty$. Then the weak derivative, when $p > 1$, and the approximate derivative, when $p = 1$, of the maximal function $\mathcal{M}f$ satisfy*

$$\begin{aligned} \nabla\mathcal{M}f(x) &= \fint_{B(x,r_x)} \nabla|f(y)|\, dy \quad \textit{for a.e. } x \in E, \\ \nabla\mathcal{M}f(x) &= \nabla|f(x)| \quad \textit{for a.e. } x \in P. \end{aligned} \tag{3}$$

Remark 3. If $x \in E$, then $r_x > 0$ is not necessarily uniquely defined and (3) holds for all such r_x.

In the next result we deal with differentiability properties of $\mathcal{M}f$ for any $f \in L^1(\mathbb{R}^n)$.

Theorem 4. *If $f \in L^1(\mathbb{R}^n)$, then any open set $\Omega \subset \mathbb{R}^n$ contains a subset $E \subset \Omega$ of positive Lebesgue measure such that $\mathcal{M}f$ is approximately differentiable a.e. in E.*

Again, Lemma 5 implies that for any open set $\Omega \subset \mathbb{R}^n$ there is a function $g \in C^1(\mathbb{R}^n)$ such that the set $\{x \in \Omega : f(x) = g(x)\}$ has positive measure.

In view of Theorem 4 it is natural to inquire whether for every $f \in L^1(\mathbb{R}^n)$ the maximal function $\mathcal{M}f$ is approximately differentiable a.e. Unfortunately the answer is in the negative, as an example presented at the end of the paper shows.

While the proofs of Theorems 1 and 2 are completely elementary, the proof of Theorem 4 requires some advanced potential theory.

Let us also mention that the result of Kinnunen [10] has been applied and generalized by many authors ([2], [3], [6], [7], [8], [11], [12], [14], [15], [16], [17], [18], [20], [21], [25]).

The notation used in the paper is pretty standard. The volume of the unit ball in $\mathbb{R}^n$ is denoted by ω_n, and we use a barred integral to denote the integral average

$$\fint_{B(x,r)} f(y)\, dy = \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)\, dy\,.$$

By C we will denote a generic positive constant whose actual value may change even in a single string of estimates.

2. Approximate differentiability

Let f be a real-valued function defined on a set $E \subset \mathbb{R}^n$. We say that f is *approximately differentiable* at $x_0 \in E$ if there is a vector $L = (L_1, \dots, L_n)$ such that for any $\varepsilon > 0$ the set

$$A_\varepsilon = \left\{ x : \frac{|f(x) - f(x_0) - L(x - x_0)|}{|x - x_0|} < \varepsilon \right\}$$

has x_0 as a density point. If this is the case, then x_0 is a density point of E and L is uniquely determined. The vector L is called the *approximate differential* of f at x_0 and is denoted by $\nabla f(x_0)$.

In what follows we will need the following theorem of Whitney [26], which provides several characterizations of a.e. approximate differentiability of a function. We state it as a lemma.

Lemma 5. *Let $f : E \to \mathbb{R}$ be measurable, $E \subset \mathbb{R}^n$. Then the following conditions are equivalent.*

(a) *f is approximately differentiable a.e.*
(b) *For any $\varepsilon > 0$ there is a closed set $F \subset E$ and a locally Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ such that $f|_F = g|_F$ and $|E \setminus F| < \varepsilon$.*
(c) *For any $\varepsilon > 0$ there is a closed set $F \subset E$ and a function $g \in C^1(\mathbb{R}^n)$ such that $f|_F = g|_F$ and $|E \setminus F| < \varepsilon$.*

Remark 6. To illustrate the relevance of the maximal function in this part of the real analysis, let us mention a useful Lipschitz type estimate valid for Sobolev functions:

$$|f(x) - f(y)| \leq C|x - y|(\mathcal{M}|\nabla f|(x) + \mathcal{M}|\nabla f|(y)) \quad \text{a.e.}; \tag{4}$$

see [1], [4], [5]. As an almost immediate consequence of (4) one obtains a well known result that each $f \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$ is approximately differentiable a.e.; cf. [22].

