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Traces of Sobolev Functions on Fractal Type Sets and Characterization of Extension Domains*

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We describe traces of Sobolev functions $u \in W^{1,p}(\mathbb{R}^n)$, $1 < p \leqslant \infty$, on certain subsets of $\mathbb{R}^n$ in terms of Sobolev spaces on metric spaces [7]. Our results apply to smooth submanifolds, fractal subsets, as well as to open subsets of $\mathbb{R}^n$. In particular if $\Omega \subset \mathbb{R}^n$ is a John domain, then we characterize those $W^{1,p}(\Omega)$ functions which can be extended to $W^{1,p}(\mathbb{R}^n)$. If Ω is uniform, then this result implies Jones' extension theorem [14]. In the case of traces on fractal subsets our results are related to those of Jonsson and Wallin [16].

1. INTRODUCTION

If $\Omega \subset \mathbb{R}^n$ is an open set and $1 \leqslant p \leqslant \infty$, then the Sobolev space is defined as

$$W^{1,p}(\Omega) = \{u \in L^p(\Omega) \colon \nabla u \in L^p(\Omega)\}.$$

Here the gradient is in the distributional sense. This space is equipped with the norm $\|u\|_{1,p} = \|u\|_p + \|\nabla u\|_p$. The subset of continuous Sobolev functions is dense in $W^{1,p}(\Omega)$.

In the paper we are concerned with the problem of description of traces of Sobolev functions $u \in W^{1,p}(\mathbb{R}^n)$ on compact subsets of $\mathbb{R}^n$. If $K \subset \mathbb{R}^n$ is

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a compact set, then for $u \in W^{1,p}(\mathbb{R}^n) \cap C^0(\mathbb{R}^n)$ we define the trace operator as a restriction $T(u) = u|_K$. Assume that the set K supports a finite Borel measure μ. The problem is to find a Banach space $X(K, \mu)$ of μ-measurable functions on K (convergence in the norm of X implies convergence in measure, moreover two functions are identified in $X(K, \mu)$ if and only if they are equal except a set of μ-measure 0) such that the trace operator extends to a bounded linear operator

$$\mathrm{T}\colon W^{1,p}(\mathbb{R}^n) \to X(K, \mu). \tag{1}$$

The converse problem is the problem of extension. Given a Banach space $X(K, \mu)$ of measurable functions on K, such that the subset of continuous functions $C^0(K) \cap X(K, \mu)$ is dense, we want to find a bounded linear operator

$$\mathrm{E}\colon X(K, \mu) \to W^{1,q}(\mathbb{R}^n) \tag{2}$$

such that $\mathrm{E}(u)$ is continuous on $\mathbb{R}^n$ whenever u is continuous on K (and hence $(\mathrm{E}u)|_K = u$ for such u).

If one can construct operators (1) and (2) with $p = q$, then we say that the space $X(K, \mu)$ characterizes traces on (K, μ) of functions in $W^{1,p}(\mathbb{R}^n)$. It is important to note the following elementary uniqueness result.

LEMMA 1. *If both spaces $X(K, \mu)$ and $Y(K, \mu)$ characterize traces of $W^{1,p}(\mathbb{R}^n)$ on (K, μ), then $X(K, \mu) = Y(K, \mu)$ as a set and the norms are equivalent.*

Proof. Denote the trace and extension operators corresponding to X and Y by a suitable subscript. Since the trace operators are surjective, the set W of restrictions of $C_0^\infty(\mathbb{R}^n)$ functions to K, is dense in both spaces X and Y. Note that $T_Y E_X\colon X \to Y$ is bounded and $T_Y E_X|_W = \mathrm{id}|_W$, hence $\mathrm{id}\colon X \to Y$ is bounded. In the same way we prove that $\mathrm{id}\colon Y \to X$ is bounded. This ends the proof.

If K is a smooth submanifold of $\mathbb{R}^n$, then the characterization of traces is well known. The theorem of Gagliardo, [5], [18, Theorems 8.3.13 and 6.9.2], states that if $M^{n-1} \subset \mathbb{R}^n$ is a sufficiently smooth, compact, $(n-1)$-dimensional submanifold, and $1 < p < \infty$, then there exist trace and extension operators

$$\mathrm{T}\colon W^{1,p}(\mathbb{R}^n) \to W^{1-1/p,p}(M^{n-1}),$$

$$\mathrm{E}\colon W^{1-1/p,p}(M^{n-1}) \to W^{1,p}(\mathbb{R}^n)$$

where the Slobodeckiĭ space $W^{1-1/p,p}(M^{n-1})$ consists of all functions u such that

$$\|u\|_S^p = \int_{M^{n-1}} \int_{M^{n-1}} \frac{|u(x)-u(y)|^p}{|x-y|^{n+p-2}}\, dx\, dy < \infty.$$

Here the integration is with respect to the Hausdorff measure H^{n-1}. The norm in the Slobodeckiĭ space is $\|u\|_S + \|u\|_{L^p(M^{n-1}, H^{n-1})}$.

Concerning the smoothness condition for M^{n-1}, it suffices to assume that M^{n-1} is locally a graph of a Lipschitz function. The theorem of Gagliardo generalizes to the case of lower dimensional submanifolds. The proof of Gagliardo's theorem strongly involves the fact that M^{n-1} is a regular submanifold of $\mathbb{R}^n$. In the paper we are concerned with traces on much more general subsets which include many fractals. Gagliardo's theorem gives a sharp description of traces on smooth submanifolds, while our results lead to a "nearly sharp" description of traces on much more general subsets.

Our approach simultaneously applies to the problem of traces on lower dimensional subsets as well as to the problem of traces on open subsets of $\mathbb{R}^n$. Extensions from lower dimensional subsets of $\mathbb{R}^n$ (fractals and submanifolds) and from open subsets of $\mathbb{R}^n$ also get a unified treatment. Previously these problems have been treated independently. In one case our methods lead to a sharp characterization of traces: For an open set $\Omega \subset \mathbb{R}^n$ with the $A(c)$ property (see Section 2 for definition) we characterize the subspace consisting of those $W^{1,p}(\Omega)$ functions which can be extended to $W^{1,p}(\mathbb{R}^n)$ (Theorem 9). In a particular case when Ω is a uniform domain our result implies the celebrated Jones extension theorem, [14] (Theorem 11). For a discussion of the trace and extension properties of Sobolev functions with examples and historical remarks see the monograph of Maz'ya [23].

We interprete the trace space $X(K, \mu)$ as a Sobolev space in a very general setup of Sobolev spaces on metric spaces introduced by the first author [7]. It was suggested to us by Paweł Strzelecki that this generalized approach may be useful for the problem of description of traces.

The approach to traces of Besov spaces on fractal type subsets was developed by Jonsson and Wallin, [16], and in a more general form by Jonsson, [15]. Their results apply to the Sobolev space $W^{1,2}$. Although their approach involves different ideas, concerns Besov spaces rather than Sobolev spaces, and is much more technical, their results are strongly related to ours.

The paper is organized as follows. In Section 2 we recall some basic definitions and results that will be used in the sequel. We are concerned there with Whitney decomposition, the $A(c)$ condition, John and uniform domains, classical Sobolev spaces and Sobolev spaces on metric spaces. In Section 3 we reformulate the classical trace theorem of Gagliardo in terms of Sobolev spaces on metric spaces. In Section 4 we prove a general trace theorem. Section 5 is devoted to the construction of the extension operators.

2. PRELIMINARIES

In the paper C will denote a general constant which may change even in a single string of estimates. Writing $C=C(n,p,\alpha)$ we mean that the constant depends on n, p, and α only. We will write $u\approx v$ to express that there are two positive constants C_1 and C_2 such that $C_1 u\leqslant v\leqslant C_2 u$. The average of u with respect to a measure μ will be denoted by $u_K=\fint_K u\,d\mu=\mu(K)^{-1}\int_K u\,d\mu$. By H^k we will denote the k-dimensional Hausdorff measure. Balls will be denoted by $B(x,r)$. Symbol Q will be reserved for a cube in $\mathbb{R}^n$ and $l(Q)$ will denote the side length of Q. Moreover Ω will always stand for a domain, and K a compact subset of $\mathbb{R}^n$. The Lebesgue measure of A will be simply denoted by $|A|$.

For a compact set $K\subset\mathbb{R}^n$, we will use the Whitney decomposition of $\mathbb{R}^n\setminus K$ into closed dyadic cubes Q_j^k, that is

$$\mathbb{R}^n\setminus K=\bigcup_{k=-\infty}^{\infty}\bigcup_{j=1}^{N_k} Q_j^k, \tag{3}$$

where each of the cubes Q_j^k has the edges parallel to the coordinate axes and side length 2^{-k}. Moreover interiors of these cubes are pairwise disjoint and

$$\operatorname{diam} Q_j^k\leqslant \operatorname{dist}(Q_j^k,K)\leqslant 4\operatorname{diam} Q_j^k.$$

Note that a simple packing argument shows that

$$N_k\leqslant C2^{kn} \tag{4}$$

for all $k\geqslant 0$ where $C=C(K,n)$.

