DIFFERENTIABILITY OF p-HARMONIC FUNCTIONS ON METRIC MEASURE SPACES

JASUN GONG AND PIOTR HAJŁASZ

Abstract. We study p-harmonic functions on metric measure spaces, which are formulated as minimizers to certain energy functionals. For spaces supporting a p-Poincaré inequality, we show that such functions satisfy an infinitesmal Lipschitz condition almost everywhere. This result is essentially sharp, since there are examples of metric spaces and p-harmonic functions that fail to be locally Lipschitz continuous on them.

As a consequence of our main theorem, we show that p-harmonic functions also satisfy a generalized differentiability property almost everywhere, in the sense of Cheeger's measurable differentiable structures.

1. Introduction

A celebrated theorem of Ural'tseva [Ura68] states that p-harmonic functions, for $1 < p < \infty$, are locally $C^{1,\alpha}$-smooth (cf. [Lew83]). Recall that for an open set $\Omega \subset \mathbb{R}^n$, a function $u_0 \in W^{1,p}_{loc}(\Omega)$ is called *p-harmonic* if it is a local minimizer of the p-energy functional

$$\mathcal{F}[u] := \int_\Omega |\nabla u|^p \, dx. \tag{1.1}$$

Since ∇u_0 is only locally p-integrable, it may happen that $\mathcal{F}[u_0] = \infty$. We therefore define local minimizers u_0 as follows: for every open set $U \subset\subset \Omega$ and $v \in W^{1,p}_0(U)$,

$$\int_U |\nabla u|^p \, dx \;\leq\; \int_U |\nabla(u+v)|^p \, dx.$$

Equivalently, $u_0 \in W^{1,p}_{loc}(\Omega)$ is a local minimizer of $\mathcal{F}$ if and only if u_0 is a solution to the *p-harmonic equation*

$$\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) \;=\; 0. \tag{1.2}$$

This regularity result has been extended to more general equations of the form

$$\operatorname{div} \mathbf{A}(x,u,\nabla u) \;=\; B(x,u,\nabla u) \tag{1.3}$$

under certain structure conditions on $\mathbf{A} : \Omega\times\mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ and $B : \Omega\times\mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$; see [LU68], [Uhl77], [DiB83]. As a direct consequence of Ural'tseva's result, we see that p-harmonic functions are locally Lipschitz continuous.

Our aim is to investigate the regularity of p-harmonic functions in the setting of metric measure spaces, i.e. metric spaces equipped with a Borel measure. Analysis on such spaces has been a subject of much investigation for the last two decades.

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In particular, a suitable notion of (upper) gradient, Sobolev space, and p-harmonic function has been developed in this setting.

General metric measure spaces are too rough for such analysis to be interesting, but the class of spaces that support a p-Poincaré inequality turns out to be particularly rich. Many results related to analysis of first-order derivatives, known in the Euclidean setting, can be generalized to spaces that support a p-Poincaré inequality. This class of spaces was introduced by Heinonen and Koskela [HK98] in their study of quasiconformal mappings between metric spaces. Subsequently, Cheeger [Che99] proved that such spaces admit a measurable differentiable structure. For the theory of Sobolev spaces on metric spaces supporting Poincaré inequalities, see [HK95], [HK98], [HK00] [Che99], [Sha00], [Haj03] and the theory of p-harmonic functions has been investigated in [KS01], [Sha01], [KM03], [Sha03], [BBS03a], [BBS03b], [BM06], and [GMP10].

Here the relevant question is: *which regularity results of p-harmonic functions, known for Euclidean spaces, also hold true for spaces supporting Poincaré inequalities?* The Moser iteration technique and the De Giorgi method can be adapted to such spaces. This implies that on spaces that support a p-Poincaré inequality, p-harmonic functions satisfy the Harnack inequality and are therefore locally Hölder continuous [KS01], [BM06]. The $C^{1,\alpha}$-regularity, on the other hand, cannot be generalized; the differentiable structure of Cheeger is only measurable and we do not consider continuity properties of differentials. However, it does make sense to ask whether p-harmonic functions are locally Lipschitz continuous.