Investigating the positive and negative parts of a function separately, one can easily prove

Lemma 7. *A measurable function $f : E \to \mathbb{R}$ is a.e. approximately differentiable if and only if $|f|$ is a.e. approximately differentiable.*

3. Proof of Theorem 1

We consider a restricted version of the maximal function

$$\mathcal{M}^\varepsilon f(x) = \sup_{r \geq \varepsilon} ⨍_{B(x,r)} |f(y)|\, dy\,.$$

Lemma 8. *If $f \in L^1(\mathbb{R}^n)$, then*

$$|\mathcal{M}^\varepsilon f(x) - \mathcal{M}^\varepsilon f(y)| \leq \frac{n}{\varepsilon}\, |x - y|\, (\mathcal{M}^\varepsilon f(x) + \mathcal{M}^\varepsilon f(y)) \leq \frac{2n}{\omega_n \varepsilon^{n+1}}\, \|f\|_1\, |x - y|\,.$$

Proof. The second inequality of the lemma is obvious because

$$\mathcal{M}^\varepsilon f(x) \leq \frac{1}{\omega_n \varepsilon^n} \int_{\mathbb{R}^n} |f(y)|\, dy = \frac{1}{\omega_n \varepsilon^n} \|f\|_1 .$$

Thus we are left with the proof of the first inequality. For $a, r > 0$ the function $\varphi(r) = r/(r+a)$ is increasing, and hence, applying Bernoulli's inequality, we have for $r \geq \varepsilon$,

$$\left(\frac{r}{r+|x-y|}\right)^n \geq \left(\frac{\varepsilon}{\varepsilon+|x-y|}\right)^n \geq 1 - n\,\frac{|x-y|/\varepsilon}{1+|x-y|/\varepsilon} \geq 1 - \frac{n}{\varepsilon}\,|x-y| .$$

Fix $x, y \in \mathbb{R}^n$. Then for any $r \geq \varepsilon$ we have $B(y,r) \subset B(x, r+|x-y|)$ and hence

$$\mathcal{M}^\varepsilon f(x) \geq \left(\frac{r}{r+|x-y|}\right)^n \fint_{B(y,r)} |f| \geq \Big(1 - \frac{n}{\varepsilon}|x-y|\Big) \fint_{B(y,r)} |f| .$$

Passing to the supremum over $r \geq \varepsilon$ we obtain

$$\mathcal{M}^\varepsilon f(x) \geq \big(1 - \frac{n}{\varepsilon}\,|x-y|\big)\, \mathcal{M}^\varepsilon f(y) .$$

Since the inequality is also true if we replace x by y and y by x, one easily concludes the first inequality from the lemma. □

Lemma 9. *If $f \in L^1(\mathbb{R}^n)$, then*

$$\{x : \mathcal{M}f(x) > |f(x)|\} = Z \cup \bigcup_{k=1}^\infty E_k,$$

where $|Z| = 0$ and $\mathcal{M}f|_{E_k}$ is Lipschitz continuous for $k = 1, 2, \dots$. In particular $\mathcal{M}f$ is a.e. approximately diferentiable in the set $\{x : \mathcal{M}f(x) > |f(x)|\}$.

Proof. Let Z be the set of points that are not Lebesgue points of $|f|$. Clearly $|Z| = 0$. Assume that $x \in \mathbb{R}^n \setminus Z$ and $\mathcal{M}f(x) > |f(x)|$. Let $r_i > 0$ be a sequence such that

$$\fint_{B(x_i, r_i)} |f| \to \mathcal{M}f(x) .$$

The sequence r_i is bounded (because $\mathcal{M}f(x) > 0$ and $f \in L^1$), and hence we can select a subsequence (still denoted by r_i) such that $r_i \to r$. Clearly $r > 0$ as otherwise we would have $\mathcal{M}f(x) = |f(x)|$. Thus

$$\mathcal{M}f(x) = \fint_{B(x,r)} |f| \quad \text{for some } r > 0.$$

This easily implies that

$$\{x : \mathcal{M}f(x) > |f(x)|\} \subset Z \cup \bigcup_{k=1}^\infty \{x : \mathcal{M}f(x) = \mathcal{M}^{1/k} f(x)\} .$$

