THE LUSIN THEOREM AND HORIZONTAL GRAPHS IN THE HEISENBERG GROUP

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ABSTRACT. In this paper we prove that every collection of measurable functions f_α, $|\alpha| = m$ coincides a.e. with mth order derivatives of a function $g \in C^{m-1}$ whose derivatives of order $m - 1$ may have any modulus of continuity weaker than that of a Lipschitz function. This is a stronger version of earlier results of Lusin, Moonens-Pfeffer and Francos. As an application we construct surfaces in the Heisenberg group with tangent spaces being horizontal a.e.

1. INTRODUCTION

In 1917 Lusin [8] proved that for every measurable function $f : \mathbb{R} \to \mathbb{R}$ there is a continuous function $g : \mathbb{R} \to \mathbb{R}$ that is differentiable a.e. and such that $g'(x) = f(x)$ for almost all $x \in \mathbb{R}$. This result was generalized by Moonens and Pfeffer [9] to the case of functions defined in $\mathbb{R}^n$ and then by Francos [6] to the case of higher order derivatives in $\mathbb{R}^n$. He proved that if f_α, $|\alpha| = m$ are measurable functions in an open set $\Omega \subset \mathbb{R}^n$, then there is a function $g \in C^{m-1}(\Omega)$ that is m times differentiable a.e. and such that for all $|\alpha| = m$, $D^\alpha g = f_\alpha$ a.e. It is easy to see that in general one cannot require that $g \in C^{m-1,1}_{\text{loc}}$, i.e. one cannot assume that the derivatives of order $m - 1$ are Lipschitz continuous. For example in the case $m = 1$ one cannot find a locally Lipschitz continuous function g on $\mathbb{R}^2$ such that $\nabla g(x, y) = \langle 2y, -2x \rangle$. Indeed, for such a function we would have

$$g(1,1) = g(0,0) + \int_0^1 \frac{\partial g}{\partial x}(t,0)\, dt + \int_0^1 \frac{\partial g}{\partial y}(1,t)\, dt = g(0,0) - 2,$$

$$g(1,1) = g(0,0) + \int_0^1 \frac{\partial g}{\partial y}(0,t)\, dt + \int_0^1 \frac{\partial g}{\partial x}(t,1)\, dt = g(0,0) + 2$$

which is impossible.

Clearly, continuity of derivatives of order $m-1$ in Francos' theorem result from some uniform convergence and one could expect that with keeping track of estimates it should be possible to prove Hölder continuity of derivatives of order $m-1$. However, as we will see, a much stronger result is true. Namely we shall prove that it is possible to construct a function g with *any* modulus of continuity of derivatives of order $m-1$ which is worse than that of a Lipschitz function.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ be open, $m \geq 1$ an integer, and let $f = (f_\alpha)_{|\alpha|=m}$, be a collection of measurable functions $f_\alpha : \Omega \to \mathbb{R}$, $|\alpha| = m$. Let $\sigma > 0$ and let $\mu : [0, \infty) \to [0, \infty)$ be a continuous function with $\mu(0) = 0$ and $\mu(t) = O(t)$ as $t \to \infty$. Then there is a function $g \in C^{m-1}(\mathbb{R}^n)$ that is m-times differentiable a.e., and such that*

  (i) *$D^\alpha g = f_\alpha$ a.e. on Ω for all $|\alpha| = m$;*
  (ii) *$\|D^\gamma g\|_{L^\infty(\mathbb{R}^n)} < \sigma$ for all $0 \leq |\gamma| \leq m-1$;*
  (iii)
$$|D^\gamma g(x) - D^\gamma g(y)| \leq \sigma |x - y|$$
  for all $x, y \in \mathbb{R}^n$ and all $0 \leq |\gamma| \leq m-2$;
  (iv)
$$|D^\gamma g(x) - D^\gamma g(y)| \leq \frac{|x - y|}{\mu(|x - y|)}$$
  for all $x, y \in \mathbb{R}^n$ and all $|\gamma| = m-1$.

In particular, we can take g such that the derivatives $D^\gamma g$, $|\gamma| = m-1$ are λ-Hölder continuous simultaneously for all $\lambda \in (0, 1)$.

Here $\mu(t) = O(t)$ as $t \to \infty$ means that $\mu(t) \leq Ct$ for all $t \geq t_0$.