The difficulty here is that standard proofs of Lipschitz continuity of p-harmonic functions involve estimates for second-order derivatives. In contrast, it is not possible to define second-order derivatives in the general setting of spaces supporting Poincaré inequalities. A new technique is therefore needed. We know of two such approaches: Koskela, Rajala, and Shanmugalingam [KRS03] proved that if the space supports a 2-Poincaré inequality and a certain heat kernel estimate, then 2-harmonic functions are locally Lipschitz continuous. Petrunin [Pet03] proved that 2-harmonic functions on Alexandrov spaces are also locally Lipschitz continuous.

Neither of the two approaches, however, can be generalized to the general case of spaces supporting a p-Poincaré inequality. An example provided in [KRS03, p. 150] shows a space that supports a 2-Poincaré inequality and a 2-harmonic function which fails to be Lipschitz continuous at one point.

Let u be a p-harmonic function on a metric measure space (X, d, μ) that supports a p-Poincaré inequality. Although u is not necessarily (locally) Lipschitz continuous, in our main result (Theorem 3.1) we nonetheless prove that the *pointwise Lipschitz constant* is finite μ-almost everywhere. More precisely, for μ-a.e. $x \in X$,

$$\operatorname{Lip}[u](x) \;:=\; \limsup_{r\to 0} \sup_{y\in B(x,r)} \frac{|u(y)-u(x)|}{r} \;<\; \infty.$$

Together with a generalization of the Stepanov theorem [BRZ04], this implies that p-harmonic functions are μ-a.e. Cheeger differentiable (Corollary 3.2). In fact, our argument is not only restricted to p-harmonic functions; we also prove the result for quasi-minimizers of the functionals

$$\mathcal{F}[u] \;:=\; \int_X F(x, u, g_u)\, d\mu$$

where g_u is the minimal p-weak upper gradient of u and F satisfies certain growth conditions, as discussed in §2.3.

In fact, Bojarski [Boj85] and Reshetnyak [Res87] proved that under certain growth and structure conditions, solutions to the nonlinear elliptic equation (1.3) are also a.e. differentiable, even if the equation does not allow for the estimates of second-order derivatives. For other Euclidean generalizations of this result, see [Str92a], [Str92b], [HS93], [Jež94], [KR97], [Bjö01], and [HM02]. Our proof is an adaptation of the technique presented in [HS93] to the metric measure space setting.

The paper is organized as follows. In Section 2 we recall basic facts about upper gradients and Newtonian spaces, an analogue of Sobolev spaces on metric spaces. We also discuss several weak differentiability theorems, as well as several useful properties of quasi-minimizers. We state our main result in Section 3. To prove it, we will use a rescaling principle to control the "difference quotients" of quasi-minimizers, which reduces the setting to that of existing differentiability theorems.

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2. Preliminaries: Analysis on Metric Spaces

In this section, we discuss a generalization of Sobolev spaces to metric spaces, as introduced in [Sha00]; an equivalent formulation can be found in [Che99]. Other standard references are [Haj96], [HK98], [HK00], [Hei01], and [Haj03].

To fix terminology, a *metric measure space* (X, d, μ) consists of a metric space (X, d) equipped with a Borel measure μ on X. Given a ball $B = B(x, r)$ and $\lambda > 0$, we write $\lambda B = B(x, \lambda r)$. We write $L^p(X)$ for the space of all p-intergrable functions on X with respect to μ, and $\|\cdot\|_p$ denotes the usual norm on $L^p(X)$. For a locally integrable function $u : X \to \mathbb{R}$, its average value over a ball B is

$$u_B \;:=\; \fint_B u\,d\mu \;=\; \frac{1}{\mu(B)}\int_B u\,d\mu.$$

(By a locally integrable function, we are referring here to functions that are integrable on every ball in their domains.)

Unless otherwise stated, the letter C will denote a positive constant that depends on certain parameters to be specified. The exact value of C may change, even within the same line.

2.1. **Generalized Sobolev spaces.**

We begin with line integrals on metric spaces and proceed to upper gradients, which are defined in terms of line integrals and a generalized Fundamental Theorem of Calculus.