Since the function $\mathcal{M}^{1/k} f$ is Lipschitz continuous by Lemma 8, the first part of the result follows. The second part is a direct consequence of Lemma 5. □

Now we can complete the proof of Theorem 1. Let $f \in L^1(\mathbb{R}^n)$ be approximately differentiable a.e. Then also $|f|$ is approximately differentiable a.e. (Lemma 7). According to Lemma 9,

$$\mathbb{R}^n = \{x : \mathcal{M}f(x) = |f(x)|\} \cup Z \cup \bigcup_{k=1}^\infty E_k,$$

where $|Z| = 0$ and $\mathcal{M}f|_{E_k}$ is Lipschitz continuous. Since $\mathcal{M}f|_{E_k}$ is approximately differentiable a.e. and $\mathcal{M}f = |f|$ is approximately differentiable a.e. in the set $\{x : \mathcal{M}f(x) = |f(x)|\}$, the theorem follows. $\square$

4. Proof of Theorem 2

Since $\mathcal{M}f(x) = |f(x)|$ in P, clearly $\nabla\mathcal{M}f(x) = \nabla|f(x)|$ a.e. in P. Thus let $x \in E$ and $r_x > 0$ be such that equality (1) holds. Assume also that $\mathcal{M}f$ is approximately differentiable at x. Note that the function

$$\varphi(y) = \mathcal{M}f(y) - \fint_{B(y,r_x)} |f(z)|\,dz = \mathcal{M}f(y) - \fint_{B(0,r_x)} |f(y+z)|\,dz$$

is approximately differentiable at x and

$$\begin{aligned} \nabla\varphi(x) &= \nabla\mathcal{M}f(x) - \fint_{B(0,r_x)} \nabla|f|(x+z)\,dz \\ &= \nabla\mathcal{M}f(x) - \fint_{B(x,r_x)} \nabla|f(z)|\,dz. \end{aligned}$$

Indeed, $\mathcal{M}f$ is approximately differentiable at x, and since $f \in W^{1,p}$ we can differentiate in the second term under the sign of the integral. Note also that $\varphi \geq 0$ and $\varphi(x) = 0$, so φ attains a minimum at x, and hence its approximate derivative at x must be equal to 0, which is the claim we wanted to prove. $\square$

5. Proof of Theorem 4

This proof requires some results from potential theory. We say that a locally integrable function $u : \Omega \to [0,\infty]$ defined on an open set $\Omega \subset \mathbb{R}^n$ is *superharmonic* if it is lower semicontinuous and

$$u(x) \geq \fint_{B(x,r)} u(y)\,dy \tag{5}$$

whenever $B(x,r) \Subset \Omega$.

The following regularity result has been established in the setting of weak solutions of the p-Laplace equation in [19]. For the convenience of the reader, we include a short proof based only on a knowledge of classical potential theory.

Lemma 10. *If a locally integrable function $u : \Omega \to [0,\infty]$, $\Omega \subset \mathbb{R}^n$, is superharmonic, then $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ for all $1 \leq p < n/(n-1)$. In particular u is a.e. approximately differentiable.*

Proof. Let $u : \Omega \to [0,\infty]$ be superharmonic and let $U \Subset \Omega$. According to the Riesz decomposition theorem [23], [9], u restricted to U can be represented as

$$u(x) = h(x) - \int_{\mathbb{R}^n} \Phi(x-y)\,d\mu(y), \quad \text{for } x \in U, \tag{6}$$

where h is harmonic, Φ is the fundamental solution to the Laplace equation and μ is a finite positive measure supported in U. It is easy to see that we can compute the weak first-order partial derivatives of u in U by differentiating the right-hand side of (6) under the sign of the integral

$$\frac{\partial u}{\partial x_i}(x) = \frac{\partial h}{\partial x_i}(x) - \frac{1}{n\omega_n}\int_{\mathbb{R}^n} \frac{(x_i - y_i)}{|x-y|^n}\,d\mu(y)\,.$$