As an application of this theorem we construct horizontal graphs in the Heisenberg group, see Theorem 3.2. For a related construction, see also [2].

The paper is organized as follows. In Section 2 we prove Theorem 1.1. In Section 3 we provide a brief introduction to the Heisenberg group and then we provide a construction of horizontal graphs based on Theorem 1.1. The notation is pretty standard. By C we will denote a general constant whose value may change within a single string of estimates. By writing $C(n, m)$ we mean that the constant depends on parameters n and m only. The symbol $C_c^m(\Omega)$ will stand for the class of compactly supported C^m functions.

2. Proof of Theorem 1.1

Parts (1)-(4) of the next lemma are due to Francos [6, Theorem 2.4] and it is a generalization of an earlier result of Alberti [1, Theorem 1]. Estimates (5) and (6) are new.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^n$ be open with $|\Omega| < \infty$. Let $m \geq 1$ be an integer, and let $f = (f_\alpha)_{|\alpha|=m}$, be a collection of measurable functions $f_\alpha : \Omega \to \mathbb{R}$, $|\alpha| = m$. Let $\mu : [0, \infty) \to [0, \infty)$ be a continuous function with $\mu(0) = 0$ and $\mu(t) = O(t)$ as $t \to \infty$. Let $\varepsilon, \sigma > 0$. Then there is a function $g \in C_c^m(\Omega)$ and a compact set $K \subset \Omega$ such that*

(1) $|\Omega \setminus K| < \varepsilon$;
(2) $D^\alpha g(x) = f_\alpha(x)$ *for all* $x \in K$ *and* $|\alpha| = m$;
(3)
$$\|D^\alpha g\|_p \leq C(n, m)(\varepsilon/|\Omega|)^{\frac{1}{p}-m}\|f\|_p$$

for all $|\alpha| = m$ *and* $1 \leq p \leq \infty$;
(4) $\|D^\gamma g\|_\infty < \sigma$ *for all* $0 \leq |\gamma| < m$;
(5)
$$|D^\gamma g(x) - D^\gamma g(y)| \leq \sigma|x - y|$$

for all $x, y \in \mathbb{R}^n$ *and all* $0 \leq |\gamma| \leq m - 2$;
(6)
$$|D^\gamma g(x) - D^\gamma g(y)| \leq \frac{|x - y|}{\mu(|x - y|)}$$

for all $x, y \in \mathbb{R}^n$ *and all* $|\gamma| = m - 1$.

Proof. For the proof of existence of $g \in C_c^m(\Omega)$ with properties (1)-(4), see [6, Theorem 2.4]. We need to prove that g can be modified in such a way that (5) and (6) are also satisfied.

Let $K' \subset \Omega$ be a compact set such that $|\Omega \setminus K'| < \varepsilon/2$ and $f|_{K'}$ is bounded. Let $\tilde{f} = f\chi_{K'}$, where $\chi_{K'}$ is the characteristic function of K'. Clearly $\|\tilde{f}\|_\infty < \infty$. By continuity of μ we can find $\delta > 0$ such that

$$\mu(t) \leq \frac{\varepsilon^m}{\sqrt{n}C(n, m)|\Omega|^m\|\tilde{f}\|_\infty} \quad \text{for all } 0 \leq t \leq \delta.$$

Here $C(n, m)$ is the constant from the inequality at (3). In particular if $0 < |x - y| \leq \delta$, then

$$\sqrt{n}C(n, m)\varepsilon^{-m}|\Omega|^m\|\tilde{f}\|_\infty \leq \frac{1}{\mu(|x - y|)}.$$

Let $M = \sup\{\mu(t)/t : t \geq \delta\}$. M is finite, because $\mu(t) = O(t)$ as $t \to \infty$. Applying (1)-(4) to $\tilde{f}$ we can find $g \in C_c^m(\mathbb{R}^n)$ and a compact set $K'' \subset \Omega$ such that

(1') $|\Omega \setminus K''| < \varepsilon/2$;
(2') $D^\alpha g(x) = f_\alpha(x)$ for all $x \in K' \cap K''$ and $|\alpha| = m$;
(3')
$$\|D^\alpha g\|_p \leq C(n,m)(\varepsilon/|\Omega|)^{\frac{1}{p}-m}\|\tilde{f}\|_p$$
for all $|\alpha| = m$ and $1 \leq p \leq \infty$;
(4')
$$\|D^\gamma g\|_\infty < \min\left\{\frac{\sigma}{\sqrt{n}}, \frac{1}{2M},\right\}$$
for all $0 \leq |\gamma| < m$.

