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Approximation in Sobolev spaces of nonlinear expressions involving the gradient

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Abstract. We investigate a problem of approximation of a large class of nonlinear expressions $f(x, u, \nabla u)$, including polyconvex functions. Here $u: \Omega \to \mathbf{R}^m$, $\Omega \subset \mathbf{R}^n$, is a mapping from the Sobolev space $W^{1,p}$. In particular, when $p=n$, we obtain the approximation by mappings which are continuous, differentiable a.e. and, if in addition $n=m$, satisfy the Luzin condition. From the point of view of applications such mappings are almost as good as Lipschitz mappings. As far as we know, for the nonlinear problems that we consider, no natural approximation results were known so far. The results about the approximation of $f(x, u, \nabla u)$ are consequences of the main result of the paper, Theorem 1.3, on a very strong approximation of Sobolev functions by locally weakly monotone functions.

1. Introduction

We are interested in approximation of mappings $u: \Omega \to \mathbf{R}^m$ from the Sobolev space $W^{1,p}(\Omega)^n$ by a sequence $\{u_k\}_{k=1}^{\infty}$ of "more regular" mappings taking care of the convergence

$$\int_{\Omega} f(x, u_k, \nabla u_k)\, dx \to \int_{\Omega} f(x, u, \nabla u)\, dx$$

for a large class of nonlinear integrands.

Here and in what follows $\Omega \subset \mathbf{R}^n$ is an open set and m is the dimension of the target space for functions considered. Some notation, terminology and conventions, mostly standard, are explained at the beginning of Section 2 (although they may be used before then) so as not to disturb the course of the introduction too much.

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A standard way to approximate a function is by mollifying with a convolution kernel. There is also another familiar method of approximation of $u\in W^{1,p}(\Omega)$, $1\le p<\infty$, by Lipschitz continuous (or even continuously differentiable) functions u_k which yields both $u_k\to u$ in $W^{1,p}(\Omega)$ and $|\{x:u_k(x)\neq u(x)\}|\to 0$. See e.g. [1], [5], [12], [23], [37], [43], [69] and also [24] and [39] for related approximations by Hölder continuous functions.

When the two methods above fail, the approximation problem becomes difficult because of the lack of other powerful and elegant tools.

This is the case of approximation in nonlinear $\mathcal{A}_{\boldsymbol{p}}$ spaces of functions with gradient minors in L^{p_j} spaces. This problem will be described in Section 3, but now for illustration we start with a very particular case.

Let $u:\Omega\to\mathbf{R}^n$ be a mapping in the Sobolev class $W^{1,p}(\Omega)^n$, $1\le p<\infty$. Assume that in addition $\det\nabla u\in L^q(\Omega)$ for some $p/n<q<\infty$. The L^p-integrability of ∇u implies only the $L^{p/n}$-integrability of the Jacobian, so the L^q integrability of the Jacobian is a strong additional condition. Now we ask:

Does there exist a sequence of "more regular" mappings $u_k:\Omega\to\mathbf{R}^n$ such that

$$u_k\to u \textit{ in } W^{1,p}(\Omega)^n \quad \textit{and} \quad \det\nabla u_k\to\det\nabla u \textit{ in } L^q(\Omega)? \tag{1}$$

There are many other related problems of approximation of determinants and minors, see e.g. [3], [11], [16], [18], [19], [22], [28], [38], [51], [52] and [53]. However very few results are in the positive direction. This was our main motivation for considering problems like (1).

It was not clarified here what we shall mean by "more regular" mappings. Observe that if $q=p/n$, then we can get the desired approximation using the approximation by convolution and the resulting sequence $\{u_k\}_{k=1}^\infty$ is C^∞ smooth, so this is not a very interesting case and in what follows we shall assume that $q>p/n$. Then trying to approximate u by convolution or any other related method we immediately lose information about the integrability of the Jacobian above the exponent p/n. The problems with the approximation are caused by the high nonlinearity of the determinant.

We would be happy to have a sequence of smooth mappings, but, in general, when $n-1\le p<n$ it is not even possible to have a sequence of continuous mappings with the Luzin property (defined below); see Proposition 3.3. The counterexample is based on the fact that the radial projection mapping $x\mapsto x/|x|:B\to S\subset\mathbf{R}^n$ belongs to the Sobolev space $W^{1,p}(B)^n$ for all $p<n$. This forms a "topological" obstacle for the approximation. There are no such obvious obstacles when $p=n$, so one may hope to have a nice approximation. And indeed we prove that when $p=n$ and $1\le q<\infty$ there is a sequence $\{u_k\}_{k=1}^\infty$ as in (1) which consists of mappings which are continuous, differentiable a.e. and have the Luzin property ((7), and Lemma 1.2). Although the case $p=n$ was the main motivation for our research the method applies also to the case $p<n$. Also then the approximating mappings are more regular than generic mappings, but in a sense which is not so transparent and will be explained below.

In the case $p<n$ there are alternative methods of approximation that will be explained in our forthcoming paper [25]. These methods are based on completely different ideas and the class of situations where they apply is neither wider nor narrower in comparison with the methods presented here.

Actually in the main theorem of the present paper, Theorem 1.3, we succeed in approximating a given function $u\in W^{1,p}(\Omega)$ by a sequence $\{u_k\}_{k=1}^{\infty}$ of locally weakly monotone functions (defined below) such that the approximating functions are "very close to u"—so close that when the approximation is applied to each coordinate of the mapping $u\in W^{1,p}(\Omega)^m$ separately, surprisingly, nonlinear expressions involving the gradient (including the determinant) also converge. As we will see, locally weakly monotone functions are more regular than generic functions in $W^{1,p}(\Omega)$. If $p=n$, in fact, from the point of view of applications, their properties are almost as good as properties of Lipschitz mappings. To our knowledge, for such nonlinear problems that we consider, no natural approximation results were known at all.

Definition. Let $u\in W^{1,p}(\Omega)$. We say that u obeys the *weak maximum principle* on Ω if the implication

$$(u-l)^+ \in W_0^{1,p}(\Omega) \quad \Longrightarrow \quad u\leq l \text{ a.e. in } \Omega$$

holds for each $l\in\mathbf{R}$. Similarly we define the *weak minimum principle.* We say that u is *weakly monotone* if u satisfies both the weak maximum and the weak minimum principles on all subdomains $\Omega'\Subset\Omega$.

If there is $R>0$ such that u is weakly monotone on all open subsets of Ω of diameter less than or equal to R, then we say that u is *locally weakly monotone.*

We say that a mapping $u=(u_1,\dots,u_m)\in W^{1,p}(\Omega)^m$ is (locally) weakly monotone if each of the coordinate functions u_i is (locally) weakly monotone.

Weakly monotone functions were introduced by Manfredi [44] (cf. [15], [33] and [45]) as a generalization of monotone functions in the sense of Lebesgue [36], [50].

Locally weakly monotone functions obey some regularity properties that we next describe.

Lemma 1.1. *Let $u\in W^{1,p}(\Omega)^m$, $\Omega\subset\mathbf{R}^n$, $n-1<p\leq n$ be locally weakly monotone. Then u is locally bounded, differentiable a.e. and there exists a set Z with $\mathcal{H}^{n-p}(Z)=0$, such that u is continuous at each point of $\Omega\setminus Z$.*

Recall that a Borel mapping $u:\Omega\to\mathbf{R}^n$, $\Omega\subset\mathbf{R}^n$, has the *Luzin property* if the image of any set of Lebesgue measure zero is of Lebesgue measure zero.

Lemma 1.2. *Let $u\in W^{1,n}(\Omega)^m$, $\Omega\subset\mathbf{R}^n$, be locally weakly monotone, then u is continuous and differentiable a.e. If $m=n$, u also has the Luzin property.*

Remarks. 1. We refer to Section 4 for a more complete exposition of the theory of weakly monotone functions including proofs of the above two lemmas.

2. Here and in what follows, by differentiability we mean differentiability in the classical sense.

3. The Luzin property is very important as it allows one to apply the change of variables formula, see e.g. [6], [15], [21], [42], [47] and [56].

4. The method of the proof works for $u\in W^{1,p}(\Omega)^n$, $p>n$, as well. Then, however, we do not gain anything interesting as every Sobolev mapping in $W^{1,p}(\Omega)^n$ is Hölder continuous, differentiable a.e. and has the Luzin property, see e.g. [6], [15] and [46].

5. In the case $p<n$, (local) weak monotonicity does not imply continuity. Indeed, the coordinate functions of the radial projection

$$u_0: B\longrightarrow S, \quad u_0(x)=\frac{x}{|x|},$$

belong to the Sobolev space $W^{1,p}(B)$ for all $p<n$. They are weakly monotone with the discontinuity at the origin. For other examples see [44].

6. Lemma 1.1 does not extend to the case $p\le n-1$. Indeed, if $\varphi\in W^{1,n-1}(Q^{n-1})$ is essentially discontinuous everywhere, then the function given by

$$u(x_1,\dots,x_{n-1},x_n)=\varphi(x_1,\dots,x_{n-1})$$

is weakly monotone and essentially discontinuous everywhere in Q^n.