We will need the following lemma, see [22, Lemma 3.4].

LEMMA 2. *If $K\subset\mathbb{R}^n$ is a compact set with $|K|=0$ and if there exist $C\geqslant 1$ and $s<n$ such that $N_k\leqslant C2^{ks}$ for all $k\geqslant 0$, then $H^s(K)<\infty$.*

Remark. The assumption $|K|=0$ is necessary. Indeed, it suffices to consider $K=[0,1]^2\subset\mathbb{R}^2$ and $s=1$.

Following Trotsenko, [27], we say that a bounded domain $\Omega \subset \mathbb{R}^n$ satisfies the $A(c)$-condition, $0 < c < 1$, if for every $x \in \partial\Omega$ and every $0 < r \leqslant \operatorname{diam} \Omega$ there exists $y \in \Omega$ such that $B(y, cr) \subset \Omega \cap B(x, r)$. Our definition is slightly different from that of [27]. Roughly speaking, the $A(c)$ condition says that Ω cannot be "thin" close to $\partial\Omega$.

If Ω satisfies the $A(c)$ condition, then the estimate (4) can be improved in the spirit of Lemma 2.

Lemma 3 ([22, Lemma 2.8], [27]). *If $K = \bar{\Omega}$ where $\Omega \subset \mathbb{R}^n$ is a bounded domain with the $A(c)$ property, then there exist $C \geqslant 1$ and $s < n$ such that*

$$N_k \leqslant C 2^{ks}$$

for all $k \geqslant 0$. Moreover $H^s(\partial\Omega) < \infty$.

For stronger results, see [22].

Now we give important examples of domains with the $A(c)$ condition.

We say that a bounded domain $\Omega \subset \mathbb{R}^n$ is John if there is a constant $C \geqslant 1$, and a distinguished point $x_0 \in \Omega$, so that each point $x \in \Omega$ can be joined to x_0 (inside Ω) by a rectifiable curve $\gamma: [0, l] \to \Omega$, $\gamma(0) = x$, $\gamma(l) = x_0$, parametrized by arc-length (l depends on x), and such that the distance to the boundary satisfies

$$\operatorname{dist}(\gamma(t), \partial\Omega) \geqslant C^{-1} t$$

for all $t \in [0, l]$.

An important class of John domains is formed by uniform domains. We say that a bounded domain Ω is uniform if there exists a constant $c \geqslant 1$ such that for any pair x, y of points in Ω we can find a curve $\gamma: [0, l] \to \Omega$ parametrized by arc-length and such that $\gamma(0) = x$, $\gamma(l) = y$, $l \leqslant c|x - y|$, and

$$\operatorname{dist}(\gamma(t), \partial\Omega) \geqslant c^{-1} \min\{t, l - t\}.$$

The definition of uniform domain can be extended also to the case of unbounded domains, but, for simplicity, we will restrict our attention only to bounded domains.

Evidently John and uniform domains satisfy the $A(c)$ condition for suitable c, and hence Lemma 3 applies.

Now we recall some results from the theory of Sobolev spaces on metric spaces introduced in the paper of the first author [7].

First, to see the motivation, we start with the results concerning classical Sobolev spaces. The following lemma appeared in [8]. Various versions of the lemma, with $\lambda = 0$, have appeared independently.

LEMMA 4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary and $0 \leqslant \lambda < 1$. Then there exist constants $C_1 = C_1(\Omega, \lambda)$ and $C_2 = C_2(\Omega)$ such that for every $u \in W^{1,1}(\Omega)$ the inequality*

$$|u(x) - u(y)| \leqslant C_1 \, |x-y|^{1-\lambda} \, (M^{\lambda}_{C_2 |x-y|} \, |\nabla u| \, (x) + M^{\lambda}_{C_2 |x-y|} \, |\nabla u| \, (y)) \tag{5}$$

holds almost everywhere. Here $M^{\lambda}_R g(x) = \sup_{r<R} r^{\lambda} \fint_{B(x,r)} |g(z)| \, dz$ is the fractional maximal operator, and we put $|\nabla u| = 0$ outside Ω.

Since the statement of the lemma slightly differs from that given in [8], we present here a proof. We will combine arguments from [8] with that of [20] (cf. [2, Section 6]).

In Section 5 we will present a method (based on a different idea) which leads to the much more general result, see (23).

Proof of Lemma 4. Let $u \in W^{1,p}(\Omega)$. It is easy to see that there is a constant L, which depends on Ω only, such that for every two points x, $y \in \Omega$ there is a L-bi-Lipschitz mapping from a ball $T \colon B(0, |x-y|) \to \Omega$ such that $x, y \in T(B(0, |x-y|))$.

If $B \subset \mathbb{R}^n$ is a ball, then the inequality

$$|w(x) - w_B| \leqslant C(n) \int_B \frac{|\nabla w(z)|}{|x-z|^{n-1}} \, dz \tag{6}$$

holds for all $w \in W^{1,1}(B)$, and almost every $x \in B$, see [6, Theorem 7.16].

Let $B = B(0, |x-y|)$ and $A = T(B)$. Applying (6) to $w = u \circ T$ and then changing the variables in the resulting inequality we obtain that

$$|u(x) - (u \circ T)_B| \leqslant C(n) \, L^{2n} \int_A \frac{|\nabla u(z)|}{|x-z|^{n-1}} \, dz$$

for almost all $x \in A$.

We recall the following elementary lemma [8, Lemma 2], [28, Lemma 2.8.3].

LEMMA 5. *If $B(R) \subset \mathbb{R}^n$ is a ball with the radius R, $0 \leqslant \lambda < 1$ and $g \in L^1(B(R))$, then*

$$\int_{B(R)} \frac{|g(z)|}{|x-z|^{n-1}} \, dz \leqslant C(n, \lambda) \, R^{1-\lambda} M^{\lambda}_{2R} \, g(x),$$

fort all $x \in B(R)$. Here we set $g = 0$ on $\mathbb{R}^n \setminus B(R)$.

Since the mapping T is L-bi-Lipschitz, A is contained in a certain ball D with the radius $L\,|x-y|$. Hence Lemma 4 follows from the estimates:

$$\begin{aligned}|u(x)-u(y)| &\leqslant |u(x)-(u\circ T)_B| + |u(y)-(u\circ T)_B| \\ &\leqslant C\left(\int_D \frac{|\nabla u(z)|}{|x-z|^{n-1}}\,dz + \int_D \frac{|\nabla u(z)|}{|y-z|^{n-1}}\,dz\right) \\ &\leqslant C\,|x-y|^{1-\lambda}\,(M^{\lambda}_{2L|x-y|}\,|\nabla u|\,(x) + M^{\lambda}_{2L\,|x-y|}\,|\nabla u|(y)).\end{aligned}$$

As a corollary we obtain that if $u\in W^{1,p}(\Omega)$, where Ω is a bounded domain with the Lipschitz boundary or $\Omega=\mathbb{R}^n$, and $1\leqslant p\leqslant\infty$, then

$$|u(x)-u(y)| \leqslant C\,|x-y|\,(M\,|\nabla u|\,(x)+M\,|\nabla u|\,(y)). \tag{7}$$

Here $Mg(x)=\sup_{r>0} \fint_{B(x,r)} |g(z)|\,dz$ is a Hardy–Littlewood maximal operator and we put $|\nabla u|=0$ outside Ω. Note that if $1<p\leqslant\infty$, then according to the Hardy–Littlewood maximal theorem $M\,|\nabla u|$ belongs to $L^p(\Omega)$.

Another consequence of Lemma 4 is the following: If $\Omega\subset\mathbb{R}^n$ is an arbitrary domain and $K\subset\Omega$ is a compact subset, then there exist constants $C\geqslant 1$ and $h\geqslant \operatorname{diam} K$, such that for $u\in W^{1,p}(\Omega)$,

$$|u(x)-u(y)| \leqslant C\,|x-y|^{1-\lambda}\,(M^{\lambda}_h\,|\nabla u|\,(x)+M^{\lambda}_h\,|\nabla u|\,(y)), \tag{8}$$

holds for almost all $x, y\in K$. Here as before $|\nabla u|=0$ *on* $\mathbb{R}^n\backslash\Omega$.

The following result was proved in [9, Proposition 1]. A weaker version was obtained earlier in [7, Theorem 1].

LEMMA 6. *If $\Omega\subset\mathbb{R}^n$ is an arbitrary open set and $u\in L^p(\Omega)$ satisfies*

$$|u(x)-u(y)| \leqslant |x-y|\,(g(x)+g(y))\ a.e. \tag{9}$$

with $g\in L^p(\Omega)$, $g\geqslant 0$, where $1\leqslant p\leqslant\infty$, then $u\in W^{1,p}(\Omega)$ and $|\nabla u|\leqslant 2\sqrt{n}\,g$ a.e.

Remarks. (1) When we say that the inequality of the type (9) holds a.e., we mean (now and in the sequel) that there exists a set $F\subset\Omega$ of measure zero, such that (9) holds for all $x, y\in\Omega\backslash F$.

