How to Measure Volume with a Thread

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1. A NAIVE QUESTION. Has it ever occurred to you that one could use a thread to measure volume? Or area? We do not mean imprecise guesses of the following sort: "*This parcel must be horribly large, as 23.5 yards of cord were needed to wrap it up*." The answer "*Sure, I can use a thread to measure the length and width of my bedroom; it is a rectangle 15 feet by 16 feet, so its area is 240 square feet,*" is also unsatisfactory. What we mean is a sort of black box that helps you to determine the volume of *every* reasonable object once you know the length of a certain piece of thread.

Here is the idea. Imagine an ideal, perfectly elastic thread, say of unstretched length one, imprinted with a scale to measure length. Imagine further that this thread is very densely packed into a unit cube C—some Brave Little Tailor, much braver than the Brothers Grimm hero, stretches it (in the process distorting the length scale), pulls it, and twists it so that finally the thread passes through every tiniest speck of the cube, through every cubic millimeter. You can measure the volume of a subset A of C as follows: color all points of A (and hence all points of the thread that lie in A) with a red dye, unpack the thread from the cube, and use the scale to record the total length (we stress: this means total *original* length, before stretching!) of the colored red pieces of the thread. This total length is *equal* to the volume of A! No matter where the set A lies in the cube, no matter what shape it has! This sounds almost magical, yet it is possible, at least in an abstract sense.

Before you start questioning our sanity, let us abandon our analogy and formulate our question in purely mathematical terms, just to avoid misunderstanding and misinterpretation. We ask: Does there exist a continuous, one-to-one mapping $\varphi\colon [0, 1) \to (0, 1)^n$ (this φ gives the rule for packing the thread into the cube; the condition that φ be one-to-one is quite natural—after all, a thread should have no self-intersections) with the following crucial property: for *each* Borel subset A of $(0, 1)^n$ the one-dimensional measure of its preimage $\varphi^{-1}(A)$ should be *equal* to the Lebesgue measure of A (i.e., standard volume for $n = 3$). (In the previous paragraph A was described as a set whose points were stained with "red dye," and the Lebesgue measure of the preimage $\varphi^{-1}(A)$ corresponded, of course, to "total original length of the colored red pieces of the thread before stretching.")

2. THE ANSWER. Here is the answer to our question; some of you might find it unexpected. In what follows $\mathcal{L}^k$ denotes k-dimensional Lebesgue measure.

Thread Theorem. *For each $n \geq 2$ there exists a continuous, one-to-one mapping $\varphi : [0, 1) \to (0, 1)^n$ such that $\mathcal{L}^1(\varphi^{-1}(A)) = \mathcal{L}^n(A)$ for all Borel subsets A of* $[0, 1]^n$.

We postpone the proof of the theorem for a second and quickly mention a few properties of φ. First of all, the image of φ fills almost the whole cube in the measure-theoretic sense (i.e., $\mathcal{L}^n([0, 1]^n \setminus \varphi([0, 1))) = 0$). Otherwise there would be a compact subset K of $[0, 1]^n \setminus \varphi([0, 1))$ with $\mathcal{L}^n(K) > 0$, which would contradict the theorem. However—unlike the classical Peano curve!—φ does not map $[0, 1)$ *onto* $(0, 1)^n$, because it is one-to-one. Of course, we also have $\mathcal{L}^1(B) = \mathcal{L}^n(\varphi(B))$ for each Borel subset B of $(0, 1)$. Thus, if $0 < a < b < 1$, then $\varphi|_{[a,b]}$ parametrizes an arc of positive measure. Finally, since φ is measure-preserving, we immediately obtain a result that is well known to functional analysts: for each n and each p satisfying $1 \leq p \leq \infty$, $L^p([0, 1]^n)$ is isometric to $L^p([0, 1])$ (see also the last corollary). The desired isometry is simply defined by the correspondence $u \mapsto u \circ \varphi$.