Now we can formulate our main theorem.

Theorem 1.3. *Let $u\in W^{1,p}(\Omega)$, where $\Omega\subset\mathbf{R}^n$ is open and $1\le p<\infty$. Then there is a sequence $\{u_k\}_{k=1}^{\infty}$ of functions from $W^{1,p}(\Omega)$ such that*

(a) *each u_k is locally weakly monotone*;

(b) *$\nabla u_k=0$ a.e. in the set where $u_k\neq u$*;

(c) *$u_k\to u$ in $W^{1,p}(\Omega)$ as $k\to\infty$.*

The essential novelty in the above approximation is the property (b). Observe that in the set where $u_k=u$ we have $\nabla u_k=\nabla u$ a.e., and thus we could write (b) in the equivalent form

$$\nabla u_k=\nabla u\chi_{\{x:u_k(x)=u(x)\}}.$$

Properties (b) and (c) do not necessarily imply that $|\{x:u_k(x)\neq u(x)\}|\to 0$. However, it follows from (b) and (c) that, passing eventually to a subsequence,

$$\chi_{\{x:u_k(x)\neq u(x) \text{ and } \nabla u(x)\neq 0\}}\to 0 \quad \text{a.e.} \tag{2}$$

and

$$\chi_{\{x:\nabla u_k(x)\neq \nabla u(x)\}}\to 0 \quad \text{a.e.} \tag{3}$$

The conditions (b) and (c), and their consequences (2) and (3) are so strong that they imply unexpected convergence of nonlinear expressions involving the gradient in most of the interesting cases.

As we have already mentioned, weakly monotone functions need not be continuous, and indeed, as we will see in Section 3, in general, when $n-1\leq p<n$, it is not possible to find an approximation of $u\in W^{1,p}(\Omega)$ by continuous functions that would satisfy the conditions (b) and (c) at the same time (Corollary 3.4). This is in contrast with another method, mentioned above, which provides an approximation of $u\in W^{1,p}$ by Lipschitz, or even C^1, functions such that $|\{x:u_k(x)\neq u(x)\}|\to 0$ and $u_k\to u$ in $W^{1,p}$, but, as we see now, the condition that the gradient of u_k vanishes on the set where u_k differs from u cannot be achieved.

Recently Vodop′yanov [65] found some applications of Theorem 1.3 in the context of the change of variables formula on Carnot groups.

The method of approximation in Theorem 1.3 seems to be based on new ideas which are of independent interest. Some other ideas may however be traced back to the old work of Lebesgue [36] on monotone functions. In order to construct locally weakly monotone functions we truncate the function u infinitely many times. After each truncation we obtain a better function, looking more like a monotone one. Since we make infinitely many truncations of a Sobolev function it requires quite delicate arguments to ensure that the constructed sequence satisfies all the desired properties.

We promised to state our result as a result on approximation of a nonlinear expression involving the gradient. The way of convergence of u_k to u allows many variants of convergence results for nonlinear integrands which may be easily derived by the reader from the properties (b) and (c). A paradox of the result consists in observing that, although the theorem is scalar in nature, the whole strength appears when applied coordinate-wise to vector-valued functions. Below we mention one result in the setting of functionals involving gradient minors. Other results will be mentioned in Section 3.

For a $W^{1,p}_{\text{loc}}$-function u we define the *ith adjugate* $\operatorname{adj}_i \nabla u$ as the collection of all $i\times i$ minors of ∇u.

Observe that for a mapping $u:\Omega\to\mathbf{R}^m$, $\Omega\subset\mathbf{R}^n$, the minors $\operatorname{adj}_i\nabla u$ are defined only for $1\le i\le\min\{n,m\}$. Obviously $\operatorname{adj}_1\nabla u=\nabla u$ and if $m=n$, then $\operatorname{adj}_n\nabla u=\det\nabla u$ and $\operatorname{adj}_{n-1}\nabla u=\operatorname{adj}\nabla u$.

The so called polyconvex integrals (see [2], [8], [9], [10], [17], [54] and [55]) arise in the context of nonlinear elasticity or in Skyrme's problem. Some typical polyconvex functionals lead naturally to a class of mappings $u:\Omega\to\mathbf{R}^n$ with prescribed integrability of minors, i.e., $\operatorname{adj}_i\nabla u\in L^{p_i}(\Omega)$, where $i=1,2,\dots,n$.

We may immediately state the result to cover the case of Orlicz type integrability.

Theorem 1.4. *Let $u\in W^{1,p}(\Omega)^m$, where $\Omega\subset\mathbf{R}^n$ is open and $1\le p<\infty$. Let $g:\Omega\times\mathbf{R}^m\times\mathbf{R}^{nm}\to\mathbf{R}$ be given by*

$$g(x,\zeta,\xi)=h(x,\zeta,\xi)+a_1(x)\Phi_1(|\operatorname{adj}_1\xi|)+\dots+a_N(x)\Phi_N(|\operatorname{adj}_N\xi|),$$

where $N=\min\{n,m\}$, $\Phi_i:[0,\infty)\to[0,\infty)$, $i=1,2,\dots,N$, are nondecreasing, $a_1,\dots,a_N$ are nonnegative measurable functions on Ω, and $h:\Omega\times\mathbf{R}^m\times\mathbf{R}^{nm}\to\mathbf{R}$ is a nonnegative function such that the nonlinear operator

$$u\longmapsto h(x,u,\nabla u):W^{1,p}(\Omega)^m\longrightarrow L^1(\Omega) \tag{4}$$

is well defined and continuous. Assume that

$$\int_\Omega g(x,u,\nabla u)\,dx<\infty. \tag{5}$$

Then for any Carathéodory function $f:\Omega\times\mathbf{R}^m\times\mathbf{R}^{nm}\to\mathbf{R}$ such that

$$|f(x,\zeta,\xi)|\le g(x,\zeta,\xi), \tag{6}$$

there exists a sequence $\{u_k\}_{k=1}^\infty\subset W^{1,p}(\Omega)^m$ of locally weakly monotone mappings such that

$$\int_\Omega f(x,u_k,\nabla u_k)\,dx\to\int_\Omega f(x,u,\nabla u)\,dx.$$

Remarks. 1. We will actually prove that the claim holds with a sequence $\{u_k\}_{k=1}^\infty$ as in Theorem 1.3. We want to emphasize that the approximating sequence $\{u_k\}_{k=1}^\infty$ is independent of the integrand $f(x,u,\nabla u)$.

2. The proof and applications of the theorem to more particular problems will be presented in Section 3.

3. A typical application is $\Phi_i(t)=t^{p_i}$, see Theorem 3.2.

4. We may consider some $a_i=0$ if we do not want to assume any integrability of the ith adjugate.

5. For a general theory of operators of the type (4), called Nemytskiĭ operators, see e.g. [14] and [34]. However for most of the applications it suffices to assume that h has a much simpler structure, for example h may be an integrable function independent of (ζ,ξ), or h may equal $|\xi|^p$ or $|\zeta|^q$, where q is an embedding exponent, or $b(x)|\zeta|^q$ where $b^{1/q}$ is a "multiplier" in the sense of Maz′ya and Shaposhnikova [48] etc.

6. The most important case of the theorem is when $p=n$, since in the case $p<n$ there are alternative methods of approximation of similar problems, see [25].

Now observe that the theorem applies to problem (1). Indeed, assume that $u\in W^{1,p}(\Omega)^n$ and that $\det\nabla u(x)\in L^q(\Omega)$. Then if we take

$$f(x,\zeta,\xi)=|\zeta-u(x)|^p+|\xi-\nabla u(x)|^p+|\det\xi-\det\nabla u(x)|^q,$$

Theorem 1.4 easily implies that there is a sequence of locally weakly monotone mappings $\{u_k\}_{k=1}^{\infty}\subset W^{1,p}(\Omega)^n$ such that

$$\int_\Omega (|u_k-u|^p+|\nabla u_k-\nabla u|^p+|\det\nabla u_k-\det\nabla u|^q)\,dx\to 0. \tag{7}$$

In particular when $p=n$ we obtain an approximation by mappings which are continuous, differentiable a.e. and satisfy the Luzin condition (Lemma 1.2). This result is sharp as in the case $n-1\le p<n$ such an approximation is not possible in general (Proposition 3.3).

The paper is organized as follows. The next section contains some auxiliary results needed in the sequel. In Section 3 we prove how Theorem 1.4 follows from Theorem 1.3, then we show some applications of Theorem 1.4 to more particular situations and prove Proposition 3.3 and Corollary 3.4 demonstrating sharpness of the result on the Luzin property and of Theorem 1.3 respectively. In Section 4 we recall the definition of the class of weakly monotone functions. We slightly improve known results and add some new observations. This will prove Lemmas 1.1 and 1.2. Section 5 is devoted to the proof of the main result of our paper, Theorem 1.3.