For a rectifiable curve $\gamma : [a, b] \to X$ (with respect to the metric d on X) and a Borel function $\rho : X \to [0, \infty]$, we define the line integral of ρ on γ to be

$$\int_\gamma \rho\,ds \;:=\; \int_0^{\ell(\gamma)} \rho(\hat{\gamma}(t))\,dt$$

where $\ell(\gamma)$ is the length of γ and $\hat{\gamma} : [0, \ell(\gamma)] \to [0, \infty]$ is the arc-length parametrization of γ. For $u : X \to \mathbb{R}$ we say that a Borel function $g : X \to [0, \infty]$ is an *upper gradient* of u if

$$|u(\gamma(b)) - u(\gamma(a))| \;\le\; \int_\gamma g\,ds \tag{2.1}$$

holds for every rectifiable curve $\gamma : [a, b] \to X$.

As examples, every smooth function $u : \mathbb{R}^n \to \mathbb{R}$ admits $|\nabla u|$ as an upper gradient (with respect to the Euclidean metric and the Lebesgue measure). More generally, for each Lipschitz function f on a metric space X, the *pointwise Lipschitz constant*

$$\mathrm{Lip}[f](x) \;:=\; \limsup_{r\to 0} \sup_{y\in B(x,r)} \frac{|f(y)-f(x)|}{r}$$

is an upper gradient of f [Sem95, Lem 1.20] (see also [Che99, Prop 1.11]).

For technical reasons we also consider *p-weak upper gradients*, which are defined similarly but allow an exceptional class of curves to the condition in (2.1). This in turn requires a means of measuring families of rectifiable curves. Unlike upper gradients, p-weak upper gradients enjoy the property that their μ-a.e. representatives are also p-weak upper gradients [Haj03, Lem 6.2].

Motivated by this, let Γ be a collection of non-constant rectifiable curves on X. For $p \geq 1$, the *p-modulus* of Γ is defined as

$$\mathrm{mod}_p(\Gamma) \;:=\; \inf_{\rho} \int_X \rho^p \, d\mu$$

where the infimum is taken over all Borel functions $\rho : X \to [0,\infty]$ satisfying $\int_\gamma \rho \, ds \geq 1$ for all $\gamma \in \Gamma$. It is well known that for each $p \geq 1$, the p-modulus is an outer measure on $\mathcal{M}$, the family of all rectifiable curves on X.

Definition 2.1. For $u : X \to \mathbb{R}$, we say that a Borel function $g : X \to [0,\infty]$ is a *p-weak upper gradient* of u if Equation (2.1) holds for mod_p*-a.e. curve* $\gamma \in \mathcal{M}$ — that is, if the sub-collection of curves in Γ for which the property fails has p-modulus zero.

We now define analogue of Sobolev spaces on metric measure spaces.

Definition 2.2. Let $p \geq 1$. We say that a function $u : X \to \mathbb{R}$ lies in the space $\hat{N}^{1,p}(X)$ if $u \in L^p(X)$ and if the quantity

$$\|u\|_{1,p} \;:=\; \|u\|_p \;+\; \inf_g \|g\|_p \tag{2.2}$$

is finite, where the infimum is taken over all p-weak upper gradients g of u. Furthermore, for $u, v \in \hat{N}^{1,p}(X)$, we write $u \cong v$ whenever $\|u - v\|_{1,p} = 0$.

The *Newtonian space* $N^{1,p}(X)$ consists of equivalence classes of functions in $\hat{N}^{1,p}(X)$ with respect to the equivalence relation $(\cong)$.

For a domain $\Omega \subset X$, the space $N^{1,p}(\Omega)$ is defined similarly, and $N^{1,p}_0(\Omega)$ denotes the functions in $N^{1,p}(\Omega)$ with zero boundary values; for details, see [Sha00], [KS01].

We conclude this section with several basic properties of the Newtonian space. They are, respectively, [Sha00, Thm 3.7] and [Haj03, Thm 7.16].