The thread theorem is a direct consequence of a beautiful old theorem of von Neumann, Oxtoby, and Ulam (see [**4**, Theorem 2, p. 886] or [**2**, p. 89]). One of our purposes here is to illustrate the power of this surprising result. In what follows we denote the closed unit cube $[0, 1]^n$ in $\mathbb{R}^n$ by Q^n.

Homeomorphic Measures Theorem (von Neumann, Oxtoby, Ulam). *Let μ be a Borel measure on Q^n such that $\mu(Q^n) = 1$, $\mu(\{x\}) = 0$ for all points x of Q^n, $\mu(\partial Q^n) = 0$, and $\mu(U) > 0$ for all nonempty open subsets U of Q^n. Then there exists a homeomorphism $h : Q^n \to Q^n$ such that $\mathcal{L}^n(A) = \mu(h(A))$ for every Borel set A in Q^n. Moreover, any such h must fix each point of ∂Q^n.*

We now apply this result to prove the thread theorem.

Proof of the thread theorem. Let $\widetilde{\varphi} : [0, 1) \to (0, 1)^n$ be an arbitrary continuous one-to-one mapping whose image is dense in Q^n. (If you are not sure whether such mappings $\widetilde{\varphi}$ exist, please read the remark following this proof.) The measure μ defined on the collection of Borel subsets of Q^n by

$$\mu(A) = \mathcal{L}^1\big(\widetilde{\varphi}^{-1}(A)\big)$$

satisfies the assumptions of the homeomorphic measures theorem. Set $\varphi = h^{-1} \circ \widetilde{\varphi}$, where h is a homeomorphism with the properties stipulated in that result. Because $\mathcal{L}^1(\varphi^{-1}(A)) = \mathcal{L}^1(\widetilde{\varphi}^{-1}(h(A))) = \mu(h(A)) = \mathcal{L}^n(A)$, we see that φ satisfies the claim of the thread theorem. ■

Remark. There are uncountably many such $\widetilde{\varphi}$. We give a quick sketch of how to construct one. Let the sequence $a_1, a_2, \ldots$ be dense in $(0, 1)^n$. Set $\widetilde{\varphi}(0) = a_1$, and extend $\widetilde{\varphi}$ to a linear mapping of $[0, 1/2]$ onto the line segment A_1 from a_1 to a_2 so that $\widetilde{\varphi}(1/2) = a_2$. Assume that for some $m \geq 1$ the map $\widetilde{\varphi}$ has already been defined on $I_m := [0, 1 - 2^{-m}]$ so that $\widetilde{\varphi}\,|_{I_m}$ is a smooth embedded curve that passes through $a_1, \ldots, a_m$ (and possibly through lots of other a_j). Pick the first point a_{j_m} in the sequence (a_i) that is not in the compact set $\widetilde{\varphi}(I_m)$. Set $\widetilde{\varphi}(1 - 2^{-m-1}) = a_{j_m}$, join the points $\widetilde{\varphi}(1 - 2^{-m-1})$ and $\widetilde{\varphi}(1 - 2^{-m})$ with a smooth arc A_m that lies except for the point $\widetilde{\varphi}(1 - 2^{-m})$ in $(0, 1)^n \setminus \widetilde{\varphi}(I_m)$, and extend $\widetilde{\varphi}$ to a smooth embedding of I_{m+1} that maps $[1 - 2^{-m}, 1 - 2^{-m-1}]$ to A_m. Take care to avoid a cusp at $\widetilde{\varphi}(1 - 2^{-m})$. All other details are left to you, the reader. Proceeding inductively, one extends $\widetilde{\varphi}$ to $\bigcup_m I_m = [0, 1)$.