Let $K = K' \cap K''$. Then $|\Omega \setminus K| < \varepsilon$ and it is easy to see that the function g has the properties (1)-(4) from the statement of the lemma. We are left with the proof of the properties (5) and (6).

If $0 \leq |\gamma| \leq m-2$, then (4') yields
$$|D^\gamma g(x) - D^\gamma g(y)| \leq \|\nabla D^\gamma g\|_\infty |x-y| \leq \sigma|x-y|.$$

Let now $|\gamma| = m-1$. If $|x-y| \geq \delta$, then
$$|D^\gamma g(x) - D^\gamma g(y)| \leq 2\|D^\gamma g\|_\infty \leq \frac{1}{M} \leq \frac{|x-y|}{\mu(|x-y|)}.$$
If $0 < |x-y| < \delta$, then (3') with $p = \infty$ yields
$$|D^\gamma g(x) - D^\gamma g(y)| \leq \|\nabla D^\gamma g\|_\infty|x-y|$$
$$\leq \sqrt{n}C(n,m)\varepsilon^{-m}|\Omega|^m\|\tilde{f}\|_\infty|x-y| \leq \frac{|x-y|}{\mu(|x-y|)}.$$
The proof is complete. $\square$

Now we can complete the proof of Theorem 1.1. We follow the argument used in [6] and [9], and the only main modification is that we are using improved estimates from Lemma 2.1.

Let $U_1 = \Omega \cap B(0,1)$. Let $V_1 \subset\subset U_1$ be open with with $|U_1 \setminus V_1| < 1/4$. Using Lemma 2.1, we can find a compact set $K_1 \subset V_1$ with $|V_1 \setminus K_1| < 1/4$ and a function $g_1 \in C_c^m(V_1)$ such that

(a) $D^\alpha g_1(x) = f_\alpha(x)$ for all $|\alpha| = m$ and $x \in K_1$;
(b) $|D^\gamma g_1(x)| < 2^{-1}\sigma \min\{\text{dist}^2(x, U_1^c), 1\}$, for all $x \in \mathbb{R}^n$ and $|\gamma| < m$;

(c)
$$|D^\gamma g_1(x) - D^\gamma g_1(y)| \le 2^{-1}\sigma|x - y|$$

for all $x, y \in \mathbb{R}^n$ and all $0 \le |\gamma| \le m - 2$;

(d)
$$|D^\gamma g_1(x) - D^\gamma g_1(y)| \le 2^{-1}\frac{|x - y|}{\mu(|x - y|)}$$

for all $x, y \in \mathbb{R}^n$ and all $|\gamma| = m - 1$.

We now proceed with an inductive definition. Suppose that the sets $K_1, \ldots, K_{k-1}$, and the functions $g_1, \ldots, g_{k-1}$ are defined, for some $k \ge 2$. Let $U_k = \Omega \cap B(0, k) \setminus (K_1 \cup \ldots \cup K_{k-1})$. Let $V_k \subset\subset U_k$ be open with with $|U_k \setminus V_k| < 2^{-k-1}$. Using Lemma 2.1, we find a compact set $K_k \subset V_k$ with $|V_k \setminus K_k| < 2^{-k-1}$ and a function $g_k \in C_c^m(V_k)$ such that

(a') $D^\alpha g_k(x) = f_\alpha(x) - \sum_{j=1}^{k-1} D^\alpha g_j(x)$, for all $|\alpha| = m$ and $x \in K_k$;
(b') $|D^\gamma g_k(x)| < 2^{-k}\sigma \min\left\{\operatorname{dist}^2(x, U_k^c), 1\right\}$, $x \in \mathbb{R}^n$, $|\gamma| < m$;
(c')
$$|D^\gamma g_k(x) - D^\gamma g_k(y)| < 2^{-k}\sigma|x - y|$$

for all $x, y \in \mathbb{R}^n$ and all $0 \le |\gamma| \le m - 2$;

(d')
$$|D^\gamma g_k(x) - D^\gamma g_k(y)| \le 2^{-k}\frac{|x - y|}{\mu(|x - y|)}$$

for all $x, y \in \mathbb{R}^n$ and all $|\gamma| = m - 1$.