2. Notation and auxiliary results

The notation used in this paper is standard. The k-dimensional Hausdorff measure will be denoted by $\mathcal{H}^k$ and the average value by

$$u_E=\fint_E u\,d\mu=\frac{1}{\mu(E)}\int_E u\,d\mu.$$

The Lebesgue measure of a set E will be denoted by $|E|$. The characteristic function of the set E will be denoted by χ_E. The L^p norm of a function u will be denoted by $\|u\|_p$. The oscillation of a measurable function over a measurable set is defined as

$$\operatorname*{osc}_E u = \operatorname*{ess\,sup}_E u - \operatorname*{ess\,inf}_E u.$$

Positive and the negative parts of a function u are $u^+=\max\{u,0\}$, $u^-=-\min\{u,0\}$. By Ω we will always denote an open subset of $\mathbf{R}^n$, even if not stated explicitly. The number n will always denote the dimension of the Euclidean space in which we consider domains of the studied functions.

By $B=B(x,r)$ we will denote an n-dimensional ball centered at x with radius r. Spheres will be denoted by $S(x,r)=\partial B(x,r)$. By a Carathéodory function we mean a function $f:\Omega\times\mathbf{R}^m\times\mathbf{R}^{nm}\to\mathbf{R}$ such that $f(\,\cdot\,,\zeta,\xi)$ is measurable on Ω for all $(\zeta,\xi)\in\mathbf{R}^m\times\mathbf{R}^{nm}$ and $f(x,\,\cdot\,,\cdot\,)$ is continuous in $\mathbf{R}^m\times\mathbf{R}^{nm}$ for almost every $x\in\Omega$. By C we will denote a general constant whose value is not important and so the same symbol C may denote different constants even in the same line. If we write $C=C(n,p)$ we mean that the constant C depends on n and p only.

The gradient ∇u is understood in the distributional sense. Given $1\le p<\infty$, we denote by $W^{1,p}(\Omega)$ the usual Sobolev space on Ω consisting of the functions u such that both $u\in L^p(\Omega)$ and $|\nabla u|\in L^p(\Omega)$. The space is equipped with the norm $\|u\|_{1,p}=\|u\|_p+\|\nabla u\|_p$. By $W^{1,p}_0(\Omega)$ we will denote the closure of $C_0^\infty(\Omega)$ in the norm $\|\cdot\|_{1,p}$. By $W^{1,p}(\Omega)^m$ we denote the class of mappings $u:\Omega\to\mathbf{R}^m$ such that the coordinate functions belong to the function space $W^{1,p}(\Omega)$. A similar convention is used also for other function spaces.

Now we collect some standard results that will be used in the sequel. We suggest that the reader skip reading this section and jump to Section 3. Then the reader can consult Section 2 whenever necessary.

The following lemma on differentiation of absolutely continuous measures can be found e.g. in [43, Corollary 1.19] or in [15, Proposition 4.37]. It is a consequence of a standard covering argument.

Lemma 2.1. *If $v\in L^1_{\mathrm{loc}}(\Omega)$, then*

$$\mathcal{H}^{n-p}\biggl(\biggl\{x\in\Omega:\limsup_{r\to0} r^p \fint_{B(x,r)} |v|\,dx>0\biggr\}\biggr)=0.$$

The following result is due to Stepanov, see [41], [60] and [61].

Lemma 2.2. *A function $u:\Omega\to\mathbf{R}$ is differentiable a.e. if and only if*

$$\limsup_{y\to x}\frac{|u(y)-u(x)|}{|y-x|}<\infty \quad \textit{a.e.}$$

The next results concern properties of functions from the Sobolev space $W^{1,p}(\Omega)$. First we recall that both $W^{1,p}(\Omega)$ and $W^{1,p}_0(\Omega)$ have the *lattice property*, i.e. they are closed under taking pointwise max and min. For the following result, see e.g. [27, Lemma 1.25].

Lemma 2.3. *If* $u\in W^{1,p}(\Omega)$, $v\in W^{1,p}_0(\Omega)$ *and* $|u|\le|v|$ *a.e., then* $u\in W^{1,p}_0(\Omega)$.

Proof. Assume first that $0\le u\le v$. Let $\varphi_k\in C^\infty_0(\Omega)$ be such that $\varphi_k\to v$ in $W^{1,p}(\Omega)$. Then the functions $\min\{\varphi_k,u\}$ have compact support and it is easy to prove that $\min\{\varphi_k,u\}\to\min\{v,u\}=u$, so $u\in W^{1,p}_0(\Omega)$. Now we pass to the general case $|u|\le|v|$. It is easy to see that $v^+,v^-\in W^{1,p}_0(\Omega)$. Hence $|v|=v^++v^-\in W^{1,p}_0(\Omega)$. Now the inequalities $0\le u^\pm\le|v|$ imply that $u^+,u^-\in W^{1,p}_0(\Omega)$. Hence also $u=u^+-u^-\in W^{1,p}_0(\Omega)$. □

Corollary 2.4. *Let* $u,v\in W^{1,p}(\Omega)$ *and* $E\subset\Omega$. *If* $0\le u\le v$ *a.e. on* E *and* $v\chi_E\in W^{1,p}_0(\Omega)$, *then* $u\chi_E\in W^{1,p}_0(\Omega)$.

Proof. The claim follows from the previous lemma and the observation that $u\chi_E=\min\{u^+,v\chi_E\}\in W^{1,p}(\Omega)$. □

Lemma 2.5. *Assume that* $u\in W^{1,p}(\Omega)$ *and* $t>0$. *Then* $u\in W^{1,p}_0(\Omega)$ *if and only if* $\min\{u,t\}\in W^{1,p}_0(\Omega)$.

Proof. The implication from the left to the right follows from Lemma 2.3. Suppose now that $\min\{u,t\}\in W^{1,p}_0(\Omega)$. Since $u^-=\min\{u,t\}^-\in W^{1,p}_0(\Omega)$ it remains to prove that $u^+\in W^{1,p}_0(\Omega)$ and thus we can assume that $u\ge0$. For each positive integer k, the function

$$u_k=\frac{ktu}{kt+u}$$

is in $W^{1,p}(\Omega)$ by the chain rule. Since $u_k\le k\min\{u,t\}$, appealing to Lemma 2.3, we have that $u_k\in W^{1,p}_0(\Omega)$. A routine argument shows that u belongs to $W^{1,p}_0(\Omega)$ as the limit of the sequence $\{u_k\}_{k=1}^\infty$. □

We need also consider infima and suprema of infinite families of Sobolev functions. If $\mathcal{U}$ is a family of measurable functions on Ω, we define the lattice supremum $\bigvee\mathcal{U}$ as the supremum with respect to the ordering, neglecting sets of measure zero. Thus, $\bigvee\mathcal{U}$ is an a.e. majorant of each element of $\mathcal{U}$ and an a.e. minorant of each a.e. majorant of $\mathcal{U}$. Similarly we introduce the lattice infimum $\bigwedge\mathcal{U}$.

It will be convenient for us to allow the lattice supremum to be $+\infty$ and the lattice infimum to be $-\infty$ on sets of positive measure.

If $\mathcal{U}$ is a countable family, then $\bigvee\mathcal{U}$ can be obtained as the pointwise supremum of $\mathcal{U}$. However, if $\mathcal{U}$ is uncountable, we must distinguish between the lattice supremum $\bigvee\mathcal{U}$ and the pointwise supremum

$$\sup\mathcal{U}\colon x\longmapsto\sup\{u(x):u\in\mathcal{U}\}.$$

The latter one heavily depends on the choice of representatives.

The following accessibility property can be found e.g. in [49, Lemma 2.6.1]. For the sake of completeness we provide a short proof.

Lemma 2.6. *Let $\mathcal{U}$ be a class of measurable functions defined in a measurable set $E\subset\mathbf{R}^n$. Then $\bigvee\mathcal{U}$ exists and there is a countable subfamily $\mathcal{V}\subset\mathcal{U}$ such that*

$$\bigvee\mathcal{U}=\bigvee\mathcal{V}=\sup\mathcal{V}.$$

Proof. First observe that we may assume that the family $\mathcal{U}$ is bounded in L^∞ and consists of nonnegative functions, otherwise we replace $\mathcal{U}$ by a family of functions $\frac{1}{2}\pi+\arctan u$, where $u\in\mathcal{U}$. We can also assume that the functions are defined in a set of finite measure, otherwise we make a diffeomorphic change of variables which maps E onto a bounded set. Let

$$s=\sup\biggl\{\int_E\max\{u_1,\dots,u_k\}\,dx:u_1,\dots,u_k\in\mathcal{U}\text{ for some }k\biggr\}<\infty.$$

Now there exists a sequence $\{u_k\}_{k=1}^\infty\subset\mathcal{U}$ such that $v_k=\max\{u_1,\dots,u_k\}$ satisfies $\lim_{k\to\infty}\int_E v_k\,dx=s$. Since $\{v_k\}_{k=1}^\infty$ is nondecreasing we have the a.e. convergence $\lim_{k\to\infty}v_k=v$. Obviously $\int_E v\,dx=s$. This easily implies that $v=\bigvee\mathcal{U}$ and so we can take $\mathcal{V}=\{u_1,u_2,\dots\}$. $\square$

Now, we mention a folklore theorem on suprema of infinite families of Sobolev functions.