We end this section with the following result:

Corollary. *If $k \geq n$ and $1 \leq p \leq \infty$, then there exists an isometric isomorphism $\Phi : L^p([0, 1]^k) \to L^p([0, 1]^n)$ such that $\Phi(u)$ is continuous in $(0, 1)^n$ for each u in $L^p([0, 1]^k)$ that is continuous in $(0, 1)^k$.*

Sketch of proof. Take a continuous, one-to-one mapping $\widetilde{\varphi}: (0, 1)^n \to (0, 1)^k$ whose image is dense in Q^k. (The construction of such $\widetilde{\varphi}$ should present no difficulties to a mathematically mature reader who has mastered the preceding remark.) As in the proof of the thread theorem, take $\varphi = h^{-1} \circ \widetilde{\varphi}$, where h is given by the homeomorphic measures theorem for an appropriate μ. The desired isometry Φ is defined by $u \mapsto u \circ \varphi$. If u is continuous in $(0, 1)^k$, then $\Phi(u)$ is plainly continuous in $(0, 1)^n$. ■

Does the corollary also hold true when $k < n$? We do not know.

3. EXTENSIONS. The thread theorem is in fact the tip of an iceberg. First of all, it provides merely one example of a more general phenomenon. The proof uses two ingredients: the existence of a one-to-one mapping $\varphi : [0, 1) \to (0, 1)^n$ such that $\varphi([0, 1))$ is dense in Q^n, and the homeomorphic measures theorem. Let us now assume that a topological space X is endowed with a nonatomic Borel probability measure μ that is positive on open sets. It follows easily from the homeomorphic measures theorem that if X can be densely and injectively mapped into $(0, 1)^n$, then there also exists an injective mapping $\varphi: X \to (0, 1)^n$ that is measure preserving (i.e., $\mu(\varphi^{-1}(A)) = \mathcal{L}^n(A)$ for all Borel subsets A of $[0, 1]^n$). Thus, one could just as well use μ to measure the volume of Borel subsets of the cube.

Oxtoby and Prasad extended the homeomorphic measures theorem to the Hilbert cube $Q = [0, 1]^{\aleph_0}$ (see [**3**]). The sufficient conditions for a complete Borel measure μ to be obtainable from Lebesgue product measure λ on Q via a homeomorphism $h: Q \to Q$ are even simpler than in the finite dimensional version: μ should be a probability measure that is nonatomic and positive on nonempty open sets. One does not have to assume that μ vanishes on the boundary (if μ vanishes on a sum S of *finitely* many faces of Q, then $h(x) = x$ for each x in S). Thus, it is also possible to use a thread to measure the infinite dimensional volume—one just has to mimick the proof of the thread theorem (i.e., one should (a) construct an injective continuous mapping $\varphi : [0, 1) \to Q$ with $\varphi([0, 1))$ dense in Q and (b) invoke [**3**] instead of the classical n-dimensional homeomorphic measures theorem).

A. H. Stone [**5**] posed the question: For which metric measure spaces (X, μ) does there exist a measure-preserving *homeomorphism* from X onto some topological subspace Q_0 of the Hilbert cube (Q, λ). He was able to obtain a partial answer: if X is a finite dimensional separable metric space endowed with a nonatomic complete regular Borel measure μ such that $\mu(X) < 1$, then there exists a homeomorphism h from (X, μ) onto a subspace of (Q, λ) such that $\lambda(A \cap h(X)) = \mu(h^{-1}(A))$ for all Borel sets A in Q.

To compare Stone's result with the thread theorem, note that we obtain a mapping $\varphi : [0, 1) \to (0, 1)^n$ that is *not* a homeomorphism onto $\varphi([0, 1))$. However, the same mapping restricted to any compact subinterval of $[0, 1)$ becomes a homeomorphism. This suggests that the assumption $\mu(X) < 1$ in Stone's theorem is natural. On the other hand, the assumption that μ be positive on open sets is not present in Stone's result. This is maybe not so surprising (for example, if $X = [a, b]$ with $\mu \equiv 0$, then it is pretty easy to find a homeomorphism of X *onto a subspace* that has the additional desired property).

Finally, one could also wander in a different direction and ask which measures $\mu = f\,dx$ on $[0, 1]^n$ can be obtained from Lebesgue measure by a homeomorphism with preordained differentiability properties. This amounts to looking for maps with prescribed Jacobians (see, for example, the work of Dacorogna and Moser [**1**]) and belongs to the realm of partial differential equations.

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