By Young's convolution inequality (cf. [24, II.1.1, p. 27]), the convolution is as integrable as the kernel. Since the measure μ has a bounded support and the function $x \mapsto \frac{x}{|x|^n}$ is clearly in $L^p_{\mathrm{loc}}(\mathbb{R}^n)$, we deduce that $\nabla u \in L^p(U)$. □

For an open set $\Omega \subset \mathbb{R}^n$ and $f \in L^1_{\mathrm{loc}}(\Omega)$ we define a local maximal function

$$\mathcal{M}_\Omega f(x) = \sup \fint_{B(x,r)} |f(y)|\, dy,$$

where the supremum is over all balls $B(x,r) \Subset \Omega$.

The following characterization of superharmonic functions will be very useful; see [13].

Lemma 11. *A locally integrable function $u : \Omega \to [0,\infty]$, $\Omega \subset \mathbb{R}^n$, is superharmonic if and only if*

$$\mathcal{M}_\Omega u(x) = u(x) \quad \text{for all } x \in \Omega.$$

Proof. If u is superharmonic, then taking the supremum over all balls in (5) gives $u(x) \geq \mathcal{M}_\Omega u(x)$ for all $x \in \Omega$. On the other hand, lower semicontinuity of u yields

$$\mathcal{M}_\Omega u(x) \geq \limsup_{r\to 0} \fint_{B(x,r)} u(y)\, dy \geq \liminf_{y\to x} u(y) \geq u(x)$$

for all $x \in \Omega$. Hence $u(x) = \mathcal{M}_\Omega u(x)$ for all $x \in \Omega$.

Now suppose that $\mathcal{M}_\Omega u(x) = u(x)$ for all $x \in \Omega$. Since the maximal function is lower semicontinuous we conclude lower semicontinuity of u. The superharmonicity of u follows from the inequality

$$\fint_{B(x,r)} u(y)\, dy \leq \mathcal{M}_\Omega u(x) = u(x),$$

which is satisfied on every ball $B(x,r) \Subset \Omega$. □

Corollary 12. *If $f \in L^1(\mathbb{R}^n)$ and $|f(x)| = \mathcal{M}f(x)$ a.e. in an open set $\Omega \subset \mathbb{R}^n$, then we can redefine f on a set of measure zero in such a way that $|f|$ becomes superharmonic in Ω.*

Proof. It follows from the Lebesgue differentiation theorem that $|f(x)| \leq \mathcal{M}_\Omega |f|(x)$ a.e. in Ω. Hence

$$\mathcal{M}_\Omega |f|(x) \leq \mathcal{M}|f|(x) = \mathcal{M}f(x) = |f(x)| \leq \mathcal{M}_\Omega |f|(x)$$

a.e. in Ω and thus $|f(x)| = \mathcal{M}_\Omega |f|(x)$ a.e. in Ω. Now it is clear that we can modify f on a set of measure zero in such a way that

$$|f(x)| = \mathcal{M}_\Omega |f|(x) \quad \text{everywhere in } \Omega,$$

which makes the function $|f|$ superharmonic. □

Now we can complete the proof of the theorem. Let $f \in L^1(\mathbb{R}^n)$ and let $\Omega \subset \mathbb{R}^n$ be open. According to Lemma 9, $\mathcal{M}f$ is a.e. approximately differentiable in the set

$$\{x \in \Omega : \, |f(x)| < \mathcal{M}f(x)\}.$$

If this set has positive measure, the theorem follows. If it has measure zero, then

$$|f(x)| = \mathcal{M}f(x) \quad \text{a.e. in } \Omega,$$

and hence $|f(x)|$ coincides a.e. with a superharmonic function in Ω; see Corollary 12. Now Lemma 10 gives a.e. approximate differentiability of $|f|$ in Ω and hence that of f; see Lemma 7. The proof is complete.