We now take $g = \sum_{k=1}^\infty g_k$. We will prove that g satisfies claim of the theorem. First, to see that $g \in C^{m-1}(\mathbb{R}^n)$, we observe that, by (b'),

$$\sum_{k=1}^\infty \|D^\gamma g_k\|_{L^\infty(\mathbb{R}^n)} < \sigma$$

for all $|\gamma| \le m-1$, which implies C^{m-1} differentiability, and this proves (ii). Properties (iii) and (iv) now follow immediately from (c') and (d'). Let $C = \bigcup_{k=1}^\infty K_k$. We have $|\Omega \setminus C| = 0$. We are left with the proof that g is m-times differentiable at all points of C and that $D^\alpha g = f_\alpha$, $|\alpha| = m$ on C. Fix $x \in C$. Then $x \in K_k$ for some k. We write $g = p + q$ where $p = \sum_{j=1}^k g_j$ and $q = \sum_{j=k+1}^\infty g_j$. Now by (a'), we have $D^\alpha p(x) = f_\alpha(x)$ for $|\alpha| = m$, so we are left to show that q is m-times differentiable at x and $D^\alpha q(x) = 0$ for $|\alpha| = m$. Fix $|\gamma| = m - 1$ and consider, for $0 \ne h \in \mathbb{R}^n$, the difference quotient

$|D^\gamma q(x + h) - D^\gamma q(x)| \, |h|^{-1}$. We actually have $D^\gamma q(x) = 0$, because $x \in K_k$ and supp $D^\gamma g_j \cap K_k = \emptyset$ for $j > k$. Hence

$$\frac{|D^\gamma q(x + h) - D^\gamma q(x)|}{|h|} \leq \frac{1}{|h|} \sum_{j=k+1}^\infty |D^\gamma g_j(x + h)| \, .$$

If $D^\gamma g_j(x + h) \neq 0$, $j \geq k + 1$, then $x + h \in U_j$. In this case, since also $x \in K_k \subset U_j^c$, we must have $\text{dist}(x + h, U_j^c) \leq \text{dist}(x + h, x) = |h|$. Hence by (b') we have $|D^\gamma g_j(x + h)| \leq 2^{-j}\sigma|h|^2$. Thus

$$\frac{|D^\gamma q(x + h) - D^\gamma q(x)|}{|h|} \leq \frac{1}{|h|} \sum_{j=k+1}^\infty 2^{-j}\sigma|h|^2 \leq \sigma|h| \to 0 \quad \text{as } h \to 0$$

which proves that the derivative $DD^\gamma q(x) = 0$ equals zero for any $|\gamma| = m - 1$. This also completes the proof of (i).

In particular, if we define

$$\mu(t) = \begin{cases} 0 & t = 0 \\ |\log t|^{-1} & 0 < t \leq e^{-1} \\ et & t > e^{-1}, \end{cases}$$

then evidently μ satisfies the hypotheses above of the theorem, and for every $\lambda \in (0, 1)$ there is number $C_\lambda > 0$ such that

$$\mu(t) > C_\lambda t^{1-\lambda}, \quad t > 0$$

In that case derivatives $D^\gamma g$, $|\gamma| = m - 1$ satisfy

$$|D^\gamma g(x) - D^\gamma g(y)| \leq C_\lambda^{-1}|x - y|^\lambda \quad \text{for all } x, y \in \mathbb{R}^n \text{ and } \lambda \in (0, 1).$$

The proof is complete. $\square$

3. The Heisenberg group

In this section we will show how to use Theorem 1.1 to construct horizontal graphs in the Heisenberg group. While our construction works for groups $\mathbb{H}_n$, for the sake of simplicity of notation we will restrict to the group $\mathbb{H}_1$; the generalization to the case of $\mathbb{H}_n$ is straightforward. For more information about the Heisenberg group and for references to results that are quoted here without proof, see for example [4].