Theorem 2.3. *Let (X, d, μ) be a metric measure space and let $p \geq 1$.*

1. *$\big(N^{1,p}(X), \|\cdot\|_{1,p}\big)$ is a Banach space.*
2. *For each $u \in N^{1,p}(X)$, there is a unique p-weak upper gradient $g_u \in L^p(X)$ with the following property: for every p-weak upper gradient $g \in L^p(X)$ of u, we have $g_u \leq g$ μ-a.e. on X.*

We refer to g_u as the *minimal p-weak upper gradient* of u.

For Euclidean spaces, we have $W^{1,p}(\mathbb{R}^n) = N^{1,p}(\mathbb{R}^n)$ as sets and their norms are equal [Sha00, Thm 4.5]. Each $u \in W^{1,p}(\mathbb{R}^n)$ satisfies $g_u = |\nabla u|$.

2.2. **Poincaré Inequalities & Generalized Differentiability.** For metric measure spaces under certain hypotheses, a generalized version of Rademacher's theorem holds. We begin with the first hypothesis: a growth condition for measures.

Definition 2.4. We say that a Borel measure μ on X is *doubling* if balls have finite and positive μ-measure, and there is a constant $\kappa \geq 1$ so that the inequality

$$\mu(B(x,2r)) \;\leq\; \kappa\mu(B(x,r)) \tag{2.3}$$

holds, for all $x \in X$ and $0 < r < \operatorname{diam}(X)$.

As a shorthand, we call $Q := \log_2 \kappa$ the *doubling exponent* of X.

The Lebesgue measure on $\mathbb{R}^n$ is clearly doubling. On a Riemannian manifold with non-negative Ricci curvature, the volume element is also doubling; this follows from the Bishop-Gromov comparison theorems [CE75].

As a consequence of the doubling condition (2.3), the μ-measures of balls are locally bounded from below by powers of their radii [Haj03, Lem 4.7].

Lemma 2.5. *Let (X,d) be a metric space and let μ be a κ-doubling measure, for some $\kappa \geq 1$. Then for each ball $B_0 := B(x_0,r_0)$ in X,*

$$4^{-s}\Big(\frac{r}{r_0}\Big)^s \;\leq\; \frac{\mu(B(x,r))}{\mu(B_0)} \tag{2.4}$$

for all $x \in B(x_0,r_0)$ and all $0 < r \leq r_0$.

On Euclidean spaces, the Poincaré inequality states that the mean oscillation of a smooth function is controlled by the average of its gradient. Below, we formulate this property in the case of metric measure spaces.

Definition 2.6. Let $p \geq 1$. A complete metric measure space (X,d,μ) is said to support a *p-Poincaré inequality* if there exist $C > 0$ and $\Lambda \geq 1$ so that

$$\fint_B |u - u_B|\, d\mu \;\leq\; C \operatorname{diam}(B) \Big(\fint_{\Lambda B} g_u^p \, d\mu\Big)^{1/p} \tag{2.5}$$

holds for all $u \in N^{1,p}(X)$ and all balls B in X.

Standard Hypotheses 2.7. Here and in what follows, we assume that all metric spaces (X,d) are complete and support both a doubling measure μ and a p-Poincaré inequality; that is, Equations (2.3) and (2.5) hold with fixed parameters κ, C, Λ.

We also say that a constant $C > 0$ *depends on the parameters of the space* and write $C = C(X)$, if it depends on the parameters κ, C, Λ from the above hypotheses.

The following result, due to Cheeger [Che99], extends the classical Rademacher theorem to metric measure spaces. With respect to certain choices of "coordinate" functions, Lipschitz functions satisfy a generalized differentiability property.

Theorem 2.8 (Cheeger, 1999)**.** *For a metric measure space (X,d,μ) satisfying Standard Hypotheses 2.7, there exist $N = N(X) \in \mathbb{N}$, and*