6. Example

In this section we will construct a bounded integrable function $f \in L^1(\mathbb{R})$ such that the set of points where the maximal function $\mathcal{M}f$ is not approximately differentiable is of positive measure. In our construction $\mathcal{M}f$ will coincide with f on a contact set P of positive length, and f will not be approximately differentiable on P. This will imply the lack of approximate differentiability of $\mathcal{M}f$ at the Lebesgue points of P.

In the first step we will construct a bounded periodic function f with period 1 such that $\mathcal{M}f$ is not approximately differentiable a.e., and then it will be clear that also for $\tilde{f} = f\chi_{[0,1]} \in L^1(\mathbb{R})$ the maximal function is not approximately differentiable a.e.

6.1. **Construction.** We denote

$$r_k = 3^{-k(k+1)}, \qquad \alpha_k = \exp(-9^{-k-2}).$$

For $k = 1, 2, \dots$, on the interval $\big[0, r_{k-1}\big)$ we define

$$g_k(x) = \begin{cases} 1, & x \in \big[(i-1)r_k, ir_k\big), \quad i \in \{2, 4, \dots, 9^k - 1\}, \\ \alpha_k, & x \in \big[(i-1)r_k, ir_k\big), \quad i \in \{3, 5, \dots, 9^k - 2\}, \\ 0, & x \in \big[(i-1)r_k, ir_k\big), \quad i \in \{1, 9^k\}. \end{cases}$$

We extend g_k to $\mathbb{R}$ periodically with the period r_{k-1}. Finally we set

$$f_0 = 1, \quad f_n = \prod_{k=1}^{n} g_k, \qquad f = \lim_{n\to\infty} f_n.$$

Observe that the function g_k is constant on the intervals $[(i-1)r_k, ir_k)$, $i \in \mathbb{Z}$, and hence f_n is constant on the intervals $[(i-1)r_n, ir_n)$, $i \in \mathbb{Z}$.

6.2. **Maximal function.** We will now estimate the maximal function of f. We denote

$$P = \bigcap_k \{f_k > 0\}.$$

Let $x \in \mathbb{R}$ and $\rho > 0$. We consider the smallest $n \in \mathbb{N}$ such that $M_n(x, \rho) = (x - \rho, x + \rho) \cap r_n\mathbb{Z} \neq \emptyset$. In this situation $(x - \rho, x + \rho)$ is contained in one of the intervals $[(i-1)r_{n-1}, ir_{n-1})$, and hence f_{n-1} equals a constant β on $(x - \rho, x + \rho)$. Let us write $M = M_n(x, \rho)$. Now we will distinguish two cases.

Case 1. Let $\rho \le r_{n+1}$. Then there is only one point $z \in M$. By reason of symmetry, we may assume that $x \ge z$. Then $x \in [z, z + r_{n+1})$. Since $g_{n+1} = 0$ on $[z, z + r_{n+1})$, $f_{n+1} = 0$ on that interval and hence $x \notin P$.

Case 2. Let $\rho > r_{n+1}$. We split $(x - \rho, x + \rho)$ into intervals $(x - \rho, x + \rho) \cap [(i-1)r_n, ir_n)$. For each interval I of the partition, with an endpoint $z \in M$, either $[z, z + r_{n+1}) \subset I$, $[z - r_{n+1}, z) \subset I$ or $I \subset (z - r_{n+1}, z + r_{n+1})$. Since $f_{n+1} = 0$ on $(z - r_{n+1}, z + r_{n+1})$ we have $(z - r_{n+1}, z + r_{n+1}) \cap P = \emptyset$. In each case

$$|I \cap P| \le \Big(1 - \frac{r_{n+1}}{r_n}\Big)|I|.$$

Summing over I we obtain

$$|(x - \rho, x + \rho) \cap P| \le 2\rho\Big(1 - \frac{r_{n+1}}{r_n}\Big).$$

It follows that

$$\fint_{x-\rho}^{x+\rho} f \le \frac{\beta}{2\rho}\,|(x-\rho,x+\rho)\cap P| \le \beta\,\Big(1-\frac{r_{n+1}}{r_n}\Big).$$