Lemma 2.7. *Let $\mathcal{U}\subset W_0^{1,p}(\Omega)$ be bounded (in the Sobolev norm) and closed under finite maxima. Then $\bigvee\mathcal{U}\in W_0^{1,p}(\Omega)$. Moreover $\bigvee\mathcal{U}$ is a pointwise limit of an increasing sequence of functions from $\mathcal{U}$.*

Proof. Let $u=\bigvee\mathcal{U}$. Then, using that $\mathcal{U}$ is closed under finite maxima, we infer from Lemma 2.6 that there is an (a.e.) increasing sequence $\{u_k\}_{k=1}^\infty$ of functions from $\mathcal{U}$ such that $u_k\to u$ a.e. The sequence $\{u_k\}_{k=1}^\infty$ is bounded in $W_0^{1,p}(\Omega)$, so that by [27, Theorem 1.32], $u\in W_0^{1,p}(\Omega)$ and in fact u is a weak limit of $\{u_k\}_{k=1}^\infty$ in $W_0^{1,p}(\Omega)$. $\square$

We shall need the following elementary Poincaré lemma, see e.g. [12] or [69].

Lemma 2.8. *If* $u\in W_0^{1,p}(B(x,r))$, *then* $\int_{B(x,r)}|u|^p\,dx\le Cr^p\int_{B(x,r)}|\nabla u|^p\,dx$.

The next lemma is a direct consequence of the Sobolev embedding theorem into the space of Hölder continuous functions. It is often called Gehring's oscillation lemma. For a proof see e.g. [43, Lemma 2.10].

Lemma 2.9. *If* $u\in W^{1,p}(S(x_0,t))$, $p>n-1$, *then* u *is Hölder continuous and*

$$\operatorname*{osc}_{S(x_0,t)} u\le C(n,p)t\left(⨍_{S(x_0,t)}|\nabla u|^p\,d\mathcal{H}^{n-1}\right)^{1/p}.$$

The following two lemmas are special cases of more general results. The proof of the first lemma can be found in [35, Section 8.3.3].

Lemma 2.10. *If* $\Omega\subset\mathbf{R}^n$ *is a bounded domain with a smooth boundary, then there exists a bounded linear extension operator* $E\colon W^{1,n-1}(\partial\Omega)\to W^{1,n}(\Omega)$ *with the additional properties that* $Eu\in C^\infty(\Omega)$ *for any* $u\in W^{1,n-1}(\partial\Omega)$ *and* $Eu\in C(\bar\Omega)$ *for any* $u\in C(\partial\Omega)\cap W^{1,n-1}(\partial\Omega)$.

The proof of the next lemma can be found in [13], [15, Theorem 5.6], [19] and [21].

Lemma 2.11. *If* $u\in W^{1,1}(B)^n$ *is continuous and satisfies the Luzin property, then*

$$|u(B)|\le\int_B|\det\nabla u|\,dx.$$

The last lemma is proved in [54, Theorem 3.2].

Lemma 2.12. *If* $v\in W^{1,p}(\Omega)^n$ *and* $|\operatorname{adj}\nabla v|\in L^q(\Omega)$, $p\ge n-1$, $q\ge n/(n-1)$, *then* $d(v^1dv^2\wedge\ldots\wedge dv^n)=dv^1\wedge dv^2\wedge\ldots\wedge dv^n$ *in the sense of distributions, i.e. for every* $\psi\in C_0^\infty(\Omega)$ *the following identity is true*

$$-\int_\Omega v^1\,d\psi\wedge dv^2\wedge\ldots\wedge dv^n=\int_\Omega\psi\,dv^1\wedge dv^2\wedge\ldots\wedge dv^n.$$

3. Convergence of nonlinear expressions

In this section we prove the assertions mentioned in the introduction concerning convergence of expressions of the type

$$\int_\Omega f(x,u,\nabla u)\,dx.$$

First we will show how to obtain Theorem 1.4 as a consequence of Theorem 1.3. Then we will provide some applications to more particular problems. The proofs in this section are not hard, because the deep step is done in Theorem 1.3. The proof of Theorem 1.3 is postponed to Section 5.

Proof of Theorem 1.4. Assume that $u \in W^{1,p}(\Omega)^m$ satisfies (5). Let $\{u_k\}_{k=1}^\infty$ be an approximating sequence as in Theorem 1.3. More precisely $\{u_k\}_{k=1}^\infty$ is obtained by applying Theorem 1.3 to each coordinate of u separately. Let us write

$$\begin{aligned} \varphi_k(x) &= f(x, u_k(x), \nabla u_k(x)), & \varphi(x) &= f(x, u(x), \nabla u(x)), \\ \psi_k(x) &= g(x, u_k(x), \nabla u_k(x)), & \psi(x) &= g(x, u(x), \nabla u(x)), \\ \eta_k(x) &= h(x, u_k(x), \nabla u_k(x)), & \eta(x) &= h(x, u(x), \nabla u(x)). \end{aligned}$$

First we notice that if we denote the ith coordinate by a superscript, each ∇u_k^i is either 0 or ∇u^i. This has the consequence that each gradient minor of u_k either vanish or is equal to the corresponding minor for u in a given point. Hence

$$|\mathrm{adj}_j \nabla u_k| \le |\mathrm{adj}_j \nabla u|,$$

and since Φ_j is nondecreasing, also

$$\Phi_j(|\mathrm{adj}_j \nabla u_k|) \le \Phi_j(|\mathrm{adj}_j \nabla u|).$$

It follows that

$$\psi_k \le \psi + \eta_k, \tag{8}$$

and thus

$$|\varphi_k - \varphi| \le \psi_k + \psi \le \eta_k + 2\psi \le |\eta_k - \eta| + 3\psi.$$

Hence

$$\int_\Omega |\varphi_k - \varphi| \, dx \le \int_\Omega |\eta_k - \eta| \, dx + \int_\Omega \min\{|\varphi_k - \varphi|, 3\psi\} \, dx. \tag{9}$$

The first integral on the right tends to zero by the assumptions on h. The convergence of the second integral to zero follows from the Lebesgue dominated convergence theorem as soon as we show that from any subsequence of $\{\varphi_k\}_{k=1}^\infty$ we can extract a subsequence that converges to φ a.e. This follows from the fact that $u_k \to u$ in $W^{1,p}$ and that f is a Carathéodory function. □

Although Theorem 1.4 is very general, it does not cover all possible applications of approximation from Theorem 1.3. We state here one typical consequence of Theorem 1.3 which is of a slightly different nature. It is based on a direct application of the fact that for $u \in W^{1,p}(\Omega)^n$ either $\nabla u_k(x) = \nabla u(x)$ and $u_k(x) = u(x)$, or $\det \nabla u_k(x) = 0$. This allows one to have no growth condition for the integrand depending on (x, ζ, ξ) outside the set where $\det \xi = 0$.

Corollary 3.1. *Let $u \in W^{1,p}(\Omega)^n$ and let $f\colon \Omega\times\mathbf{R}^n\times\mathbf{R}\to\mathbf{R}$ be any function such that $f(\cdot,\cdot,0)=0$ and $f(x,u(x),\det\nabla u(x))\in L^q(\Omega)$, $1\le q<\infty$. Let $\{u_k\}_{k=1}^\infty$ be an approximating sequence satisfying the properties* (b) *and* (c) *of Theorem* 1.3 *coordinate-wise. Then we have*

$$\int_\Omega |f(x,u(x),\det\nabla u(x))-f(x,u_k(x),\det\nabla u_k(x))|^q\,dx\to 0,\quad \text{as } k\to\infty.$$

Proof. The integrand equals $|f(x,u(x),\det\nabla u(x))|^q$ multiplied by a characteristic function of the set $\{x:u_k(x)\neq u(x) \text{ and } \nabla u(x)\neq 0\}$. Hence the convergence follows from (2) and the Lebesgue dominated convergence theorem. $\square$

Now we state the most typical application in terms of nonlinear function spaces introduced by Ball [2], see also [17].

Definition. Let $\boldsymbol{p}=(p_1,\dots,p_N)$ be a multi-index, where $1\le p_1<\infty$, $0\le p_j<\infty$, $j=2,3,\dots,N$, and $N=\min\{m,n\}$. We denote by $\mathcal{A}_{\boldsymbol{p}}(\Omega)$ the class of mappings $u\in W^{1,p_1}(\Omega)^m$ such that $|\operatorname{adj}_j\nabla u|\in L^{p_j}$ for all $j\in\{1,\dots,N\}$ with $p_j\neq 0$. This, in particular, means that we pose no assumptions about the integrability of the jth adjugate when $p_j=0$. We will say that u_k approximates u in $\mathcal{A}_{\boldsymbol{p}}$ or that $u_k\to u$ in $\mathcal{A}_{\boldsymbol{p}}$, if $u_k\to u$ in $W^{1,p_1}(\Omega)^m$ and $|\operatorname{adj}_j\nabla u_k-\operatorname{adj}_j\nabla u|\to 0$ in $L^{p_j}(\Omega)$ for all $j\in\{1,\dots,N\}$ with $p_j\neq 0$.