(2) The proof in [9] gives the estimate $|\nabla u|\leqslant 4\sqrt{n}\,g$. The better estimate $|\nabla u|\leqslant 2\sqrt{n}\,g$ can be obtained by a minor modification of the proof or by an application of Lemma 13.

Since the Hardy–Littlewood maximal operator is bounded in L^p for $1<p\leqslant\infty$, inequality (7) and Lemma 6 lead to the following characterization of the Sobolev space.

THEOREM 1 ([7]). *If* $\Omega \subset \mathbb{R}^n$ *is a bounded domain with a Lipschitz boundary and* $1 < p \leqslant \infty$, *then* $u \in W^{1,p}(\Omega)$ *if and only if there exists* $g \in L^p(\Omega)$, $g \geqslant 0$, *such that* (9) *holds a.e. in* Ω. *Moreover*

$$\|\nabla u\|_{L^p(\Omega)} \approx \inf_g \|g\|_{L^p(\Omega)},$$

where the infimum is taken over all $g \geqslant 0$ *which satisfy* (9).

Remark. This characterization is not valid for $p = 1$, see [9].

Since this characterization of the Sobolev space does not involve the notion of the derivative, the Sobolev space can be introduced on an arbitrary metric space with a measure μ. The following definition is taken from [7].

Let (X, d, μ) be a metric space (X, d) equipped with a Borel measure μ. Assume that diam $X < \infty$ and $\mu(X) < \infty$.

For $1 < p \leqslant \infty$ we define the Sobolev space $W^{1,p}(X, d, \mu)$ on the triple (X, d, μ) to be the space of all functions $u \in L^p(X, \mu)$ such that (9), with $|x - y| = d(x, y)$, holds μ–a.e. for some nonnegative $g \in L^p(X, \mu)$. Every function $g \in L^p(X, \mu)$, $g \geqslant 0$ which satisfies the inequality (9) will be called *a generalized gradient* of u.

Moreover we set $\|u\|_{W^{1,p}(X,d,\mu)} = \|u\|_{L^p(X,\mu)} + \|u\|_{L^{1,p}(X,d,\mu)}$ where $\|u\|_{L^{1,p}(X,d,\mu)} = \inf_g \|g\|_{L^p(X,\mu)}$. Here the infimum is taken over all $g \geqslant 0$ which satisfy the inequality (9) in the definition of $W^{1,p}(X, d, \mu)$.

If $\Omega \subset \mathbb{R}^n$ is a bounded domain, and $1 < p \leqslant \infty$, then besides the space $W^{1,p}(\Omega)$ we consider the space $W^{1,p}(\Omega, |\cdot|, H^n)$ defined as a Sobolev space on a metric space Ω with the Euclidean metric $|x - y|$ and the Lebesgue measure H^n. Theorem 1 states that if Ω is a bounded domain with the Lipschitz boundary, then

$$W^{1,p}(\Omega) = W^{1,p}(\Omega, |\cdot|, H^n). \tag{10}$$

Lemma 6 implies that for an arbitrary bounded domain $W^{1,p}(\Omega, |\cdot|, H^n) \subset W^{1,p}(\Omega)$. However as we will see, in general, $W^{1,p}(\Omega, |\cdot|, H^n) \neq W^{1,p}(\Omega)$.

It is natural to ask when (10) holds. One of the main results of the paper (Theorem 10) states that for the class of bounded domains with the $A(c)$ condition, (10) is equivalent to the existence of an extension operator $\mathrm{E}\colon W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$.

THEOREM 2 ([7, Theorems 3 and 5]). *If* (X, d, μ) *is as above and* $1 < p \leqslant \infty$, *then* $W^{1,p}(X, d, \mu)$ *is a Banach space and the set of Lipschitz functions* Lip(X) *is dense in* $W^{1,p}(X, d, \mu)$.

In the classical Sobolev imbedding theorem the dimension of the space plays a role of critical exponent. In order to extend the imbedding theorem into the metric setting we impose a condition on the measure μ.

Let (X, d, μ) be as above. We say that the measure μ is s-regular $(s > 0)$ if there exists a constant $C > 0$ such that for all $x \in X$ and all $r \leqslant \operatorname{diam} X$

$$\mu(B(x, r)) \geqslant Cr^s.$$

THEOREM 3 ([7, Theorem 6]). *Let (X, d, μ) be as above. Assume that μ is s-regular and $1 < p < s$. Then there is a bounded imbedding*

$$W^{1,p}(X, d, \mu) \subset L^{p^*}(X, \mu),$$

where $p^* = sp/(s-p)$.

For further properties and applications of the metric approach to Sobolev spaces, see [7], [9], [4], [17], [11], [12].

The purpose of the paper is the description of the traces of $W^{1,p}(\mathbb{R}^n)$ functions on compact subsets $K \subset \mathbb{R}^n$. As we already said we will describe these traces in terms of Sobolev spaces on metric spaces. Roughly speaking our theorems state that under certain assumptions there exist bounded trace and extension operators

$$\mathrm{T}\colon W^{1,p}(\mathbb{R}^n) \to W^{1,r}(K, |\cdot|^{1-\lambda}, \mu),$$

$$\mathrm{E}\colon W^{1,r}(K, |\cdot|^{1-\lambda}, \mu) \to W^{1,q}(\mathbb{R}^n),$$

where $|\cdot|^{1-\lambda}$, $0 \leqslant \lambda < 1$ denotes the metric $d(x, y) = |x-y|^{1-\lambda}$, and μ is a positive, finite Borel measure supported on K.

3. CLASSICAL TRACE THEOREM

In this section we interpret the classical trace theorem (stated in the introduction) in terms of Sobolev spaces on metric spaces.

THEOREM 4. *If Ω is a bounded domain with a Lipschitz boundary, then*

$$W^{1-1/p,p}(\partial\Omega) \subset W^{1,p}(\partial\Omega, |\cdot|^{1-1/p}, H^{n-1}) \subset W^{1-1/(p-\varepsilon),p-\varepsilon}(\partial\Omega), \tag{11}$$

for any $\varepsilon > 0$.

Remark. The theorem still holds (with the same proof) if we replace $\partial\Omega$ by a sufficiently regular, compact submanifold $M^{n-1} \subset \mathbb{R}^n$.

The space $W^{1,p}$ in the middle of (11) is the Sobolev space on the metric space $\partial\Omega$ with the metric $d(x, y) = |x-y|^{1-1/p}$, and with respect to the measure H^{n-1}. The spaces on the left and the right hand side of (11) are

Slobodeckiĭ spaces. Theorem 4 together with the theorem of Gagliardo lead to the trace and extension operators

$$T\colon W^{1,p}(\Omega) \to W^{1,p}(\partial\Omega, |\cdot|^{1-1/p}, H^{n-1}) \tag{12}$$

$$E\colon W^{1,p}(\partial\Omega, |\cdot|^{1-1/p}, H^{n-1}) \to W^{1,p-\varepsilon}(\Omega) \tag{13}$$

for any $\varepsilon > 0$. Hence $W^{1,p}(\partial\Omega, |\cdot|^{1-1/p}, H^{n-1})$ almost characterizes traces of Sobolev functions from $W^{1,p}(\Omega)$.

The space $W^{1-1/p,p}(\partial\Omega)$ gives a sharp characterization of traces, but its definition is of essentially different character than that of the classical Sobolev space $W^{1,p}$. The "metric" approach is a unified approach to Sobolev spaces and trace spaces, but it does not lead to a sharp characterization of traces—this is the price one has to pay.

Proof of Theorem 4. First we prove the second inclusion. If $u \in W^{1,p}(\partial\Omega, |\cdot|^{1-1/p}, H^{n-1})$, then there exists $g \in L^p(\partial\Omega)$ such that

$$|u(x) - u(y)| \leqslant |x-y|^{1-1/p}(g(x) + g(y)) \tag{14}$$

for almost every $x, y \in \partial\Omega$. Hence

$$\begin{aligned}\int_{\partial\Omega}\int_{\partial\Omega} \frac{|u(x)-u(y)|^{p-\varepsilon}}{|x-y|^{n+p-\varepsilon-2}}\,dx\,dy &\leqslant C\int_{\partial\Omega}\int_{\partial\Omega} \frac{g(x)^{p-\varepsilon}+g(y)^{p-\varepsilon}}{|x-y|^{n-1-\varepsilon/p}}\,dx\,dy\\ &= 2C\int_{\partial\Omega}\left(\int_{\partial\Omega}\frac{dx}{|x-y|^{n-1-\varepsilon/p}}\right) g(y)^{p-\varepsilon}\,dy\\ &\leqslant C'\int_{\partial\Omega} g(y)^{p-\varepsilon}\,dy < \infty.\end{aligned}$$