On the other hand, if $x\in P$, then

$$f(x)\ge \beta\alpha_n\alpha_{n+1}\cdots.$$

Since

$$1-\frac{r_{n+1}}{r_n} = 1-9^{-n-1} \le e^{-9^{-n-1}} < e^{-9^{-n-2}-9^{-n-3}-9^{-n-4}-\cdots} = \alpha_n\alpha_{n+1}\dots,$$

we obtain

$$\mathcal{M}f(x)\le f(x) \qquad \text{on } P,$$

and hence $\mathcal{M}f(x)=f(x)$ a.e. in P.

6.3. **The contact set.** On the set P we have

$$f(x)\ge \beta_\infty := \alpha_1\alpha_2\alpha_3\cdots = \exp(-9^{-3}-9^{-4}-9^{-5}-\dots)>0.$$

We will estimate the size of the set $P\cap[0,1]$. We see that

$$\begin{aligned}
&\big|\{f_1>0\}\cap[0,1]\big| = 1-2r_1,\\
&\big|\{f_2>0\}\cap[0,1]\big| = \big|\{f_1>0\}\cap[0,1]\big|\,\Big(1-2\frac{r_2}{r_1}\Big) = (1-2r_1)\Big(1-2\frac{r_2}{r_1}\Big),\\
&\qquad\qquad\dots,
\end{aligned}$$

so that

$$|P\cap[0,1]| = (1-2r_1)\Big(1-2\frac{r_2}{r_1}\Big)\Big(1-2\frac{r_3}{r_2}\Big)\cdots>0 \tag{7}$$

as

$$\sum_k\frac{r_{k+1}}{r_k} = \sum_k 9^{-k-1} < +\infty.$$

6.4. **Differentiability.** Let us consider $x\in P$, $k\in\mathbb{N}$ and an interval $[z,z+r_k)$ such that $z\in r_k\mathbb{Z}$ and

$$x\in[z,z+r_k)\subset\{f_k>0\}\,.$$

This interval is contained in an interval $[(i-1)r_{k-1},ir_{k-1})$, where the function f_{k-1} has constant value $\beta\in(\beta_\infty,1]$. Since $z\ge(i-1)r_{k-1}+r_k$ (otherwise $f_k(z)=0$) we have that

$$[z-r_k,z+r_k)\subset[(i-1)r_{k-1},ir_{k-1})$$

and hence $f_{k-1}=\beta$ on $[z-r_k,z+r_k)$. There are three possibilities:

$$f_k=\begin{cases}0 & \text{on } [z-r_k,z)=I,\\ \beta & \text{on } [z,z+r_k)=J,\end{cases}$$

$$f_k=\begin{cases}\beta & \text{on } [z-r_k,z)=J,\\ \alpha_k\beta & \text{on } [z,z+r_k)=I,\end{cases}$$

$$f_k=\begin{cases}\alpha_k\beta & \text{on } [z-r_k,z)=I,\\ \beta & \text{on } [z,z+r_k)=J.\end{cases}$$

In each case

$$f\le\beta\alpha_k=\beta\exp(-9^{-k-2}) \quad \text{on } I,$$

whereas

$$f \geq \beta\alpha_{k+1}\alpha_{k+2}\alpha_{k+3}\ldots = \beta \exp\Big(-\frac{9^{-k-2}}{8}\Big) \quad \text{on } J\cap P.$$