As special cases we consider the *John Ball class* $A_{p,q}$ with $\boldsymbol{p}=(p,0,\dots,0,q,0)$, see [2], [64] and [54], and spaces $B_{p,q}$ with $\boldsymbol{p}=(p,0,\dots,0,q)$, both for $N=m=n$ and $1\le p,q<\infty$.

Thus the John Ball class $A_{p,q}(\Omega)$ consists of all mappings in $W^{1,p}(\Omega)^n$ such that $\operatorname{adj}\nabla u\in L^q(\Omega)$, $1\le p,q<\infty$, and the space $B_{p,q}(\Omega)$ of the mappings $u\in W^{1,p}(\Omega)^n$ with $\det\nabla u\in L^q(\Omega)$, $1\le p,q<\infty$.

Theorem 3.2. *Let $u\in\mathcal{A}_{\boldsymbol{p}}(\Omega)$ and let $\boldsymbol{p}$ be a multi-index with $1\le p_1<\infty$ and $0\le p_j<\infty$, $j=2,\dots,N$. Let $\{u_k\}_{k=1}^\infty$ be an approximating sequence satisfying the properties* (b) *and* (c) *of Theorem* 1.3 *coordinate-wise. Then $u_k\to u$ in $\mathcal{A}_{\boldsymbol{p}}(\Omega)$.*

Proof. We set

$$f(x,\zeta,\xi)=|\zeta-u(x)|^{p_1}+\sum_{j=1}^N a_j|\operatorname{adj}_j\xi-\operatorname{adj}_j\nabla u(x)|^{p_j},$$

where

$$a_j=\begin{cases}1, & p_j>0,\\ 0, & p_j=0.\end{cases}$$

If we let $q_j = \max\{p_j, 1\}$, we have

$$|\operatorname{adj}_j \xi - \operatorname{adj}_j \nabla u(x)|^{p_j} \le 2^{q_j-1}(|\operatorname{adj}_j \xi|^{p_j} + |\operatorname{adj}_j \nabla u(x)|^{p_j}).$$

We see that the assumptions (5) and (6) of Theorem 1.4 are satisfied with

$$h(x,\zeta,\xi) = |\zeta - u(x)|^{p_1} + \sum_{j=1}^{N} a_j 2^{q_j-1} |\operatorname{adj}_j \nabla u(x)|^{p_j} \quad \text{and} \quad \Phi_j(t) = 2^{q_j-1} t^{p_i}.$$

Hence the convergence $u_k \to u$ in $\mathcal{A}_{\boldsymbol{p}}(\Omega)$ follows directly from Theorem 1.4. □

Let $1 \le q < \infty$. Recall that by Theorem 1.3, Lemma 1.2 and Theorem 3.2, each mapping $u \in B_{n,q}(\Omega)$ can be approximated in $B_{n,q}$ by a sequence $\{u_k\}_{k=1}^{\infty}$ of continuous mappings with the Luzin property. We are going to show that n is a borderline exponent for such approximation. This shows the sharpness of our results.

Proposition 3.3. *The radial projection mapping $u: B(0,1) \to S(0,1)$ given by $u(x) = x/|x|$ belongs to $B_{p,q}(B)$ for all $1 \le p < n$ and $1 \le q < \infty$. If $n-1 \le p < n$ and $1 \le q < \infty$, then u cannot be approximated by continuous mappings $u_k \in B_{p,q}(B)$ with the Luzin property in the metric of $B_{p,q}$. In particular u_k cannot be Lipschitz continuous.*

Proof. We argue by contradiction. Let $n-1 \le p < n$, $1 \le q < \infty$ and suppose that a sequence $\{u_k\}_{k=1}^{\infty} \subset B_{p,q}(B)$ of continuous mappings with the Luzin property converges to u in the metric of $B_{p,q}$. Then applying a version of the Fubini theorem valid for Sobolev spaces we conclude that for almost all $r \in (0,1)$ after taking a subsequence we have the convergence

$$u_k|_{S(0,r)} \to u|_{S(0,r)} \quad \text{in } W^{1,p}(S(0,r)),$$

where the restrictions to spheres are understood in the sense of traces. Fix one such $r \in (0,1)$. By Lemma 2.10 there exists a bounded linear extension operator

$$E: W^{1,n-1}(\partial B(0,r) \cup \partial B(0,1))^n \longrightarrow W^{1,n}(B(0,1) \setminus B(0,r))^n$$

such that Ev is smooth in $B(0,1) \setminus \overline{B}(0,r)$. Let

$$v_k = \begin{cases} 0 & \text{on } \partial B(0,1), \\ u_k - u & \text{on } \partial B(0,r), \end{cases}$$

and define

$$w_k = \begin{cases} u + Ev_k & \text{on } \overline{B}(0,1) \setminus B(0,r), \\ u_k & \text{on } \overline{B}(0,r). \end{cases}$$

It is easy to see that $w_k:\overline{B}\to\mathbf{R}^n$ is continuous and identity on the boundary. Hence by the Brouwer fixed point theorem $B\subset w_k(B)$, and thus $|w_k(B)|\ge|B|$.

Moreover, $w_k\in W^{1,p}(B)^n$ satisfies the Luzin property (the Luzin property is satisfied in the annulus $B(0,1)\setminus\overline{B}(0,r)$ because w_k is smooth there). Hence invoking Lemma 2.11 we obtain

$$\begin{aligned}|B|\le|w_k(B)|&\le\int_{B(0,1)}|\det\nabla w_k|\,dx\\&=\int_{B(0,r)}|\det\nabla u_k|\,dx+\int_{B(0,1)\setminus B(0,r)}|\det\nabla w_k|\,dx.\end{aligned}$$

The second integral in the right-hand side converges to zero. Indeed,

$$v_k\to0\quad\text{in } W^{1,n-1}(\partial B(0,1)\cup\partial B(0,r)),$$

so $w_k=u+Ev_k\to u$ in $W^{1,n}(B(0,1)\setminus B(0,r))$ which implies that

$$\int_{B(0,1)\setminus B(0,r)}|\det\nabla w_k|\,dx\to\int_{B(0,1)\setminus B(0,r)}|\det\nabla u|\,dx=0.$$

Hence

$$\liminf_{k\to\infty}\int_{B(0,r)}|\det\nabla u_k|\,dx\ge|B| \tag{10}$$

which contradicts the convergence $\det\nabla u_k\to\det\nabla u\equiv0$ in L^q. □

Remarks. 1. When $n-1<p<n$ the proof can be slightly simplified. Indeed, by the Sobolev embedding theorem $u_k\to u$ uniformly on the sphere $S(0,r)$. Then the construction invoking the extension operator is not needed as inequality (10) follows rather easily from the Brouwer theorem.

2. The obstacle for the existence of the approximation has a topological nature, it is essential that we create a hole in the image of the mapping. If $p=n$, then we cannot construct counterexamples as above simply because $u:x\mapsto x/|x|$ does not belong to $W^{1,n}(B)^n$.

Corollary 3.4. *The function $u^i(x)=x_i/|x|\in W^{1,p}(B)$, $i\in\{1,\dots,n\}$, $n-1\le p<n$, cannot be approximated by a sequence of continuous functions which satisfy the conditions* (b) *and* (c) *of Theorem* 1.3.

Proof. We argue by contradiction. If there is such a sequence for one i, then it can be found for each i and thus there is an approximating sequence $\{v_k\}_{k=1}^\infty$ for $u(x)=x/|x|$ satisfying the properties (b) and (c) coordinate-wise. Since $\nabla v_k^i(x)\in\{\nabla u^i(x),0\}$ a.e., it follows that the functions v_k are locally Lipschitz continuous outside the origin and thus satisfy the Luzin property. Hence by Corollary 3.1, v_k approximate u in $B_{p,q}$ which contradicts Proposition 3.3. □

4. Weakly monotone functions

We say that $u\in W^{1,p}(\Omega)$, $1\le p<\infty$, is *weakly K-pseudomonotone*, $K\ge 1$, if for every $x\in\Omega$ and a.e. $0<r<\operatorname{dist}(x,\Omega^c)$,

$$\operatorname*{osc}_{B(x,r)} u\le K \operatorname*{osc}_{S(x,r)} u,$$

where the oscillation on the left is essential with respect to the Lebesgue measure and the oscillation on the right is essential with respect to the $(n-1)$-dimensional Hausdorff measure.

This class contains a class of weakly monotone functions, see Section 1. The proof of the fact that weakly monotone functions are weakly 1-pseudomonotone is standard and left to the reader. One may, e.g., use the characterization of the Sobolev space by the absolute continuity on lines, see [12, Section 4.9.2] and [43,Theorem 1.41].

The following result is a slightly stronger version of a result due to Manfredi [44] (cf. [64]).