For the first inclusion let $u \in W^{1-1/p,p}(\partial\Omega)$. We have to find $g \in L^p(\partial\Omega)$ such that (14) holds. Fix $0 \leqslant \lambda < 1$. Let $x, y \in \partial\Omega$. Take a ball B_R with radius R such that $x, y \in B_R$, and $R \approx |x-y|$, say $R \leqslant 2|x-y|$. We have

$$\begin{aligned}|u(x)-u(y)| &\leqslant |u(x) - u_{B_R}| + |u(y) - u_{B_R}|\\ &\leqslant 2|x-y|^{1-\lambda}\,(G^\lambda_{|x-y|}\,u(x) + G^\lambda_{|x-y|}\,u(y)),\end{aligned} \tag{15}$$

where

$$G^\lambda_s u(x) = \sup\left\{\frac{|u(x)-u_{B_r}|}{r^{1-\lambda}} : x \in B_r,\ r \leqslant 2s\right\}.$$

Here, of course, the average u_B is with respect to the $(n-1)$–dimensional measure i.e., $u_B = \int_{B\cap\partial\Omega} u\,dH^{n-1}(x)/H^{n-1}(B\cap\partial\Omega)$. Let $h = \operatorname{diam}\Omega$. For $\lambda = 1/p$ we have

$$|u(x) - u(y)| \leqslant 2\,|x-y|^{1-1/p}\,(\,G^{1/p}_h u(x) + G^{1/p}_h u(y)).$$

Now it suffices to prove that $G_h^{1/p}u \in L^p(\partial\Omega)$. For $x \in \partial\Omega$ with $G_h^{1/p}u(x) < \infty$, there is a ball B_r with $x \in B_r$ and $r \leqslant 2h$, such that

$$G_h^{1/p}u(x) \leqslant 2\frac{|u(x)-u_{B_r}|}{r^{1-1/p}} \leqslant C\left(\int_{\partial\Omega}\frac{|u(x)-u(z)|^p}{|x-z|^{p+n-2}}\,dz\right)^{1/p},$$

and hence $G_h^{1/p}u \in L^p$. We have used here Hölder's inequality and the estimate $|x-z| \leqslant 2r$ for $z \in B_r$. The proof is complete.

To see how (12) works, we will apply the imbedding theorem (Theorem 3) to the right hand side of (12). First we find s, such that the triple $(\partial\Omega, |\cdot|^{1-1/p}, H^{n-1})$ is s-regular.

If $\widetilde{B}(r)$ denotes a ball (subset of $\partial\Omega$) with respect to the metric $|\cdot|^{1-1/p}$, then $\widetilde{B}(r) = B(r^{p/(p-1)})$ where the last ball is with respect to the Euclidean metric (induced from $\mathbb{R}^n$). Now $H^{n-1}(\widetilde{B}(r)) = H^{n-1}(B(r^{p/(p-1)})) \approx r^{p(n-1)/(p-1)}$, which means that the space is s–regular for $s = p(n-1)/(p-1)$. Now applying the imbedding theorem (Theorem 3) we get for $p < n$

$$W^{1,p}(\partial\Omega, |\cdot|^{1-1/p}, H^{n-1}) \subset L^{p(n-1)/n-p}(\partial\Omega),$$

and this is just a classical imbedding theorem for traces [18, Theorem 6.4.1].

Theorem 4 explains the relation between the spaces $W^{1-1/p,p}(\partial\Omega)$ and $W^{1,p}(\partial\Omega, |\cdot|^{1-1/p}, H^{n-1})$, but it does not provide a good approach to trace theorems via Sobolev spaces on metric spaces. The reason is evident, namely Theorem 4 reduces the problem to classical trace theory, so this way we will not go beyond the classical results. One of the possible "good" approaches is presented in the next sections.

4. GENERAL TRACE THEOREM

Assume that $u \in W^{1,p}(\Omega)$, $K \subset \Omega$ is a compact set, μ a finite Borel measure on K and $0 \leqslant \lambda < 1$. If for every $h > 0$

$$M_h^{\lambda}\colon L^p(\Omega) \to L^q(K,\mu), \tag{16}$$

is a bounded operator, then inequality (8) immediately implies $u \in W^{1,q}(K, |\cdot|^{1-\lambda}, \mu)$ i.e. this leads to the following trace operator

$$\mathrm{Tr} : W^{1,p}(\Omega) \to W^{1,q}(K, |\cdot|^{1-\lambda}, \mu).$$

Now Marcinkiewicz's interpolation theorem or, more directly, Adams' trace theorem, [1], provide imbedding of the form (16). This leads to the following theorem.

THEOREM 5. *If* $0<\lambda<1$, $1\leqslant(n-d)/\lambda<p\leqslant n/\lambda$ *and* μ *is a Borel measure supported on a compact set* $K\subset\mathbb{R}^n$ *such that* $\mu(B(x,R))\leqslant CR^d$ *for all* $x\in\mathbb{R}^n$ *and all* $R>0$, *then there is a bounded trace operator*

$$\mathrm{Tr}: W^{1,p}(\mathbb{R}^n)\to W^{1,\,dp/(n-\lambda p)}(K,|\cdot|^{1-\lambda},\mu).$$

Proof. According to (8) it suffices to prove that for every $h>0$

$$M_h^\lambda: L^p(\mathbb{R}^n)\to L^{dp/(n-\lambda p)}(K,\mu) \tag{17}$$

is a bounded operator. This follows from Adams' theorem [1], [23, Theorem 1.4.1/2], [28, Theorem 4.7.2]. However for the sake of completeness we prefer to give a direct proof. In fact the proof is an elementary generalization of the fractional integration theorem.

LEMMA 7. *If* μ *is as in Theorem* 5, $d>0$, $h>0$, $0<\delta\leqslant n-d$ *and* $p=(n-d)/\delta$, *then the operator* M_h^δ *is of the weak type* (p,p) *i.e., there is a constant* C *such that*

$$\mu(\{x\in K: M_h^\delta g(x)>t\})\leqslant Ct^{-p}\int_{\mathbb{R}^n}|g|^p\,dx$$

for all $g\in L^p(\mathbb{R}^n)$.

The proof is exactly the same as the proof that the standard Hardy–Littlewood maximal operator is of the weak type (1, 1), thus we skip details.

LEMMA 8. *If* μ *is a Borel measure supported on a compact set* K, $h>0$ *and* $0<\delta\leqslant n$, *then* M_h^δ *is of a strong type* $(n/\delta,\infty)$ *i.e. the operator*

$$M_h^\delta: L^{n/\delta}(\mathbb{R}^n)\to L^\infty(K,\mu)$$

is bounded.

This lemma follows directly from the definition of M_h^δ and from Hölder's inequality.

Since we established weak-strong type estimates, application of Marcinkiewicz's interpolation theorem, [26, Appendix B.1], readily leads to the following result which, in turn, implies (17).

LEMMA 9. *If* μ *is as in Theorem* 5, $d>0$, $h>0$, *and* $0<\delta\leqslant n-d$, *then for* $(n-d)/\delta<p\leqslant n/\delta$ *the operator*

$$M_h^\delta: L^p(\mathbb{R}^n)\to L^{dp/(n-\delta p)}(K,\mu)$$

is bounded.

In Theorem 5, for the clarity of the statement, we excluded the case $\lambda = 0$. We state it as a separate result.

THEOREM 6. *If $1 < p \leqslant \infty$ and μ is a Borel measure supported on a compact set $K \subset \mathbb{R}^n$, such that $\mu(B(x, R)) \leqslant CR^n$, for all $x \in \mathbb{R}^n$ and all $R > 0$, then there exists a bounded trace operator*

$$\mathrm{Tr} : W^{1,p}(\mathbb{R}^n) \to W^{1,p}(K, |\cdot|, \mu).$$

Let us compare the above results with Theorem 4. Thus assume that $1 < p < \infty$, $d = n - 1$, $\lambda = 1/p$ and $\mu = H^{n-1}$ is supported on the boundary $\partial\Omega$ of a bounded Lipschitz domain. In such a case Lemma 7 leads to the trace operator

$$\mathrm{Tr}: W^{1,p}(\Omega) \to W^{1,p}_{\mathrm{w}}(\partial\Omega, |\cdot|^{1-1/p}, H^{n-1}),$$

where the space $W^{1,p}_{\mathrm{w}}$ is defined in the same manner as the Sobolev space on a metric space, but with $L^p(\partial\Omega, H^{n-1})$ replaced by the Marcinkiewicz's space $L^p_{\mathrm{w}}(\partial\Omega, H^{n-1})$ (for the definition of the Marcinkiewicz's space see e.g., [18]). This result is weaker than (12). The proof of (12) involves Gagliardo's theorem and makes a strong use of the geometry of the boundary $\partial\Omega$, while in the above general approach (Theorem 5, Lemma 7) we only use properties of the measure. Since our method involves less information about the set K, it applies to a much more general setting, however, when specified to the classical setting, it leads to weaker results.

A version of Theorem 5 appeared implicitly in [8]. It was used to generalize the theorem of Øksendal, [24], on the support of harmonic measure.