Hence

$$|f - f(x)| \geq \frac{1}{2}\beta\Big(\exp\Big(-\frac{9^{-k-2}}{8}\Big) - \exp(-9^{-k-2})\Big) \tag{8}$$

on at least one of the sets I or $J \cap P$. Since the infinite product at (7) converges, for sufficiently large k we have

$$|J\cap P| = |J|\Big(1-2\frac{r_{k+1}}{r_k}\Big)\Big(1-2\frac{r_{k+2}}{r_{k+1}}\Big)\cdots > \frac{1}{2}|J| = \frac{r_k}{2}\,,$$

and hence inequality (8) is satisfied on a set $E_k \subset [z-r_k, z+r_k)$ of length $|E_k| > r_k/2$. To estimate the right hand side of (8), observe that $e^{-x} - e^{-y} \geq e^{-y}(e^{y-x}-1) \geq (1-y)(y-x)$, $0<x<y$, and thus

$$\exp\Big(-\frac{9^{-k-2}}{8}\Big) - \exp(-9^{-k-2}) \geq \frac{7}{8}\Big(1-9^{-k-2}\Big)9^{-k-2} > 4\cdot 9^{-k-3}\,.$$

Accordingly

$$\frac{|f(y)-f(x)|}{|y-x|} \geq \frac{4\beta\, 9^{-k-3}}{4r_k} = \beta\, 3^{k^2-k-6} \qquad \text{for } y\in E_k. \tag{9}$$

Set $E_n^* = \bigcup\limits_{k=n}^{\infty} E_k$. Since

$$\limsup_{h\to 0+} \frac{|E_n^*\cap(x-h,x+h)|}{2h} \geq \limsup_{k\to\infty}\frac{|E_k|}{4r_k} \geq \frac{1}{8},$$

the approximate limit-superior of $\frac{|f(y)-f(x)|}{|y-x|}$ as $y\to x$ is at least $\beta\, 3^{n^2-n-6}$ for each n, and thus it is ∞. Hence f cannot be approximately differentiable at x. If x is a density point of P, then also $\mathcal{M}f$ cannot be approximately differentiable at x.

6.5. **An integrable function.** Finally let $\tilde f = f\chi_{(0,1)}$. Then $\mathcal{M}\tilde f(x) \leq \mathcal{M}f(x) \leq f(x) = \tilde f(x)$ in $P\cap(0,1)$ and hence $\mathcal{M}\tilde f(x) = \tilde f(x) = f(x)$ a.e. in $P\cap(0,1)$. Since f is not approximately differentiable on $P\cap(0,1)$, $\mathcal{M}\tilde f$ cannot be approximately differentiable a.e.

References

[1] Acerbi, E., Fusco, N.: An approximation lemma for $W^{1,p}$ functions. In: *Material instabilities in continuum mechanics* (Edinburgh, 1985–1986), pp. 1–5, Oxford Sci. Publ., Oxford Univ. Press, New York, 1988. MR970512 (89m:46060)

[2] Aldaz, J. M., Pérez Lázaro, J.: Boundedness and unboundedness results for some maximal operators on functions of bounded variation. *J. Math. Anal. Appl.* 337 (2008), 130–143. MR2356061 (2008k:42050)

[3] Aldaz, J. M., Pérez Lázaro, J.: Functions of bounded variation, the derivative of the one dimensional maximal function, and applications to inequalities. *Trans. Amer. Math. Soc.* 359 (2007), 2443–2461. MR2276629 (2008f:42010)

[4] Bojarski, B.: Remarks on some geometric properties of Sobolev mappings. In: *Functional analysis & related topics (Sapporo, 1990)*, pp. 65–76, World Sci. Publ., River Edge, NJ, 1991. MR1148607 (92k:46046)

[5] Bojarski, B., Hajłasz, P.: Pointwise inequalities for Sobolev functions and some applications. *Studia Math.* 106 (1993), 77–92. MR1226425 (94h:46045)

[6] Buckley, S. M.: Is the maximal function of a Lipschitz function continuous? *Ann. Acad. Sci. Fenn. Math.* 24 (1999), 519–528. MR1724375 (2001e:42025)

[7] Carneiro, E., Moreira, D.: On the regularity of maximal operators. *Proc. Amer. Math. Soc.* 136 (2008), 4395-4404. MR2431055