(1) *a collection of μ-measurable, pairwise-disjoint subsets $\{X_k\}_{k=1}^\infty$ of X with*

$$\mu\Big(X \setminus \bigcup_{k=1}^\infty X_k\Big) \;=\; 0,$$

(2) *a collection of numbers $\{n_k\}_{k=1}^\infty$, with $0 \leq n_k \leq N$,*
(3) *a family of Lipschitz mappings $\xi_k : X \to \mathbb{R}^{n_k}$, with $k \in \mathbb{N}$,*

with the following property: for each $k \in \mathbb{N}$ and each Lipschitz function $f : X \to \mathbb{R}$, there is a unique map $D_k f \in L^\infty(X_k; \mathbb{R}^{n_k}) \subset L^\infty(X_k; \mathbb{R}^N)$ so that, for μ-a.e. $x \in X_k$,

$$\lim_{y \to x} \frac{f(y) - f(x) - D_k f(x) \cdot \big(\xi_k(y) - \xi_k(x)\big)}{d(x,y)} \;=\; 0. \tag{2.6}$$

Remark 2.9. By Keith's theorem [Kei04, Thm 2.7], the maps ξ_k can be chosen as distance functions; that is, their component functions are of the form $x \mapsto d(x, x_k)$. It follows that the family of functions $\{\xi_k\}_{k=1}^\infty$ is *uniformly* 1*-Lipschitz.*

By the uniqueness part of Theorem 2.8, the operator

$$f \;\mapsto\; Df \;:=\; \sum_{k=1}^\infty \chi_{X_k} \cdot D_k f$$

is linear, with $Df \in L^\infty(X; \mathbb{R}^N)$. We refer to Df as the *(Cheeger) differential* of f.

Combining [Sha00, Thm 4.1] with Standard Hypotheses 2.7, we see that Lipschitz functions are norm-dense in $N^{1,p}(X)$. The differential Du is therefore well-defined for all functions in $u \in N^{1,p}(X)$, as proven in [FHK99, Thm 10]; see also [BRZ04, Thm 4.4]. However, for $p < n$ not every function $f \in W^{1,p}(\mathbb{R}^n)$ is a.e. differentiable with respect to the Lebesgue measure. Similarly, Equation (2.6) may fail to hold for functions in $N^{1,p}(X)$. This leads to the following definition.

Definition 2.10. For a function $f : X \to \mathbb{R}$, we say that f is *Cheeger differentiable* at a point $x \in X_k$ if Equation (2.6) holds for f at x.

As in the case of $\mathbb{R}^n$ and the gradient map $f \mapsto \nabla f$, similar properties hold for the differential $u \mapsto Du$ on metric spaces. The first theorem gives generalizations of the classical Stepanov and Calderón theorems [BRZ04, Thm 3.1 & Cor 4.3].

Theorem 2.11 (Balogh-Rogovin-Zürcher, 2004)**.** *If (X, d, μ) is a metric measure space satisfying Standing Hypotheses 2.7, then*

(1) *each function $f : X \to \mathbb{R}$ is Cheeger differentiable at μ-a.e. point of the set*
$$S(f) \;:=\; \Big\{ x \in X \;:\; \limsup_{r \to 0} \sup_{y \in B(x,r)} \frac{|f(x) - f(y)|}{r} \;<\; \infty \Big\},$$

(2) *for $p > Q$, each $f \in N^{1,p}(X)$ has a locally Hölder continuous representative that is Cheeger differentiable μ-a.e. on X.*

The next fact states that Cheeger differentials satisfy a first-order Taylor approximation property, in terms of integral averages [Bjö00, Cor 4.6]. To clarify, the original result is formulated for the space $H^{1,p}(X)$, which is defined as the closure of locally Lipschitz functions on X with respect to the norm

$$\|f\|_{H^{1,p}(X)} \;:=\; \|f\|_{L^p(X)} + \|Df\|_{L^p(X)}.$$

However, from the hypothesis of a p-Poincaré inequality and from [Haj03, Thms 10.2 & 11.2], it follows that $H^{1,p}(X) = N^{1,p}(X)$, so the same result also holds for $N^{1,p}(X)$. We state this version below.