5. EXTENSION OPERATORS

This section is devoted to the construction of extension operators. Assume that μ is a finite Borel measure supported on a compact set $K \subset \mathbb{R}^n$, $0 \leqslant \lambda < 1$ and $1 < q, r < \infty$. We want to construct an extension operator

$$\mathrm{Ext}: W^{1,r}(K, |\cdot|^{1-\lambda}, \mu) \to W^{1,q}(\mathrm{R}^n),$$

provided μ, λ, q and r satisfy certain conditions. We construct the operator Ext as follows. To every $x \in \mathbb{R}^n \setminus K$ we associate $\tilde{x} \in K$ such that $|x - \tilde{x}| = \mathrm{dist}\,(x, K)$. Fix a Whitney decomposition of $\mathbb{R}^n \setminus K$ (see (3)). If x is a corner of a cube in the Whitney decomposition, then we set

$$\tilde{u}(x) = \fint_{B(\tilde{x}, |x - \tilde{x}|) \cap K} u(z)\, d\mu(z).$$

Then $\tilde{u}$ is defined at a finite number of points of every Whitney cube Q (we will denote this finite set by $V(Q)$—it contains at least all corners of Q) and we extend $\tilde{u}$ piecewise linearly in each Q in such a way that resulting function is continuous on $\mathbb{R}^n \setminus K$. Let $D = \bigcup_{k=0}^{\infty} \bigcup_{j=1}^{N_k} Q_j^k$ where the side length of Q_j^k is 2^{-k}. Fix $\varphi \in C_0^\infty(D \cup K)$ with $\varphi|_K = 1$. Now we set

$$\operatorname{Ext} u(x) = \begin{cases} u(x), & x \in K, \\ \varphi(x)\, \tilde{u}(x), & x \in \mathbb{R}^n \setminus K. \end{cases}$$

Note that if $|K| = 0$, then it suffices to set $\operatorname{Ext} u(x) = \varphi(x)\, \tilde{u}(x)$ as all the Sobolev functions equal outside the set of measure zero are identified. In what follows, by Ext we will *always* denote an operator defined as above.

THEOREM 7. *Assume that a finite Borel measure μ, supported on a compact set $K \subset \mathbb{R}^n$, satisfies the d-regularity condition* ($d > 0$):

$$\mu(B(x, R) \cap K) \geqslant CR^d,$$

whenever $x \in K$ and $R \leqslant \operatorname{diam} K$. Assume that $|K| = 0$ and that $0 \leqslant \lambda < 1$.

1. *If $r > 1$ and $n - d > \lambda r$, then*

$$\operatorname{Ext}\colon W^{1,r}(K, |\cdot|^{1-\lambda}, \mu) \to W^{1,r}(\mathbb{R}^n)$$

is a bounded operator.

2. *Assume that there exist $C \geqslant 1$ and $s < n$ such that $N_k \leqslant C2^{ks}$ for all $k \geqslant 0$. If $q \leqslant r$ and $n - s - q(\lambda - (s-d)/r) > 0$, then*

$$\operatorname{Ext}\colon W^{1,r}(K, |\cdot|^{1-\lambda}, \mu) \to W^{1,q}(\mathbb{R}^n)$$

is a bounded operator.

Remarks. (1) A priori we assume only that $|K| = 0$, however as we will see later (Lemma 10) the assumptions of the theorem imply $H^{n-\lambda}(K) = 0$.

(2) Note that in Theorem 5, the measure μ was supposed to satisfy the inequality which was opposite to the above d-regularity condition.

(3) We suggest the reader to apply the above theorem to $K = \partial\Omega$ where Ω is a bounded domain with the Lipschitz boundary.

Proof of Theorem 7. Assume for simplicity that $\operatorname{diam} K = 1$. We divide the proof into several steps. In the first two steps we do not employ the assumption $|K| = 0$.

Step 1. Function $\tilde{u}$ is locally Lipschitz on $\mathbb{R}^n \setminus K$ and $\|\nabla \tilde{u}\|_{L^q(D)} \leqslant C\, \|u\|_{L^{1,r}(K, |\cdot|^{1-\lambda}, \mu)}$, where $q = r$ in the case 1.

Let Q be a cube in the Whitney decomposition of $\mathbb{R}^n \setminus K$. To estimate $|\nabla \tilde{u}|$ in Q it suffices to estimate

$$\frac{|\tilde{u}(x_1) - \tilde{u}(x_2)|}{|x_1 - x_2|},$$

where x_1 and x_2 belong to $V(Q)$. Let g be a generalized gradient of u. Then the definition of $W^{1,p}(K, |\cdot|^{1-\lambda}, \mu)$ yields

$$\begin{aligned} &|\tilde{u}(x_1) - \tilde{u}(x_2)| \\ &\quad = \left| \fint_{B(\tilde{x}_1, |x_1 - \tilde{x}_1|) \cap K} u \, d\mu - \fint_{B(\tilde{x}_2, |x_2 - \tilde{x}_2|) \cap K} u \, d\mu \right| \\ &\quad \leqslant C|x_1 - x_2|^{1-\lambda} \left(\fint_{B(\tilde{x}_1, |x_1 - \tilde{x}_1|) \cap K} g \, d\mu + \fint_{B(\tilde{x}_2, |x_2 - \tilde{x}_2|) \cap K} g \, d\mu \right). \end{aligned}$$

Let $B(Q)$ be such a ball among $B(\tilde{x}_i, |x_i - \tilde{x}_i|)$ where x_i belong to $V(Q)$ that

$$g(Q) := \fint_{B(Q) \cap K} g \, d\mu = \max_i \fint_{B(\tilde{x}_i, |x_i - \tilde{x}_i|) \cap K} g \, d\mu.$$

Note that $|x_1 - x_2| \approx l(Q)$ and hence

$$\sup_Q |\nabla \tilde{u}| \leqslant C l(Q)^{-\lambda} g(Q).$$

By a simple packing argument there is a constant $C = C(n)$ such that for every $k \in \mathbb{Z}$ no point of $\mathbb{R}^n$ belongs to more than C balls from the family $\{B(Q_j^k)\}_{j=1}^{N_k}$. It is important to note that the constant C does not depend on k. Now for $q \leqslant r$ we have

$$\int_Q |\nabla \tilde{u}|^q \, dx \leqslant C\, |Q|\, l(Q)^{-\lambda q} g(Q)^q = C l(Q)^{n - \lambda q} g(Q)^q.$$

Let $A_k = \bigcup_{j=1}^{N_k} Q_j^k$ and note that $l(Q_j^k) = 2^{-k}$. For each $k \geqslant 0$

$$\begin{aligned} \int_{A_k} |\nabla \tilde{u}|^q \, dx &\leqslant C 2^{-k(n - \lambda q)} \sum_{j=1}^{N_k} (g(Q_j^k))^q \\ &\leqslant C 2^{-k(n - \lambda q)} N_k^{1 - q/r} \left(\sum_{j=1}^{N_k} (g(Q_j^k))^r \right)^{q/r} \\ &\leqslant C 2^{-k(n - \lambda q)} N_k^{1 - q/r} \left(\sum_{j=1}^{N_k} \fint_{B(Q_j^k) \cap K} g^r \, d\mu \right)^{q/r} \\ &\leqslant C 2^{-k(n - \lambda q)} N_k^{1 - q/r}\, 2^{kdq/r} \left(\int_K g^r \, d\mu \right)^{q/r}; \end{aligned}$$

in the last step we used d-regularity of the measure μ and the estimate $C(n)$ for the number of overlapping balls. Thus we have

$$\int_D |\nabla \tilde{u}|^q \, dx = \sum_{k=0}^{\infty} \int_{A_k} |\nabla \tilde{u}|^q \, dx$$
$$\leqslant C \left(\int_K g^r \, d\mu \right)^{q/r} \sum_{k=0}^{\infty} 2^{-k(n - q(\lambda + d/r))} N_k^{1 - q/r},$$

and we need to prove

$$\sum_{k=0}^{\infty} 2^{-k(n - q(\lambda + d/r))} N_k^{1 - q/r} < \infty. \tag{18}$$

In the first case we choose $q = r$. Since $n - r(\lambda + d/r) > 0$, the sum in (18) is finite. In the second case we use the estimate $N_k < C 2^{sk}$ and hence the series in (18) is convergent when

$$n - s + q\left(\frac{s-d}{r} - \lambda\right) > 0.$$

Step 2. $\|\mathrm{Ext}\, u\|_{W^{1,q}(\mathbb{R}^n \setminus K)} \leqslant C \|u\|_{W^{1,r}(K, |\cdot|^{1-\lambda}, \mu)}$, where $q = r$ in case 1.

Since $\|\tilde{u}\varphi\|_{W^{1,q}(\mathbb{R}^n \setminus K)} \leqslant C \|\tilde{u}\|_{W^{1,q}(D)}$, it suffices to prove that $\|\tilde{u}\|_{L^q(D)} \leqslant C \|u\|_{L^r(K, \mu)}$. The proof of this inequality is similar to that of step 1, so we will be sketchy.