[8] Hajłasz, P., Onninen, J.: On boundedness of maximal functions in Sobolev spaces. *Ann. Acad. Sci. Fenn. Math.* 29 (2004), 167–176. MR2041705 (2005a:42010)

[9] Hayman, W. K., Kennedy, P. B.: *Subharmonic functions.* Vol. I. London Mathematical Society Monographs, No. 9. Academic Press, London-New York, 1976. MR0460672 (57:665)

[10] Kinnunen, J.: The Hardy-Littlewood maximal function of a Sobolev function. *Israel J. Math.* 100 (1997), 117–124. MR1469106 (99a:30029)

[11] Kinnunen, J., Latvala, V.: Lebesgue points for Sobolev functions on metric spaces. *Rev. Mat. Iberoamericana* 18 (2002), 685–700. MR1954868 (2004c:46054)

[12] Kinnunen, J., Lindqvist, P.: The derivative of the maximal function. *J. Reine Angew. Math.* 503 (1998), 161–167. MR1650343 (99j:42027)

[13] Kinnunen, J., Martio, O.: Maximal operator and superharmonicity. In: *Function spaces, differential operators and nonlinear analysis* (Pudasjärvi, 1999), pp. 157–169, Acad. Sci. Czech Repub., Prague, 2000. MR1755307 (2001f:31005)

[14] Kinnunen, J., Saksman, E.: Regularity of the fractional maximal function. *Bull. London Math. Soc.* 35 (2003), 529–535. MR1979008 (2004e:42035)

[15] Kinnunen, J., Tuominen, H.: Pointwise behaviour of $M^{1,1}$ Sobolev functions. *Math. Z.* 257 (2007), 613–630. MR2328816 (2008e:46042)

[16] Korry, S.: A class of bounded operators on Sobolev spaces. *Arch. Math. (Basel)* 82 (2004), 40–50. MR2034469 (2004k:42033)

[17] Korry, S.: Extensions of Meyers-Ziemer results. *Israel J. Math.* 133 (2003), 357–367. MR1968435 (2004c:46055)

[18] Korry, S.: Boundedness of Hardy-Littlewood maximal operator in the framework of Lizorkin-Triebel spaces. *Rev. Mat. Complut.* 15 (2002), 401–416. MR1951818 (2004a:42020)

[19] Lindqvist, P.: On the definition and properties of p-superharmonic functions. *J. Reine Angew. Math.* 365 (1986), 67–79. MR826152 (87e:31011)

[20] Luiro, H.: Continuity of the maximal operator in Sobolev spaces. *Proc. Amer. Math. Soc.* 135 (2007), 243–251. MR2280193 (2007i:42021)

[21] MacManus, P.: Poincaré inequalities and Sobolev spaces. In: *Proceedings of the 6th International Conference on Harmonic Analysis and Partial Differential Equations* (El Escorial, 2000). *Publ. Mat.* 2002, Vol. Extra, pp. 181–197. MR1964820 (2004j:46049)

[22] Malý, J., Ziemer, W. P.: *Fine regularity of solutions of elliptic partial differential equations.* Mathematical Surveys and Monographs, 51. American Mathematical Society, Providence, RI, 1997. MR1461542 (98h:35080)

[23] Ransford, T.: *Potential theory in the complex plane.* London Mathematical Society Student Texts, 28. Cambridge University Press, Cambridge, 1995. MR1334766 (96e:31001)

[24] Stein, E. M.: *Singular integrals and differentiability properties of functions.* Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, N.J., 1970. MR0290095 (44:7280)

[25] Tanaka, H.: A remark on the derivative of the one-dimensional Hardy-Littlewood maximal function. *Bull. Austral. Math. Soc.* 65 (2002), 253–258. MR1898539 (2002m:42017)

[26] Whitney, H.: On totally differentiable and smooth functions. *Pacific J. Math.* 1 (1951), 143–159. MR0043878 (13:333d)

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