Theorem 4.1. *Let $u\in W^{1,p}_{\mathrm{loc}}(\Omega)$ be weakly K-pseudomonotone for some $K\ge 1$.*

(1) *If $n-1<p<n$, then u is locally bounded and*

$$\Big(\operatorname*{osc}_{B(x_0,r)} u\Big)^p\le C(n,p)K^p r^p ⨍_{B(x_0,2r)} |\nabla u|^p\,dx, \tag{11}$$

whenever $B(x_0,2r)\subset\Omega$. Moreover there exists a set $Z\subset\Omega$ with $\mathcal{H}^{n-p}(Z)=0$ and such that u is continuous on the set $\Omega\setminus Z$.

(2) *If $p=n$, then u is continuous. Moreover*

$$\Big(\operatorname*{osc}_{B(x_0,r)} u\Big)^n\le \frac{C(n)K^n}{\log(R/r)}\int_{B(x_0,R)} |\nabla u|^n\,dx, \tag{12}$$

whenever $B(x_0,R)\subset\Omega$ and $r<R$.

Remark. The estimate (12) is a generalization of the Courant–Lebesgue lemma, see [7] and [31, Lemma 8.3.5].

Proof. Let $u\in W^{1,p}(\Omega)$, $p>n-1$, be a weakly K-pseudomonotone function, and let $B(x_0,R)\subset\Omega$, $r<R$. By Fubini's theorem, $u\in W^{1,p}(S(x_0,t))$ for almost all $r<t<R$. Hence by Lemma 2.9 we get a family of inequalities

$$\operatorname*{osc}_{S(x_0,t)} u\le C(n,p)t\Big(⨍_{S(x_0,t)} |\nabla u|^p\,d\mathcal{H}^{n-1}\Big)^{1/p}. \tag{13}$$

Integrating those inequalities with respect to t yields

$$\int_r^R \Big(\operatorname*{osc}_{S(x_0,t)} u\Big)^p \frac{dt}{t^{p-n+1}} \le C(n,p) \int_{B(x_0,R)\setminus B(x_0,r)} |\nabla u|^p \, dx. \tag{14}$$

Since u is weakly K-pseudomonotone we get

$$\operatorname*{osc}_{B(x_0,r)} u \le \operatorname*{osc}_{B(x_0,t)} u \le K \operatorname*{osc}_{S(x_0,t)} u \tag{15}$$

for almost all $t\in[r,R]$.

From (14) and (15) we infer (11) and (12). Inequality (12) implies the continuity when $p=n$. The continuity outside a set Z with $\mathcal{H}^{n-p}(Z)=0$ is a direct consequence of inequality (11) and Lemma 2.1. □

Remark. In the case $n-1<p<n$, Manfredi [44] and also Šverák [64] obtained a weaker result. They proved continuity of u outside a set with vanishing p-capacity. The estimate was improved in [55, Theorem 7.4].

Yet another property of weakly K-pseudomonotone mappings has been obtained by Malý and Martio [42], see also [40, Theorems 3.4 and 4.3].

Theorem 4.2. *If $u\in W^{1,n}(\Omega)^n$ is weakly K-pseudomonotone for some $K\ge 1$, then u has the Luzin property.*

The following result is known among specialists as folklore. For the case $p=n$ see [58, Corollary, p. 341] and also [40], the general case can be found in [66].

Theorem 4.3. *If $u\in W^{1,p}_{\mathrm{loc}}(\Omega)$, $p>n-1$, is weakly K-pseudomonotone for some $K\ge 1$, then u is differentiable a.e.*

Proof. If $p>n$, then any Sobolev function is differentiable a.e., so we can assume that $p\le n$. We also can assume that $p<n$, since $W^{1,n}_{\mathrm{loc}}\subset W^{1,p}_{\mathrm{loc}}$ for $p<n$. Then by (11) the condition from the Stepanov theorem (Lemma 2.2) is satisfied, whenever x is a Lebesgue point of $|\nabla u|^p$. □

Observe that Lemmas 1.1 and 1.2 are consequences of the above results.

We close this section by recalling two classes of examples of weakly monotone functions.

Definition. We follow Koskela, Manfredi and Villamor [33]. Let $0<\alpha(x)\le\beta<\infty$ a.e. in Ω, where α is a measurable function and β is a constant. Let $1<p<\infty$ and $\mathcal{A}:\Omega\times\mathbf{R}^n\to\mathbf{R}^n$ be a Carathéodory function such that

$$|\mathcal{A}(x,\xi)|\le\beta|\xi|^{p-1} \quad\text{and}\quad \mathcal{A}(x,\xi)\cdot\xi\ge\alpha(x)|\xi|^p.$$

Weak solutions $u\in W^{1,p}_{\rm loc}(\Omega)$ to the equation $\operatorname{div}\mathcal{A}(x,\nabla u)=0$, are called *$\mathcal{A}$-harmonic functions.* The usual assumption is that $\alpha(x)>\alpha>0$ a.e., but our assumption is much weaker, so the equation may be very degenerate. It is easy to prove, [33], that $\mathcal{A}$-harmonic functions are weakly monotone. Thus the properties of weakly monotone functions yield the following result.

Theorem 4.4. *Let $\mathcal{A}$ be as above, $n-1<p\le n$. Then any $\mathcal{A}$-harmonic function u is locally bounded and continuous outside a set Z with $\mathcal{H}^{n-p}(Z)=0$. Moreover u is differentiable a.e.*

Remarks. 1. The continuity property has been observed in [33], but with a worse estimate for the size of the discontinuity set. The differentiability result generalizes that of Bojarski [4], and Reshetnyak [57]; see also [26], [30] and [32] for related elliptic results, and [62] and [63] for related parabolic results.

2. In the nondegenerate case $\alpha(x)>\alpha>0$ a.e., it is known that any $\mathcal{A}$-harmonic function is Hölder continuous, [43], and differentiable a.e., see [4] and [57].

Definition. We say that the mapping $u\in W^{1,p}(\Omega)^n$, $\Omega\subset\mathbf{R}^n$, has *finite dilatation* if there is a function K, $1\le K(x)<\infty$ a.e., such that

$$|\nabla u(x)|^n\le K(x)\det\nabla u(x)\quad\text{a.e.}$$

In other words finite dilatation means that for almost every point x either $\det\nabla u>0$ or $\nabla u(x)=0$.

Obviously mappings with $\det\nabla u>0$ a.e. have finite dilatation.

Gol′dshteĭn and Vodop′yanov [20], proved that if $p=n$, then the coordinate functions of a mapping with finite dilatation are weakly monotone. The following result is a generalization of the result of Gol′dshteĭn and Vodop′yanov [20], see also [15], [29], [44], [54] and [64].

Theorem 4.5. *If $u\in A_{p,q}(\Omega)$, $p\ge n-1$, $q\ge n/(n-1)$, is a mapping of finite dilatation, then the coordinate functions of u are weakly monotone.*

Proof. We argue by contradiction. Suppose that one of the coordinate functions, say u^1, is not weakly monotone. Then there is $\Omega'\Subset\Omega$ and $m,M\in\mathbf{R}$ such that either $(u^1-M)^+\in W^{1,p}_0(\Omega')$ and $|\{x\in\Omega':u^1(x)>M\}|>0$, or $(m-u^1)^+\in W^{1,p}_0(\Omega')$ and $|\{x\in\Omega':u^1(x)<m\}|>0$. Assume the first case. Let $\bar u^1=\min\{u^1,M\}\chi_{\Omega'}+u^1\chi_{\Omega\setminus\Omega'}$ and $\bar u=(\bar u^1,u^2,\dots,u^n)$. Obviously $\bar u\in A_{p,q}(\Omega)$. We will prove in a while that

$$\int_\Omega\det\nabla u\,dx=\int_\Omega\det\nabla\bar u\,dx. \tag{16}$$

Before we do this, however, we show how to complete the proof of the theorem. Let $E=\{x\in\Omega':u^1(x)>M\}$. Observe that $\det\nabla\bar u=0$ in E. This, identity (16) and the fact that $u=\bar u$ in $\Omega\setminus E$ imply that

$$\int_E \det\nabla u\,dx=\int_E\det\nabla\bar u\,dx=0.$$

Hence the finiteness of the dilatation implies that $\nabla u\equiv 0$ in E which is not possible.

Thus we are left with the proof of (16).

Let $\psi\in C_0^\infty(\Omega)$ such that $\psi|_{\Omega'}\equiv 1$. Employing Lemma 2.12 we obtain

$$\begin{aligned}\int_\Omega(\det\nabla u-\det\nabla\bar u)\,dx&=\int_\Omega\psi(\det\nabla u-\det\nabla\bar u)\,dx\\&=\int_\Omega\psi(du^1\wedge du^2\wedge\ldots\wedge du^n-d\bar u^1\wedge du^2\wedge\ldots\wedge du^n)\\&=-\int_\Omega(u^1-\bar u^1)\,d\psi\wedge du^2\wedge\ldots\wedge du^n=0.\quad\Box\end{aligned}$$

5. The proof of the main result

This section is devoted to the proof of Theorem 1.3. The proof of the theorem is quite difficult so we start with describing the main idea. We hope it will help to understand the steps of the proof.