If Q is a cube in the Whitney decomposition of $\mathbb{R}^n \setminus K$, then

$$\sup_Q |\tilde{u}| = \max_i |\tilde{u}(x_i)| = \max_i \left| \fint_{B(\tilde{x}_i, |x_i - \tilde{x}_i|) \cap K} u \, d\mu \right|$$

where the maximum is over the set $V(Q)$. Now repeating arguments from step 1 we obtain

$$\int_D |\tilde{u}|^q \, dx \leqslant C \left(\int_K |u|^r \, d\mu \right)^{q/r} \sum_{k=0}^{\infty} 2^{-k(n - dq/r)} N_k^{1 - q/r}. \tag{19}$$

Now it suffices to note that the convergence of the series (18) implies that of (19).

In step 2 we obtained the estimates for the Sobolev norm of Ext u, outside the set K. However, we need to know that Ext u is in the Sobolev space "through" K.

Step 3. $\mathrm{Ext}\, u \in W^{1,q}(\mathbb{R}^n)$ and $\|\mathrm{Ext}\, u\|_{W^{1,q}(\mathbb{R}^n)} \leqslant C \|u\|_{W^{1,r}(K, |\cdot|^{1-\lambda}, \mu)}$, where $q = r$ in case 1.

Now we will employ the assumption $|K| = 0$.

According to Theorem 2, the class of $C^{0,1-\lambda}$ Hölder continuous functions on K is a dense subset in the Sobolev space $W^{1,r}(K, |\cdot|^{1-\lambda}, \mu)$. Indeed, $C^{0,1-\lambda}$ functions are Lipschitz continuous with respect to the metric $|\cdot|^{1-\lambda}$. It remains to prove that $\operatorname{Ext} u \in W^{1,q}(\mathbb{R}^n)$ for every $u \in C^{0,1-\lambda}(K)$. This combined with step 2 and the fact $|K| = 0$ implies the inequality of step 3 for all $u \in C^{0,1-\lambda}(K)$, and the general case follows by the density argument. It follows from the construction that $\operatorname{Ext} u \in C^{0,1-\lambda}(\mathbb{R}^n)$ when $u \in C^{0,1-\lambda}(K)$ (we leave the proof of this fact to the reader), so the remaining step follows from the following two lemmas.

Lemma 10. *Under the assumptions of Theorem 7 we have* $H^{n-\lambda}(K) = 0$.

Proof. The d-regularity condition combined with the covering argument leads to the estimate $H^d(K) < C\mu(K) < \infty$, see [28, Lemma 3.2.1], so in case 1 of Theorem 7 our lemma follows from $d < n - \lambda r \leqslant n - \lambda$. In case 2, conditions $N_k \leqslant C2^{ks}$ and $|K| = 0$ imply $H^s(K) < \infty$, see Lemma 2. Thus the lemma follows, since the assumptions of case 2 easily imply $\min(s, d) < n - \lambda$.

Lemma 11. *If a compact set* $K \subset \mathbb{R}^n$ *satisfies* $H^{n-\lambda}(K) = 0$, *where* $0 \leqslant \lambda < 1$, *then for every* $1 \leqslant q \leqslant \infty$

$$W^{1,q}(\mathbb{R}^n \setminus K) \cap C^{0,1-\lambda}(\mathbb{R}^n) \subset W^{1,q}(\mathbb{R}^n).$$

Proof. According to the ACL characterization of the Sobolev space, [18, Theorems 5.6.2-3], [23, Section 1.1.3], it suffices to prove that every $u \in W^{1,q}(\mathbb{R}^n \setminus K) \cap C^{0,1-\lambda}(\mathbb{R}^n)$ is absolutely continuous on almost all lines parallel to coordinate directions.

Let $\pi: \mathbb{R}^n \to \mathbb{R}^{n-1}$ be an orthogonal projection along the direction of one of the coordinate axis. It follows from Eilenberg's inequality [3, Theorem 2.10.25] that for almost all $y \in \mathbb{R}^{n-1}$ we have $H^{1-\lambda}(K \cap \pi^{-1}(y)) = 0$. Thus it suffices to show that every function $v \in W^{1,q}(\mathbb{R} \setminus F) \cap C^{0,1-\lambda}(\mathbb{R})$ where F is a compact set with $H^{1-\lambda}(F) = 0$ is absolutely continuous. Let $\varepsilon > 0$ and choose a covering of F by intervals I_j such that $\sum_{j=1}^m |I_j|^{1-\lambda} < \varepsilon$. Note that by Hölder continuity of v we have $|v(I_j)| \leqslant C\, |I_j|^{1-\lambda}$. Since $v \in W^{1,q}(\mathbb{R} \setminus F)$ and v is continuous, the absolute continuity of v on $\mathbb{R}$ follows from the following estimate:

$$\sum_{j=1}^{m} |v(I_j)| \leqslant C \sum_{j=1}^{m} |I_j|^{1-\lambda} < C\varepsilon.$$

The proof of Theorem 7 is complete.

Now we will generalize the above result to the case $\lambda=0$, μ is the Lebesgue measure and $|K|>0$. First recall that the trace is a bounded operator

$$\mathrm{Tr}\colon W^{1,p}(\mathbb{R}^n)\to W^{1,p}(K,|\cdot|,H^n)$$

for every $1<p\leqslant\infty$ (see Theorem 6). Note that since μ is the Lebesgue measure, we can define the trace operator just as a restriction.

THEOREM 8. *Assume that a compact set $K\subset\mathbb{R}^n$ satisfies the condition*

$$H^n(B(x,R)\cap K)\geqslant CR^d \tag{20}$$

whenever $x\in K$ and $R\leqslant\operatorname{diam} K$. Also assume that there is $s<n$ such that $N_k\leqslant C2^{ks}$ for all $k\geqslant 0$. If $n-s+q(s-d)/r>0$, then

$$\mathrm{Ext}\colon W^{1,r}(K,|\cdot|,H^n)\to W^{1,q}(\mathbb{R}^n)$$

is a bounded operator. In particular if $d=n$, then

$$\mathrm{Ext}\colon W^{1,r}(K,|\cdot|,H^n)\to W^{1,r-\varepsilon}(\mathbb{R}^n)$$

is bounded for an arbitrary $\varepsilon>0$.

Remarks. Note that the condition (20) cannot hold for $d<n$. However it can happen that the least d for which (20) holds is strictly greater than n. For example it suffices to consider a cusp as a set K. Also the assumptions imply that $q<r$. Note that the condition $N_k\leqslant C2^{ks}$, $s<n$, does not imply $|K|=0$, see Lemma 3.

Proof. As we already noticed, in the first two steps of the proof of Theorem 7 we did not use the assumption $|K|=0$ and hence the estimate from the second step extends to our current situation. We state it in the following lemma.

LEMMA 12. *Under the assumptions of Theorem 8*

$$\|\mathrm{Ext}\, u\|_{W^{1,q}(\mathbb{R}^n\setminus K)}\leqslant C\|u\|_{W^{1,r}(K,|\cdot|,H^n)}.$$

Now it remains to prove that $\mathrm{Ext}\, u\in W^{1,q}(\mathbb{R}^n)$ and that suitable estimate of the Sobolev norm extends to the entire $\mathbb{R}^n$. We need the following result.

LEMMA 13. *Assume that $u\in W^{1,1}_{\mathrm{loc}}(\Omega)$ where $\Omega\subset\mathbb{R}^n$ is an arbitrary open set and $u|_E\in W^{1,p}(E,|\cdot|,H^n)$ where $E\subset\Omega$ is a bounded measurable set. Then for every generalized gradient g of $u|_E\in W^{1,p}(E,|\cdot|,H^n)$ we have $|\nabla u|\leqslant 2\sqrt{n}\, g$ a.e. in E.*

Remark. This lemma is related to Lemma 6, however, it does not imply Lemma 6 because we do not know a priori that the function u, in the statement of Lemma 6, belongs to $W^{1,1}_{\mathrm{loc}}(\Omega)$.

Proof. Excising a subset of measure zero we can assume that the inequality $|u(x)-u(y)| \leqslant |x-y|(g(x)+g(y))$ holds everywhere in E. We can also assume that $g>0$ everywhere in E, otherwise we replace g by $g+\varepsilon$ and pass to the limit as $\varepsilon \to 0$.

The Sobolev function $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ has a representative which is absolutely continuous on almost all lines parallel to coordinate axes. For such a representative the gradient $\nabla u = (\partial_1 u, \partial_2 u, ..., \partial_n u)$ is defined almost everywhere in the classical sense.

By e_i we will denote the unit vector parallel to ith coordinate direction. We can assume that g is defined on the entire $\mathbb{R}^n$, by putting $g=0$ outside E.

Almost all points $x \in E$ have the following properties

1. x is a point of density of E in every coordinate direction (see [26, Chapter 1, Section 2.1] for the notion of a point of density).
2. $\nabla u(x)$ exists in the classical sense.
3. $g(x) < \infty$ and

$$\lim_{t \to 0} \frac{1}{t} \int_0^t g(x+\tau e_i)\, d\tau = g(x)$$

for $i=1, 2, ..., n$.

It suffices to prove that at every point $x \in E$ which has the properties (1)–(3), the inequality $|\partial_i u(x)| \leqslant 2g(x)$ is satisfied for $i=1, 2, ..., n$.

