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ON THE DIFFERENTIABILITY OF SOLUTIONS OF QUASILINEAR ELLIPTIC EQUATIONS

BY

PIOTR HAJŁASZ AND
PAWEŁ STRZELECKI (WARSZAWA)

1. Introduction. In this note we consider weak solutions (in the Sobolev space $W^{1,q}_{\mathrm{loc}}(\Omega)$, $1 < q \le n$) of quasilinear elliptic equations of the type

$$\operatorname{div} \mathbf{A}(x,u,\nabla u) = B(x,u,\nabla u), \quad x \in \Omega \subset \mathbb{R}^n, \tag{1}$$

where $\mathbf{A}$ and B satisfy certain growth conditions given in Section 2. In [3] Yu. G. Reshetnyak proved that weak solutions of (1) are totally differentiable almost everywhere. For a linear equation $\operatorname{div}(a(x)\nabla u) = 0$ this was proved independently by B. Bojarski [1]. We will show how to simplify Reshetnyak's proof by adopting the method of Bojarski.

As was shown by Reshetnyak, the theorem on almost everywhere differentiability is a simple consequence of a difficult theorem of Serrin [5] which asserts the Hölder continuity of weak solutions of (1). We shall use instead a weaker (and much easier to prove) result on local boundedness of weak solutions. Then the final argument is provided by the classical Stepanov differentiability criterion (Theorem 4 below).

This shows that the a.e. differentiability is, in some sense, independent of Hölder continuity of weak solutions of (1). It also seems that the proof presented in this paper is simpler and more natural than the original one: it works for any class of elliptic equations in divergence form for which one is able to prove the local boundedness of weak solutions, provided the *difference quotient* $(u(x_0 + hX) - u(x_0))/h$ satisfies an equation (belonging to the class in question) whenever $u(x)$ does.

Finally, we want to stress the fact that, in the case $1 < q \le n$, Reshetnyak's theorem in fact yields some nontrivial geometric information about weak solutions of (1). First of all, the Hölder exponent provided by [5] is very close to zero. On the other hand, there exist continuous, nowhere differentiable functions $v \in W^{1,n}_{\mathrm{loc}}(\Omega)$ (see the example of Serrin [6]). (For $q > n$ the result becomes trivial: by a well-known theorem of Calderón [2], all

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elements of $W^{1,q}_{\mathrm{loc}}(\Omega)$ are differentiable a.e. The case $q=1$ is rather troublesome, mainly due to the fact that the spaces $L^1(\Omega)$ and $W^{1,1}(\Omega)$ are not reflexive; equation (1) can then admit quite irregular solutions and the results of Serrin are, in general, not valid.)

2. Assumptions and the result. $\mathbf{A}$ and B are respectively $\mathbb{R}^n$-and $\mathbb{R}$-valued functions of $(x,u,p)\in\Omega\times\mathbb{R}\times\mathbb{R}^n$. Moreover, we assume that $\mathbf{A}(x,u(x),p(x))$ and $B(x,u(x),p(x))$ are measurable for any measurable functions $u(x)$ and $p(x)$, and

$$\begin{aligned} |\mathbf{A}(x,u,p)| &\le a|p|^{q-1}+b|u|^{q-1}+e\,,\\ |B(x,u,p)| &\le c|p|^{q-1}+d|u|^{q-1}+f\,,\\ p\cdot\mathbf{A}(x,u,p) &\ge |p|^q-d|u|^q-g\,, \end{aligned}\tag{2}$$

where a is some positive constant, while b, c, d, e, f, g are positive measurable functions each in some L^s:

$$b,e\in L^{n/(q-1-\varepsilon)};\qquad c\in L^{n/(1-\varepsilon)};\qquad d,f,g\in L^{n/(q-\varepsilon)}\tag{3}$$

for some $\varepsilon\in(0,\min\{1,q-1\})$. A function $u\in W^{1,q}_{\mathrm{loc}}(\Omega)$ is called a *weak solution* of (1) if and only if

$$\int\limits_{\Omega}(\nabla\psi\cdot\mathbf{A}(x,u,\nabla u)+\psi B(x,u,\nabla u))\,dx=0\tag{4}$$

for each $\psi\in W^{1,q}_{0,\mathrm{loc}}(\Omega)$, where $W^{1,q}_{0,\mathrm{loc}}(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in $W^{1,q}_{\mathrm{loc}}(\Omega)$. In the sequel $B(x,r)$ will denote the Euclidean ball with center x and radius r; we write $B(r)$ if $x=0$. By $⨍_A f(x)\,dx$ we denote the averaged integral $|A|^{-1}\int_A f(x)\,dx$.

The result of Reshetnyak reads as follows.

THEOREM 1. *Each weak solution of* (1) *is differentiable almost everywhere with respect to the Lebesgue measure in* Ω.

Our proof is very close to the original one. We shall need three theorems. The first one is taken from Serrin [5, Theorems 1 and 2].

THEOREM 2. *Assume that* $u\in W^{1,q}_{\mathrm{loc}}(\Omega)$, $B(2)\Subset\Omega$, *solves the equation* (1). *Then*

$$\|u\|_{\infty,B(1)}\le C(\|u\|_{q,B(2)}+K)\,,$$

where the constant C *depends on* n, q, a, ε, $\|b\|$, $\|c\|$, $\|d\|$ *and*

$$K=(\|e\|+\|f\|)^{1/(q-1)}+\|g\|^{1/q}\,,$$

the norms of $b,\ldots,g$ *being taken in the appropriate* L^s *spaces.*

The next theorem is a slightly weaker version of the L^p-differentiability theorem of Calderón and Zygmund [7, Chapter VIII, Theorem 1] (see also [4]).