The *Heisenberg group* is a Lie group $\mathbb{H}_1 = \mathbb{C} \times \mathbb{R} = \mathbb{R}^3$ equipped with the group law

$$(z, t) * (z', t') = \left(z + z', t + t' + 2 \operatorname{Im} z\overline{z'}\right).$$

A basis of left invariant vector fields is given by

$$(3.1) \qquad X = \frac{\partial}{\partial x} + 2y\frac{\partial}{\partial t}, \; Y = \frac{\partial}{\partial y} - 2x\frac{\partial}{\partial t}, \text{and } T = \frac{\partial}{\partial t} \, .$$

Here and in what follows we use notation $(z, t) = (x, y, t)$. The Heisenberg group is equipped with the *horizontal distribution* $H\mathbb{H}_1$, which is defined at every point $p \in \mathbb{H}_1$ by

$$H_p\mathbb{H}_1 = \text{span}\left\{X(p), Y(p)\right\}.$$

The distribution $H\mathbb{H}_1$ is equipped with the left invariant metric $\mathbf{g}$ such that the vectors $X(p), Y(p)$ are orthonormal at every point $p \in \mathbb{H}_1$. An absolutely continuous curve $\gamma : [a, b] \to \mathbb{H}_1$ is called *horizontal* if $\gamma'(s) \in H_{\gamma(s)}\mathbb{H}_1$ for almost every s. The Heisenberg group $\mathbb{H}_1$ is equipped with the *Carnot-Carathéodory metric* d_{cc} which is defined as the infimum of the lengths of horizontal curves connecting two given points. The lengths of curves are computed with respect to the metric $\mathbf{g}$ on $H\mathbb{H}_1$. It is well known that any two points in $\mathbb{H}_1$ can be connected by a horizontal curve and hence d_{cc} is a true metric. Actually, d_{cc} is topologically equivalent to the Euclidean metric. Moreover, for any compact set K there is a constant $C \geq 1$ such that

$$(3.2) \qquad C^{-1}|p - q| \leq d_{cc}(p, q) \leq C|p - q|^{1/2}$$

for all $p, q \in K$. In what follows $\mathbb{H}_1$ will be regarded as a metric space with metric d_{cc}. The Heisenberg group is an example of a sub-Riemannian manifold [7].

It is often more convenient to work the *Korányi metric* which is bi-Lipschitz equivalent to the Carnot-Carathéodory metric, but is much easier to compute. The Korányi metric is defined by

$$d_K(p, q) = \|q^{-1} * p\|_K, \quad \text{where} \quad \|(z, t)\|_K = \left(|z|^4 + t^2\right)^{1/4} \, .$$

For nonnegative functions f and g we write $f \approx g$ if $C^{-1}f \leq g \leq Cf$ for some constant $C \geq 1$. Thus bi-Lipschitz equivalence of metrics means that $d_K \approx d_{cc}$. A straightforward computation shows that for $p = (z, t) = (x, y, t)$ and $q = (z', t') = (x', y', t')$ we have

$$(3.3) \qquad d_K(p, q) \approx |z - z'| + |t - t' + 2(x'y - xy')|^{1/2}.$$

The inequality (3.2) implies that the identity mapping from $\mathbb{H}_1$ to $\mathbb{R}^3$ is locally Lipschitz, but its inverse is only locally Hölder continuous with exponent $1/2$. One can prove that the Hausdorff dimension of any open set in $\mathbb{H}_1$ equals 4 and hence $\mathbb{H}_1$ is not bi-Lipschitz homeomorphic to $\mathbb{R}^3$, not even locally.

Gromov [7, 0.5.C] posted the following question: *Given two sub-Riemannian manifolds V and W and $0 < \alpha \leq 1$, describe the space of C^α maps $f : V \to W$.* He also asked explicitly about the existence of Hölder continuous embeddings and homeomorphisms. In particular Gromov [7, Corollary 3.1.A], proved that if $f : \mathbb{R}^2 \to \mathbb{H}_1$ is a C^α-embedding, then $\alpha \leq 2/3$ and he conjectured that $\alpha = 1/2$. This leads to a search for various surfaces in the Heisenberg group with interesting geometric properties from the perspective of the Carnot-Carathéodory metric and with suitable estimates for the Hölder continuity of a parametrization.

A problem which is related, but of independent interest, is that of finding estimates for the size of the characteristic set on a surface S in $\mathbb{H}_1$. We say that a point on a surface in the Heisenberg group is *characteristic* if the tangent plane at this point is horizontal. The characteristic set $C(S)$ is the collection of all characteristic points on S. In general the Hausdorff dimension of $C(S)$ on a regular surface is small. Denote by $\mathcal{H}_E^s$ and $\dim_E$ the Hausdorff measure and the Hausdorff dimension with respect to the Euclidean metric. Balogh [2] proved that if S is a C^2 surface in $\mathbb{H}_1$, then $\dim_E(C(S)) \leq 1$ and if S is a $C^{1,1}$ surface, then $\dim_E C(S) < 2$. On the other hand he proved that $\bigcap_{0 < \alpha < 1} C^{1,\alpha}$ surfaces may satisfy $\mathcal{H}_E^2(C(S)) > 0$. For other related results, see [5]. We should also mention the paper [3] that contains a construction of horizontal fractals being graphs of BV functions.