Fix $\varepsilon>0$. Note that (1) and (3) imply that there is $t>0$, as small as we wish, such that $x+te_i \in E$ and $g(x+te_i) \leqslant (1+\varepsilon)\, g(x)$. We have

$$|u(x)-u(x+te_i)| \leqslant t(g(x)+g(x+te_i)) \leqslant t(2+\varepsilon)\, g(x).$$

Now the lemma follows by dividing both sides of the inequality by t and letting first $t \to 0$ and then $\varepsilon \to 0$.

Now we follow the same argument as in the proof of Theorem 7. According to Theorem 2 the class of Lipschitz functions on K is dense in $W^{1,r}(K, |\cdot|, H^n)$. If $u \in \mathrm{Lip}\,(K)$, then $\mathrm{Ext}\, u \in \mathrm{Lip}\,(\mathbb{R}^n)$. Hence applying the above two lemmas and using the fact that $q<r$ we readily get the desired estimate

$$\|\mathrm{Ext}\, u\|_{W^{1,q}(\mathbb{R}^n)} \leqslant C\, \|u\|_{W^{1,r}(K, |\cdot|, H^n)}$$

whenever $u \in \mathrm{Lip}\,(K)$. Now the estimate for general u follows by the density argument. The proof for Theorem 8 is complete.

In the case $d=n$ Theorem 8 shows that the space $W^{1,r}(K,|\cdot|,H^n)$ "almost" characterizes traces on the set K. This characterization is not sharp because we "lose ε" in the estimate of the extension. Now we show that in the case when K is a domain with the $A(c)$ property, it is possible to find a sharp characterization of traces.

THEOREM 9. *Let $\Omega\subset\mathbb{R}^n$ be a bounded domain with the $A(c)$ property and let $1<p\leqslant\infty$. Then $u\in W^{1,p}(\Omega)$ is a trace of a $W^{1,p}(\mathbb{R}^n)$ function if and only if $u\in W^{1,p}(\Omega,|\cdot|,H^n)$. Moreover there exist bounded trace and extension operators*:

$$\mathrm{Tr}\colon W^{1,p}(\mathbb{R}^n)\to W^{1,p}(\Omega,|\cdot|,H^n),$$

$$\mathrm{Ext}^*\colon W^{1,p}(\Omega,|\cdot|,H^n)\to W^{1,p}(\mathbb{R}^n).$$

Remark. Since $|\partial\Omega|=0$ (see Lemma 3), it is equivalent to consider the compact set $K=\bar{\Omega}$ instead of an open set Ω. Moreover the condition (20) is satisfied for $d=n$ and, according to Lemma 3, there exists $s<n$ such that $N_k\leqslant C2^{sk}$ for all $k\geqslant 0$. Thus Theorem 8 applies and we have the bounded trace and extension operators

$$\mathrm{Tr}\colon W^{1,p}(\mathbb{R}^n)\to W^{1,p}(\Omega,|\cdot|,H^n),$$

$$\mathrm{Ext}\colon W^{1,p}(\Omega,|\cdot|,H^n)\to W^{1,p-\varepsilon}(\mathbb{R}^n) \tag{21}$$

for every $\varepsilon>0$. Theorem 9 states that we can improve the extension operator. In fact we have to change the construction, so we denote it by Ext* instead of Ext.

The following example shows that without the $A(c)$ condition a bounded domain Ω need not have an extension E: $W^{1,p}(\Omega,|\cdot|,H^n)\to W^{1,p}(\mathbb{R}^n)$. It shows even more: if we do not assume (20) for $d=n$, it may happen that there is no extension (21).

Example. Let $\Omega_s=\{(x,t)=(x_1,\dots,x_{n-1},t)\in\mathbb{R}^n\colon 0<t<1,\ |x|<t^s\}$ be a cusp of order $s>1$. For $\lambda<0$ consider a function $\varphi_\lambda(x,t)=t^\lambda$ defined on Ω_s. Since $(t_1^\lambda-t_2^\lambda)/(t_1-t_2)=\lambda t^{\lambda-1}$ for a certain t between t_1 and t_2, it follows that

$$|t_1^\lambda-t_2^\lambda|\leqslant|\lambda|\,|t_1-t_2|\,(t_1^{\lambda-1}+t_2^{\lambda-1}). \tag{22}$$

It is easy to check that $t^{\lambda-1}\in L^p(\Omega_s)$ if $p(\lambda-1)+s(n-1)>-1$, and hence, in this case the inequality (22) implies $\varphi_\lambda\in W^{1,p}(\Omega_s,|\cdot|,H^n)$. Fix large $s>1$ and $\lambda<0$ close to zero. Then for p slightly greater than n we have $p(\lambda-1)+s(n-1)>-1$. Now it is clear that in such a case φ_λ cannot be extended to $W^{1,p}(\mathbb{R}^n)$ because φ_λ is unbounded at the origin and $p>n$.

Proof of Theorem 9. Let $K=\bar{\Omega}$ and let $\mathbb{R}^n \setminus K = \bigcup_{k=-\infty}^{\infty} \bigcup_{j=1}^{N_k} Q_j^k$ be a Whitney decomposition. Set $D=\bigcup_{k=0}^{\infty} \bigcup_{j=1}^{N_k} Q_j^k$. For simplicity we assume that K is big enough to guarantee $\operatorname{dist}(x, K) \leqslant \operatorname{diam} K$ whenever $x \in D$.

Fix $x \in D$ and consider a ball $B(x, 2\operatorname{dist}(x, K))$. According to the $A(c)$ property there exists $x^* \in \Omega$ such that $B(x^*, c\operatorname{dist}(x, K)) \subset \Omega \cap B(x, 2\operatorname{dist}(x, K))$. The point x^* is not defined uniquely, however, to every $x \in D$ we can assign x^* with the above property. Let g be a generalized gradient of $u \in W^{1,p}(\Omega, |\cdot|, H^n)$.

If x is a corner of a cube $Q \subset D$ from the Whitney decomposition, then we set

$$u^*(x) = \fint_{B(x^*, (1/2)c\operatorname{dist}(x, K))} u(z)\, dH^n(z).$$

Next we extend u^* in a piecewise linear way onto D. The remaining arguments are similar to those in the proof of Theorem 8.

Put $B(x^*) = B(x^*, \frac{1}{2}c\operatorname{dist}(x, K))$. Fix a cube $Q \subset D$ in the Whitney decomposition and let $B^*(Q)$ be such a ball among $B(x_i^*)$, where x_i belong to $V(Q)$ (the set $V(Q)$ was defined at the begining of Section 5), that

$$\fint_{B^*(Q)} g\, dH^n = \max_i \fint_{B(x_i^*)} g\, dH^n.$$

The same arguments as in the proof of Theorem 7 lead to

$$\sup_Q |\nabla u^*| \leqslant C \fint_{B^*(Q)} g\, dx,$$

and hence using the fact $|Q| \approx |B^*(Q)|$ and Hölder's inequality we obtain

$$\int_Q |\nabla u^*|^p\, dx \leqslant C \int_{B^*(Q)} g^p\, dx.$$

Let $A_k = \bigcup_{j=1}^{N_k}$ and $A_k^* = \bigcup_{j=1}^{N_k} B^*(Q_j^k)$. Note that there is a constant $C=C(n, c)$ such that no point of Ω belongs to more than C sets in the family $\{A_k^*\}_{k=0}^{\infty}$. Thus we obtain

$$\int_D |\nabla u^*|^p\, dx = \sum_{k=0}^{\infty} \int_{A_k} |\nabla u^*|^p \leqslant C \sum_{k=0}^{\infty} \int_{A_k^*} g^p\, dx \leqslant C' \int_{\Omega} g^p\, dx.$$

In a similar way we prove $\int_D |u^*|^p\,dx \leqslant C \int_\Omega |u|^p\,dx$. Let $\varphi \in C_0^\infty(D \cup \bar{\Omega})$ with $\varphi|_\Omega = 1$. We set

$$\operatorname{Ext}^* u(x) = \begin{cases} u(x), & x \in \Omega, \\ \varphi(x)\, u^*(x), & x \in \mathbb{R}^n \setminus \bar{\Omega}. \end{cases}$$

Since $|\partial\Omega| = 0$, the function $\operatorname{Ext}^* u$ is defined a.e.

Our estimates immediately lead to

$$\|\operatorname{Ext}^* u\|_{W^{1,p}(\mathbb{R}^n \setminus \bar{\Omega})} \leqslant C\, \|u\|_{W^{1,p}(\Omega,\,|\cdot|,\,H^n)}.$$

Now the same arguments as in the proof of Theorem 8 show that $\operatorname{Ext}^* u \in W^{1,p}(\mathbb{R}^n)$ and that the estimate

$$\|\operatorname{Ext}^* u\|_{W^{1,p}(\mathbb{R}^n)} \leqslant C\, \|u\|_{W^{1,p}(\Omega,\,|\cdot|,\,H^n)}$$

holds. This completes the proof for Theorem 9.