The rough idea is the following. Applying infinitely many corrections "from above" to u we make the function "locally weakly upper monotone". Then applying infinitely many corrections "from below" we make it "locally weakly lower monotone". The resulting function is locally weakly monotone. Now we describe the corrections "from above". The function fails to satisfy the weak maximum principle on an open set $\Omega'\Subset\Omega$ if there is a $t\in\mathbf{R}$ such that $(u-t)^+\in W_0^{1,p}(\Omega')$ and $u>t$ on some subset of Ω' with positive Lebesgue measure.

This suggests the method of corrections "from above". We fix $R>0$. Whenever $E\subset\Omega$ is a set with diameter less than or equal to R and such that $u>t$ a.e. in E and $(u-t)\chi_E\in W_0^{1,p}(\Omega)$ we replace u with the *upper truncation* $v=t\chi_E+u\chi_{\Omega\setminus E}$.

The *upper R-correction* is defined as the infimum of all upper truncations over all t real and E as above. We prove then that the resulting function satisfies the weak maximum principle on all open sets $\Omega'\Subset\Omega$ with diameter less than or equal to R.

This is a delicate construction as a priori we take the infimum over an uncountable set of functions. Since each function is measurable, and hence defined up to a set of measure zero, such an infimum does not really make sense. Thus we have to be very careful in our constructions.

Then we apply similar corrections "from below" to the modified function to make the function satisfying the weak minimum principles on open sets $\Omega' \Subset \Omega$ with diameter less than or equal to R. The resulting function is locally weakly monotone.

Next we prove that passing to the limit as $R \to 0$ gives the desired approximation.

Observe that in each step of the correction we get a function v with $\nabla v = 0$ on the set $\{x : u(x) \neq v(x)\}$. This property is stable when we take the infimum and hence property (b) follows.

To prove (c) we use the fact that the L^p-distance of the R-correction from u is limited by the scaling constant in the Poincaré inequality which behaves as R.

The proof is quite long and difficult, so we divide it into several lemmas.

It is customary to call the symbols $\cup$ and $\cap$, cup and cap, respectively. The shape of the symbols explains the terminology that follows.

First we define the class of sets on which we will truncate the function u.

Let $v \in W^{1,p}(\Omega)$ and $a \in \mathbf{R}$. We say that a Borel set $E \subset \Omega$ is an *a-cap set* (*a-cup set*) for v if $v > a$ ($v < a$) a.e. on E and $(v-a)\chi_E \in W_0^{1,p}(\Omega)$. We say that E is a *cap set* (*cup set*) for v if E is an a-cap set (a-cup set) for v and some $a \in \mathbf{R}$.

Let E be an a-cap set or an a-cup set of positive measure for v. Since no characteristic function of a set of positive measure can be in $W_0^{1,p}(\Omega)$, the value a is uniquely determined by the function v and the set E. (In fact, it is the essential infimum (for a cap set) or essential supremum (for a cup set) of v over E.) We set

$$v^E = (v-a)\chi_E.$$

We say that v is *R-capless* (*R-cupless*) if all cap (cup) sets for v of diameter less than or equal to R have measure 0.

Lemma 5.1. *If $u \in W^{1,p}(\Omega)$ is R-capless and $\Omega' \Subset \Omega$ with* $\operatorname{diam} \Omega' \le R$, *then u satisfies the weak maximum principle on Ω'. If u is R-cupless and $\Omega' \Subset \Omega$ with* $\operatorname{diam} \Omega' \le R$, *then u satisfies the weak minimum principle on Ω'. Hence any function u that is both R-capless and R-cupless is locally weakly monotone.*

Proof. Assume that $u \in W^{1,p}(\Omega)$ is R-capless. Suppose that $\Omega' \Subset \Omega$ is an open set with $\operatorname{diam} \Omega' \le R$ and $(u-t)^+ \in W_0^{1,p}(\Omega')$. If $E = \{x \in \Omega' : u(x) > t\}$, then $(u-t)\chi_E = (u-t)^+ \chi_{\Omega'} \in W_0^{1,p}(\Omega)$ and thus E is a t-cap set for u. Since u is R-capless, $|E| = 0$ and thus $u \le t$ a.e. on Ω'. This verifies the weak maximum principle on Ω'. Similarly we can prove the statement about the weak minimum principle and the final part is obvious. $\square$

The last lemma suggests that to make the appropriate corrections, the right idea will be to "remove caps and cups".

First we need to prove some nice properties of cap sets and truncations.

Lemma 5.2. *Let* $v\in W^{1,p}(\Omega)$. *If* E *is an* a*-cap set for* v, F *is a* b*-cap set for* v *and* $a\le b$, *then* $F\setminus E$ *is a* b*-cap set for* v. *Hence* $\max\{v^E,v^F\}=\max\{v^E,v^{F\setminus E}\}=v^E+v^{F\setminus E}$.

Proof. The first assertion follows from the identity

$$(v-b)\chi_{F\setminus E}=v^F-\min\{v^F,v^E\}.$$

The second one is an obvious consequence. $\square$

Lemma 5.3. *Let* $v\in W^{1,p}(\Omega)$, $R>0$, $\mathcal{E}$ *be a subclass of the family of all cap sets* E *for* v *with* $\operatorname{diam} E\le R$ *and*

$$w=\bigvee\{v^E:E\in\mathcal{E}\}.$$

Then $w\in W_0^{1,p}(\Omega)$,

$$\nabla w=\nabla v\chi_{\{x:w(x)>0\}} \tag{17}$$

and

$$\|w\|_p\le CR\|\nabla v\|_p. \tag{18}$$

Proof. First let us assume that $\mathcal{E}$ is a finite family. Then we use Lemma 5.2 to show that we may pass to a finite disjointed family $\mathcal{F}$ of cap sets F for v with $\operatorname{diam} F\le R$, such that

$$w=\bigvee\{v^F:F\in\mathcal{F}\}=\sum_{F\in\mathcal{F}}v^F.$$

This proves that $w\in W_0^{1,p}(\Omega)$ and (17) holds. Fix $F\in\mathcal{F}$. Then there is a ball $B(z,R)$ in $\mathbf{R}^n$ which contains F. The function v^F can be extended to a function in $W_0^{1,p}(B(z,R))$ by setting $v^F=0$ outside Ω. By the Poincaré inequality (Lemma 2.8) we have

$$\int_F|v^F|^p\,dx=\int_{B(z,R)}|v^F|^p\,dx\le CR^p\int_{B(z,R)}|\nabla v^F|^p\,dx=CR^p\int_F|\nabla v|^p\,dx.$$

Summing over $F\in\mathcal{F}$ we obtain

$$\int_\Omega|w|^p\,dx\le CR^p\int_\Omega|\nabla v|^p\,dx.$$

Now we consider the general case of $\mathcal{E}$. Let

$$\mathcal{U}=\{\max\{v^{E_1},\dots,v^{E_k}\}:E_1,\dots,E_k\in\mathcal{E} \text{ for some } k\}.$$

It follows from formulas (17) and (18), proved above for finite maxima, that $\mathcal{U}$ is bounded in $W_0^{1,p}(\Omega)$. Hence Lemma 2.7 implies that $w=\bigvee\mathcal{U}\in W_0^{1,p}(\Omega)$ and there is a countable subfamily $\{E_1,E_2,\dots\}\subset\mathcal{E}$ such that $\{w_k\}_{k=1}^{\infty}$,

$$w_k=\max\{v^{E_1},\dots,v^{E_k}\},$$

is increasing and $\lim_{k\to\infty} w_k=w$. By the preceding step,

$$\nabla w_k=\nabla v\chi_{\{x:w_k(x)>0\}} \quad\text{and}\quad \|w_k\|_p\le CR\|\nabla v\|_p. \tag{19}$$

We have the a.e. pointwise convergence $w_k\to w$ and $\nabla w_k\to\nabla v\chi_{\{x:w(x)>0\}}$. Further, w^p is a majorant to $|w_k-w|^p$ and $|\nabla v|^p$ is a majorant to $|\nabla w_k-\nabla v\chi_{\{x:w(x)>0\}}|^p$. Hence by the Lebesgue dominated convergence theorem, $w_k\to w$ in $W_0^{1,p}(\Omega)$ and (17) holds. Passing to the limit in (19) we obtain (18). □

Definition. Let $v\in W^{1,p}(\Omega)$. We introduce

$$\begin{aligned}\overline{M}_a^R v&=\bigwedge\{v-v^E:E \text{ is an } a\text{-cap set for } v \text{ with } \operatorname{diam}E\le R\}\\ &=v-\bigvee\{v^E:E \text{ is an } a\text{-cap set for } v \text{ with } \operatorname{diam}E\le R\},\\ \overline{M}^R v&=\bigwedge\{v-v^E:E \text{ is a cap set for } v \text{ with } \operatorname{diam}E\le R\}\\ &=v-\bigvee\{v^E:E \text{ is a cap set for } v \text{ with } \operatorname{diam}E\le R\}\\ &=\bigwedge\{\overline{M}_a^R v:a\in\mathbf{R}\}.\end{aligned}$$

The function $\overline{M}^R v$ is called the *upper R-correction* of v. Similarly we define the *lower R-correction* of v as

$$\underline{M}^R v=\bigvee\{v-v^E:E \text{ is a cup set for } v \text{ with } \operatorname{diam}E\le R\}.$$

By Lemma 5.3, $v-\overline{M}^R v\in W_0^{1,p}(\Omega)$,

$$\nabla\overline{M}^R v=\nabla v\chi_{\{x:\overline{M}^R v(x)=v(x)\}}, \tag{20}$$

and similarly for $\overline{M}_a^R v$ and $\underline{M}^R v$. We easily observe that

$$E \text{ is an } a\text{-cap set for } v \text{ with } \operatorname{diam}E\le R \quad\Longrightarrow\quad \overline{M}_a^R v=a \text{ a.e. on } E. \tag{21}$$

It is less obvious than it perhaps seems to be that the cap sets for the upper R-correction of v with diameter less than or equal to R are cap sets for v and thus removed. Before we prove that the upper R-correction of v is R-capless (Lemma 5.6) we need some partial steps.