These questions motivated us in the construction of the example that we describe next (Theorem 3.2). In what follows we will investigate surfaces in $\mathbb{H}_1$ being graphs of continuous functions of variables (x, y). Given a function $u : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}^2$ we denote by $\Phi(x, y) = (x, y, u(x, y))$ the canonical parametrization of the graph. We regard Φ as a mapping from Ω to $\mathbb{H}_1$.

Proposition 3.1. *Suppose $\Omega \subset \mathbb{R}^2$ is bounded. Then u is α-Hölder continuous, $\alpha \in (0, 1]$ if and only if $\Phi : \Omega \to \mathbb{H}_1$ is $\alpha/2$-Hölder continuous.*

Proof. Suppose that u is α-Hölder continuous. We need to prove that

$$
\begin{aligned}
d_K(\Phi(z), \Phi(z')) &\approx |z - z'| + |u(z) - u(z') + 2(x'y - xy')|^{1/2} \\
&\leq C|z - z'|^{\alpha/2}.
\end{aligned}
$$
(3.4)

Since Ω is bounded and $t \leq C t^{\alpha/2}$ for $0 \leq t \leq t_0$, we have $|z - z'| \leq C|z - z'|^{\alpha/2}$ for all $z, z' \in \Omega$. Similarly boundedness of Ω yields

$$
\begin{aligned}
|2(x'y - xy')|^{1/2} &\leq \sqrt{2}(|y|\,|x - x'| + |x|\,|y - y'|)^{1/2} \leq C|z - z'|^{1/2} \\
\text{(3.5)} &\leq C'|z - z'|^{\alpha/2}.
\end{aligned}
$$

The above estimates and the α-Hölder continuity of u readily imply (3.4).

Suppose now that Φ is $\alpha/2$-Hölder continuous i.e., (3.4) is true. The triangle inequality, (3.5) and (3.4) yield

$$
\begin{aligned}
|u(z) - u(z')|^{1/2} &\leq |u(z) - u(z') + 2(x'y - xy')|^{1/2} + C|z - z'|^{\alpha/2} \\
&\leq C'|z - z'|^{\alpha/2}
\end{aligned}
$$

which in turn implies α-Hölder continuity of u. The proof is complete. $\square$

If $u : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}^2$ is Lipschitz continuous, then u is differentiable a.e. and hence the graph of u has the tangent plane for a.e. $z \in \Omega$. However, it cannot happen that the tangent plane to the graph is horizontal a.e. Indeed, it is well known and easy to check that the tangent plane at $(x, y, u(x, y))$ is horizontal if and only if

$$
\frac{\partial u}{\partial x} = 2y \quad \text{and} \quad \frac{\partial u}{\partial y} = -2x,
$$

but we have already checked at the beginning of this article that this system of equations admits no Lipschitz solutions. However, we have the following result.

Theorem 3.2. *Let $\mu : [0, \infty) \to [0, \infty)$ be a continuous function such that $\mu(0) = 0$ and $\mu(t) = O(t)$ as $t \to \infty$. Then there is a continuous function $u : \mathbb{R}^2 \to \mathbb{R}$ such that*

(1) $|u(x) - u(y)| \leq |x - y|/\mu(|x - y|)$ *for all $x, y \in \mathbb{R}^2$;*
(2) *u is differentiable a.e.;*
(3) *the tangent plane to the graph of u is horizontal for almost all $(x, y) \in \mathbb{R}^2$.*

This result is a straightforward consequence of Theorem 1.1 in the case of $m = 1$ and $f_1 = 2y$, $f_2 = -2x$.

Note that for almost all $(x, y) \in \mathbb{R}^n$ the corresponding points on the surface are characteristic. The result is sharp – any modulus of continuity stronger than that in (1) would mean that the function is Lipschitz continuous and for such functions there are no surfaces with

the property (3). Proposition 3.1 allows one to reinterpret the theorem in terms of Hölder continuous surfaces with horizontal tangent planes. We leave details to the reader.

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