In the rest of the paper we will *always* denote by Ext* the operator defined in the previous proof.

THEOREM 10. *If $\Omega \subset \mathbb{R}^n$ is a bounded domain with the $A(c)$ property and if $1 < p \leqslant \infty$, then the following conditions are equivalent.*

1. *For every $u \in W^{1,p}(\Omega)$ there exists $v \in W^{1,p}(\mathbb{R}^n)$ such that $v|_\Omega = u$.*
2. *The trace operator* Tr: *$W^{1,p}(\mathbb{R}^n) \to W^{1,p}(\Omega)$ is surjective.*
3. *There exists a bounded extension operator E: $W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$.*
4. *The operator Ext*: $W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$ is bounded.*
5. *$W^{1,p}(\Omega) = W^{1,p}(\Omega, |\cdot|, H^n)$.*

Remarks. (1) If $p = 2$, then conditions (1), (2) and (3) are equivalent for an arbitrary open set Ω. The equivalence (1) $\Leftrightarrow$ (2) and the implication (3) $\Rightarrow$ (2) are evident. The implication (2) $\Rightarrow$ (3) follows from the Hilbert structure of $W^{1,2}$. Namely, $\operatorname{Tr}|_{(\ker \operatorname{Tr})^\perp}$: $(\ker \operatorname{Tr})^\perp \to W^{1,2}(\Omega)$ is an isomorphism and hence we can define the extension E as

$$E = (\operatorname{Tr}|_{(\ker \operatorname{Tr})^\perp})^{-1}\colon W^{1,2}(\Omega) \to (\ker \operatorname{Tr})^\perp \subset W^{1,2}(\mathbb{R}^n).$$

(2) V. G. Maz'ya, [23, Theorem 1.5.2], has constructed an example of a planar domain Ω with a Jordan boundary and with the $A(c)$ property such that there exists an extension operator E: $W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^2)$ if and only if $1 \leqslant p < 2$. Thus as a corollary from Theorem 10 we obtain that for this particular domain $W^{1,p}(\Omega) = W^{1,p}(\Omega, |\cdot|, H^2)$ if and only if $1 < p < 2$.

(3) A result of Herron and Koskela, [13, Corollary 4.9], states that an arbitrary bounded domain is a $W^{1,p}$-extension domain if and only if it is an $L^{1,p}$-extension domain, see [13] for details.

(4) Also it might be interesting to recall one result of Peetre, [25]. According to the theorem of Gagliardo, [5], there is a bounded and surjective trace operator $\mathrm{Tr}: W^{1,1}(\mathbb{R}^n) \to L^1(\mathbb{R}^{n-1})$, and hence every $u \in L^1(\mathbb{R}^{n-1})$ admits an extension to $W^{1,1}(\mathbb{R}^n)$. However, as was proven by Peetre, [25], there is no bounded linear extension $E: L^1(\mathbb{R}^{n-1}) \to W^{1,1}(\mathbb{R}^n)$.

Proof of Theorem 10. The implications (1) $\Leftrightarrow$ (2), (3) $\Rightarrow$ (2) and (4) $\Rightarrow$ (3) are evident. The implication (5) $\Rightarrow$ (4) follows from Theorem 9. It remains to prove (1) $\Rightarrow$ (5). Now it follows from Lemma 6 that $W^{1,p}(\Omega, |\cdot|, H^n) \subset W^{1,p}(\Omega)$. For the opposite inclusion it suffices to note that $u \in W^{1,p}(\Omega)$ satisfies the inequality $|u(x) - u(y)| \leqslant C\,|x-y|\,(M\,|\nabla v|\,(x) + M\,|\nabla v|\,(y))$, and $M\,|\nabla v| \in L^p(\Omega)$, see (7). The proof is complete.

Theorem 10 gives an alternative approach to the Jones celebrated extension theorem [14]. However, our method does not cover the case $p=1$ and the case of higher order derivatives.

THEOREM 11 ([14]). *If $\Omega \subset \mathbb{R}^n$ is a bounded uniform domain and $1 < p \leqslant \infty$, then there is a bounded extension operator*

$$\mathrm{Ext}^*: W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n).$$

Proof. According to Theorem 10 it suffices to prove the following lemma.

LEMMA 14. *If $\Omega \subset \mathbb{R}^n$ is uniform and $1 < p \leqslant \infty$, then*

$$W^{1,p}(\Omega) = W^{1,p}(\Omega, |\cdot|, H^n).$$

Proof. Since $W^{1,p}(\Omega, |\cdot|, H^n) \subset W^{1,p}(\Omega)$, it suffices to prove that every $u \in W^{1,p}(\Omega)$ satisfies the inequality

$$|u(x) - u(y)| \leqslant |x-y|\,(g(x) + g(y)),$$

for almost all $x, y \in \Omega$ with some $g \in L^p(\Omega)$. We will employ the method of the proof of [10, Theorem 10].

We use a standard covering argument in uniform domains, see [10] for details. Fix $x, y \in \Omega$ and let γ be a curve joining x and y as in the definition of a uniform domain. Now there are constants d, depending only on the uniformity constant of Ω, and $C=C(n)$ together with a sequence of balls $\{B_i\}_{i=-\infty}^{\infty}$ with the following properties.

1. $|B_i \cup B_{i+1}| \leqslant C\,|B_i \cap B_{i+1}|$.

2. $\operatorname{dist}(x, B_i) \leqslant dr_i$ and $B_i \subset B(x, d\,|x-y|)$ if $i \leqslant 0$. Moreover $r_i \to 0$ as $i \to -\infty$.

3. $\operatorname{dist}(y, B_i) \leqslant dr_i$ and $B_i \subset B(y, d\,|x-y|)$ if $i \geqslant 0$. Moreover $r_i \to 0$ as $i \to +\infty$.

4. No point of Ω belongs to more than C balls B_i.

According to the version of the Lebesgue differentiation theorem given in [26, Chapter 1, Section 1.8], for almost all points $x, y \in \Omega$ and the associated chain $\{B_i\}_{i=-\infty}^{\infty}$ we have

$$\begin{aligned}
|u(x)-u(y)| &\leqslant \sum_{i=-\infty}^{\infty} |u_{B_i} - u_{B_{i+1}}| \\
&\leqslant \sum_{i=-\infty}^{\infty} \left(|u_{B_i} - u_{B_i \cap B_{i+1}}| + |u_{B_{i+1}} - u_{B_i \cap B_{i+1}}|\right) \\
&\leqslant \sum_{i=-\infty}^{\infty} \left(⨍_{B_i \cap B_{i+1}} |u - u_{B_i}| + ⨍_{B_i \cap B_{i+1}} |u - u_{B_{i+1}}| \right) \\
&\leqslant C \sum_{i=-\infty}^{\infty} ⨍_{B_i} |u - u_{B_i}| \\
&\leqslant C \sum_{i=-\infty}^{\infty} r_i ⨍_{B_i} |\nabla u|.
\end{aligned}$$

In the last step we used the Poincaré inequality. Note that by conditions 2) and 3), for each $z \in B_i$, $|x-z| \leqslant Cr_i$ when $i \leqslant 0$ and $|y-z| \leqslant Cr_i$ when $i \geqslant 0$. Hence

$$\begin{aligned}
|u(x)-u(y)| &\leqslant C \sum_{i=-\infty}^{0} \int_{B_i} \frac{|\nabla u(z)|}{|x-z|^{n-1}}\,dz + C \sum_{i=0}^{\infty} \int_{B_i} \frac{|\nabla u(z)|}{|y-z|^{n-1}}\,dz \\
&\leqslant C \left(\int_{B(x,\, d\,|x-y|)} \frac{|\nabla u(z)|}{|x-z|^{n-1}}\,dz + \int_{B(y,\, d\,|x-y|)} \frac{\nabla u(z)}{|y-z|^{n-1}}\,dz \right) \\
&\leqslant C\,|x-y|\,\left(M_{2d\,|x-y|}\,|\nabla u|\,(x) + M_{2d\,|x-y|}\,|\nabla u|\,(y)\right).
\end{aligned}$$

In the last step we used Lemma 5 with $\lambda = 0$. This completes the proof for Lemma 14 and for Theorem 11.

Remarks. (1) An argument, similar to that of Lemma 4, can be used to produce a shorter proof than above. According to the theorem of Martin, [19], every two points in the uniform domain can be joined by a bi-Lipschitz ball as in the proof of Lemma 4. However, this reasoning employs the difficult theorem of Martin.

(2) In the last step of the proof of Lemma 14 the general case of Lemma 5 can be employed to produce the inequality

$$|u(x)-u(y)| \leqslant C\,|x-y|^{1-\lambda}\,(M^{\lambda}_{d\,|x-y|}\,|\nabla u|\,(x)+M^{\lambda}_{d|x-y|}\,|\nabla u|\,(y)) \tag{23}$$

where $C=C(n,c,\lambda)$, $d=d(c)$ and c is the uniformity constant of Ω. This generalizes Lemma 4. Note that in Lemma 4 the constant C_2 cannot be choosen to depend on the Lipschitz constant of Ω only.

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