Lemma 5.4. *Let* $v \in W^{1,p}(\Omega)$ *and* $a \in \mathbf{R}$. *Then each* a*-cap set* E *for* $\overline{M}_a^R v$ *with* $\operatorname{diam} E \le R$ *has measure zero.*

Proof. Let E be an a-cap set for $\overline{M}_a^R v$ with $\operatorname{diam} E \le R$. Then $\overline{M}_a^R v > a$ a.e. on E. Obviously $\overline{M}_a^R v = v$ a.e. on $\{x : \overline{M}_a^R v(x) > a\}$. Hence $(v-a)\chi_E = (\overline{M}_a^R v - a)\chi_E \in W_0^{1,p}(\Omega)$ and thus E is an a-cap set for v. Now (21) yields $\overline{M}_a^R v = a$ a.e. in E, and hence $|E| = 0$. $\square$

Lemma 5.5. *Let* $v \in W^{1,p}(\Omega)$ *and* $a \in \mathbf{R}$. *Then*

$$\overline{M}^R v \ge a \quad \text{a.e. on } \{x : \overline{M}_a^R v(x) > a\}. \tag{22}$$

Proof. We write

$$w_b = \overline{M}_b^R v, \ b \in \mathbf{R}, \quad \text{and} \quad w = \overline{M}^R v. \tag{23}$$

Consider $s \in \mathbf{R}$. We claim that

$$w_s > a \quad \text{a.e. on } \{x : w_a(x) > a\}. \tag{24}$$

If $s > a$, then obviously

$$w_s \ge \min\{v, s\} \ge \min\{w_a, s\} > a \quad \text{a.e. on } \{x : w_a(x) > a\}.$$

There is nothing to prove for $s = a$. Suppose that $s < a$. Let E be an s-cap set for v with $\operatorname{diam} E \le R$. Observe that $\{x \in E : v(x) > a\}$ is an a-cap set for v. Indeed, this follows from Corollary 2.4 as

$$0 \le (v-a)^+ \le v - s \quad \text{a.e. on } E.$$

Thus by (21)

$$w_a = a \quad \text{a.e. on } \{x \in E : v(x) > a\}. \tag{25}$$

Passing to the lattice infimum, Lemma 2.6 and (25) imply that

$$w_a = a \quad \text{a.e. in } \{x : v(x) > \max\{a, w_s(x)\}\}.$$

Since $\{x : w_a(x) > a\} \supset \{x : v(x) > a\}$ we conclude that

$$w_s = v \ge w_a > a \quad \text{a.e. on } \{x : w_a(x) > a\}$$

which proves the claim. Passing to the lattice infimum in (24) we obtain (22). $\square$

Lemma 5.6. *Let* $v\in W^{1,p}(\Omega)$. *Then* $\overline{M}^R v$ *is R-capless.*

Proof. We continue using the notation (23). Consider $t\in\mathbf{R}$ and a t-cap set F for w with $\operatorname{diam} F\le R$. We claim that if $a>t$, then $\{x\in F:w_a(x)>a\}$ is an a-cap set for w_a. Indeed, Lemma 5.5 yields the inequality

$$0\le \min\{(w_a-a)\chi_{\{x:w_a(x)>a\}},a-t\}\le w-t \quad \text{a.e. on } F,$$

and then Corollary 2.4 implies that

$$\min\{(w_a-a)\chi_{\{x\in F:w_a(x)>a\}},a-t\}\in W_0^{1,p}(\Omega).$$

Hence Lemma 2.5 gives $(w_a-a)\chi_{\{x\in F:w_a(x)>a\}}\in W_0^{1,p}(\Omega)$, which proves the claim. Thus by Lemma 5.4, $|\{x\in F:w_a(x)>a\}|=0$. We infer that $w\le w_a\le a$ a.e. on F. Since $a>t$ was arbitrary, we conclude that $w\le t$ a.e. on F and thus $|F|=0$. The proof is complete. □

The next important step is the following lemma.

Lemma 5.7. *If* $v\in W^{1,p}(\Omega)$ *is R-cupless, then* $\overline{M}^R v$ *is R-cupless as well.*

Proof. We write

$$w=\overline{M}^R v.$$

Let F be a b-cup set for w with $\operatorname{diam} F\le R$. Then

$$0\le (b-v)^+\le b-w \quad \text{a.e. on } F.$$

By Corollary 2.4, $(b-v)\chi_{\{x\in F:v(x)<b\}}\in W_0^{1,p}(\Omega)$ and hence $\{x\in F:v(x)<b\}$ is a b-cup set for v. Since v is R-cupless, it follows that

$$v\ge b \quad \text{a.e. on } F. \tag{26}$$

Now, let E be an a-cap set for v with $\operatorname{diam} E\le R$. We claim that

$$v-v^E\ge b \quad \text{a.e. on } F\cap E. \tag{27}$$

To prove this, we distinguish two cases. If $a\ge b$, then (27) holds as

$$v-v^E=a\ge b \quad \text{a.e. on } F\cap E.$$

Let us suppose that $a<b$. Then by (26) and the fact that $w\le v-v^E=a$ a.e. on E we conclude that

$$0\le b-a\le \min\{(v-a)\chi_E,(b-w)\chi_F\}=\min\{v^E,-w^F\} \quad \text{a.e. on } F\cap E.$$

Since

$$\min\{v^E,-w^F\}\chi_{F\cap E}=\min\{v^E,-w^F\}\in W_0^{1,p}(\Omega),$$

we get from Corollary 2.4 that $(b-a)\chi_{F\cap E}\in W_0^{1,p}(\Omega)$ which implies that $|F\cap E|=0$ and (27) holds as well. By (26),

$$v-v^E=v\geq b \quad \text{a.e. on } F\setminus E.$$

This together with (27) yields

$$v-v^E\geq b \quad \text{a.e. on } F. \tag{28}$$

Passing to the lattice infimum with the aid of Lemma 2.6 we obtain

$$w\geq b \quad \text{a.e. on } F.$$

This shows that $|F|=0$. We have proved that w is R-cupless. □

Now we complete the proof of the whole theorem as follows.

Proof of Theorem 1.3. Let $R_k\searrow 0$. We set

$$w_k=\underline{M}^{R_k}u.$$

Write

$$E_k=\{x:w_k(x)\neq u(x)\} \quad \text{and} \quad E=\bigcap_{k=1}^{\infty}E_k.$$

Then from the definition of the corrections it easily follows that for $i>j$ we have $u\leq\underline{M}^{R_i}u\leq\underline{M}^{R_j}u$ a.e., and hence neglecting sets of measure zero we get inclusions

$$E_1\supset E_2\supset \dots\ .$$

By (20),

$$\nabla w_k=\nabla u\chi_{\Omega\setminus E_k}.$$

Hence

$$\nabla w_k\to\nabla u\chi_{\Omega\setminus E} \quad \text{in } L^p(\Omega)^n. \tag{29}$$

Now Lemma 5.3 yields

$$\|w_k-u\|_p\leq CR_k\|\nabla u\|_p. \tag{30}$$

We deduce from (29) and (30) that $w_k \to u$ in $W^{1,p}(\Omega)$ and

$$\nabla u = \nabla u \chi_{\Omega \setminus E} \quad \text{a.e. in } \Omega.$$

Now, we set

$$w_{k,j} = \overline{M}^{R_j} w_k.$$

Then as above we obtain that

$$\nabla w_{k,j} = \nabla u \chi_{\{x : u(x) = w_k(x) = w_{k,j}(x)\}} \tag{31}$$

and

$$\lim_{j\to\infty} \|w_{k,j} - w_k\|_{1,p} = 0. \tag{32}$$

Hence we may find $j(k) \geq k$ such that

$$\lim_{k\to\infty} \|w_{k,j(k)} - w_k\|_{1,p} = 0.$$

Set

$$u_k = w_{k,j(k)}.$$

Then u_k converges to u in $W^{1,p}(\Omega)$, which is (c). From (31) we obtain (b). By the mirror version of Lemma 5.6, w_k is R_k-cupless and by Lemma 5.6 and Lemma 5.7, u_k is both $R_{j(k)}$-capless and cupless. Hence, by Lemma 5.1, u_k is locally weakly monotone, and this is (a). The proof is finished. □

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