THEOREM 3. *Let Ω be an open domain in $\mathbb{R}^n$ and $u \in W^{1,q}_{\mathrm{loc}}(\Omega)$. Then, for $h \to 0$, and for almost all $x_0 \in \Omega$, the following function of $X \in B(2)$:*

$$\frac{u(x_0 + hX) - u(x_0)}{h} - \sum_{i=1}^{n} \frac{\partial u}{\partial x_i}(x_0) X_i$$

tends to zero in $L^q(B(2))$.

The following theorem is due to Stepanov [8] (we recall the statement from [7, Chapter VIII, Theorem 3]).

THEOREM 4 (Stepanov differentiability criterion). *Let $u : \Omega \to \mathbb{R}$ be an arbitrary function defined on an open set $\Omega \subset \mathbb{R}^n$. Define*

$$E = \left\{ a \in \Omega : \limsup_{x \to a} \frac{|u(x) - u(a)|}{|x - a|} < \infty \right\}.$$

Then E is Lebesgue measurable and u is differentiable a.e. in E.

Proof of Theorem 1. Let u be a weak solution of (1). Define the difference quotients

$$v_h(X) = \frac{u(x_0 + hX) - u_0}{h},$$

where $u_0 = u(x_0)$. For $h < \frac{1}{2}\mathrm{dist}(x_0, \partial\Omega)$ this is a well defined function of $X \in B(2)$, readily of class $W^{1,q}(B(2))$. Using the change of variables $x = x_0 + hX$ and the definition of weak solutions of (1) one easily proves that $v_h(X)$ solves the equation

$$\operatorname{div} \mathbf{A}_h(X, v, \nabla_X v) = B_h(X, v, \nabla_X v),$$

where

$$\begin{aligned} \mathbf{A}_h(X, v, p) &= \mathbf{A}(x_0 + hX, u_0 + hv, p), \\ B_h(X, v, p) &= hB(x_0 + hX, u_0 + hv, p), \end{aligned}$$

for $X \in B(2)$, $v \in \mathbb{R}$, $p \in \mathbb{R}^n$. Theorem 2 implies that

$$\sup_{X \in B(1)} \frac{|u(x_0 + hX) - u(x_0)|}{|h|} \leq C_h(\|v_h\|_{q,B(2)} + K_h). \tag{5}$$

Notice that by changing u on a set of measure zero we can actually use supremum instead of essential supremum in (5). Namely, it is enough to put

$$u(x) := \limsup_{r \to 0} \fint_{B(x,r)} u(y)\, dy.$$

We shall show that for almost all $x_0 \in \Omega$ the right hand side of (5) remains bounded when h tends to zero. This will allow us to apply the Stepanov differentiability criterion and finish the proof.

Step 1. Using the properties of $\mathbf{A}$ and B one can easily check that $\mathbf{A}_h$ and B_h satisfy the growth conditions (2) with the same constant a and $b, \ldots, g$ replaced by $b_h, \ldots, g_h$:

$$\begin{aligned}
b_h(X) &= 2^{q-1}|h|^{q-1}b(x_0 + hX)\,,\\
e_h(X) &= 2^{q-1}|u_0|^{q-1}b(x_0 + hX) + e(x_0 + hX)\,,\\
c_h(X) &= |h|c(x_0 + hX)\,,\\
d_h(X) &= 2^{q-1}|h|^{q}d(x_0 + hX)\,,\\
f_h(X) &= |h|(2^{q-1}|u_0|^{q-1}d(x_0 + hX) + f(x_0 + hX))\,,\\
g_h(X) &= 2^{q-1}|u_0|^{q}d(x_0 + hX) + g(x_0 + hX)\,.
\end{aligned}$$

Now, choose x_0 to be an s-Lebesgue point of all the functions $b, c, \ldots, g$ (for each of them take s according to (3)). Then the Lebesgue differentiation theorem implies that the norms of $b_h, c_h, \ldots, g_h$ in the respective $L^s(B(2))$ are bounded for h tending to zero. For instance, if $s = n/(q-\varepsilon)$, then

$$\begin{aligned}
\|g_h\|_s &= \Big(\int_{B(2)} [g_h(X)]^s\,dX\Big)^{1/s}\\
&\le 2^n|B(2)|\Big[2^{q-1}|u_0|^q\Big(\fint_{B(x_0,2h)} [d(y)]^s\,dy\Big)^{1/s} + \Big(\fint_{B(x_0,2h)} [g(y)]^s\,dy\Big)^{1/s}\Big]\\
&\to C_1 d(x_0) + C_2 g(x_0) \qquad \text{as } h \to 0\,,
\end{aligned}$$

and obviously the remaining cases can be treated in the same way. Hence, C_h and K_h on the right hand side of (5) are bounded (by a constant independent of h) when $|h|$ is sufficiently small.

Step 2. Theorem 3 readily implies that for $h \to 0$,

$$v_h(X) - \sum_{i=1}^{n} \frac{\partial u}{\partial x_i}(x_0)X_i$$

tends to zero in $L^q(B(2))$, hence the L^q-norm of v_h is bounded for sufficiently small $|h|$.

Putting together the conclusions of both steps we see that the left hand side of (5) is bounded by a constant independent of h, hence

$$\limsup_{\mathbb{R}^n \ni k \to 0} \frac{|u(x_0 + k) - u(x_0)|}{|k|} < \infty, \qquad \text{a.e. } x_0 \in \Omega\,.$$

Stepanov's criterion (Theorem 4) now implies that u is totally differentiable almost everywhere in Ω. The proof of Theorem 1 is complete.

Remark. One can easily show that this total differential is equal to the weak differential.

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INSTITUTE OF MATHEMATICS
UNIVERSITY OF WARSAW
BANACHA 2
02-097 WARSZAWA, POLAND

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