Theorem 2.12 (Björn, 2000)**.** *Let (X, d, μ) be a metric measure space satisfying Standard Hypotheses 2.7, with $1 < p < \infty$, and let $\{X_k\}_{k=1}^\infty$ and $\{\xi_k\}_{k=1}^\infty$ be as in Theorem 2.8. For each $u \in N^{1,p}(X)$ and μ-a.e. $x \in X_k$, we have*

$$\lim_{r \to 0} \frac{1}{r} \Big(\fint_{B(x,r)} \Big| u(y) - u(x) - D_k u(x) \cdot \big(\xi_k(y) - \xi_k(x)\big) \Big|^p \, d\mu(y) \Big)^{\frac{1}{p}} \;=\; 0.$$

2.3. **Quasi-minimizers and Local Boundedness.** We now formulate a variational problem on metric measure spaces. For a domain $\Omega \subset\subset X$ and a function $F : X \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, consider functionals on $N^{1,p}_{\mathrm{loc}}(X)$ of the form

$$\mathcal{F}\big(u;\Omega\big) \ := \ \int_\Omega F\big(x, u(x), g_u(x)\big) d\mu(x). \tag{2.7}$$

Definition 2.13. A function $u \in N^{1,p}_{\mathrm{loc}}(X)$ is called a *(K-)quasi-minimizer* (of the functional (2.7)) if there exists $K \geq 1$ so that

$$\mathcal{F}\big(u;\Omega\cap\{u\neq v\}\big) \ \leq \ K\,\mathcal{F}\big(v;\Omega\cap\{u\neq v\}\big)$$

for all $\Omega \subset\subset X$ and all $v \in N^{1,p}_{\mathrm{loc}}(X)$ with $u - v \in N^{1,p}_0(\Omega)$. If $K = 1$, then u is called a *minimizer* (of (2.7)).

For the case of $X = \mathbb{R}^n$ and $F = |\nabla u|^p$, it is well known that u is a weak solution of (1.2) if and only if it is a minimizer of the functional (2.7). The same holds true for equations (1.3) with a vector field $\mathbf{A} : \Omega\times\mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ and a non-homogeneous term $B : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, where $\mathbf{A}$ and B satisfy certain growth and structure conditions; for details, see [Giu03, Chap. 5].

Structure Conditions 2.14. Motivated by this, in what follows we will focus exclusively on functionals whose integrand F satisfies

$$|z|^p - b(x)|y|^p - a(x) \ \leq \ F(x,y,z) \ \leq \ L|z|^p + b(x)|y|^p + a(x) \tag{2.8}$$

where $L \geq 1$, $1 < p < Q$, $s > Q/p$, and $a, b \in L^s(X)$ with $a, b \geq 0$ μ-a.e. on X. (As before, Q denotes the doubling exponent of X.)

We call the parameters s, L, $a(x)$, and $b(x)$ the *data* of the functional $\mathcal{F}$. We will also say that a constant $C > 0$ *depends on data* and write $C = C(\mathrm{data})$, if C depends on s, L, $a(x)$, and $b(x)$ from (2.8).

As an example, the non-homogeneous p-energy functional

$$u \ \mapsto \ \int_\Omega \big(g_u^p + ub + a\big)\,dx \tag{2.9}$$

corresponds to the p-Laplace equation (1.2) on $\mathbb{R}^n$, with b replacing 0 on the RHS and with $a = 0$. From the elementary identity $t \leq t^p + 1$ for $t \geq 0$, we easily see that the functional satisfies Structure Conditions 2.14.

Remark 2.15 (p-harmonicity vs. Cheeger p-harmonicity)**.** Recall from [Che99] that there exists $C = C(X) \geq 1$ so that the μ-a.e. inequality

$$C^{-1}g_u \ \leq \ |Du| \ \leq \ Cg_u$$

holds for all Lipschitz functions u (and by density, for all $u \in N^{1,p}(X)$) where $|\cdot|$ indicates the standard Euclidean norm (on $\mathbb{R}^N$). As a result, for $b = 0$ the K-quasi-minimizers u of (2.9) are $C^{2p}K$-quasi-minimizers to

$$u \ \mapsto \ \int_\Omega \big(|Du|^p + a\big)\,d\mu. \tag{2.10}$$

Similarly, K-quasi-minimizers of the functional (2.10) are $C^{2p}K$-quasi-minimizers to (2.9).

For $K = 1$ and $a = b = 0$, the corresponding minimizers to (2.9) and (2.10) are known as *p-harmonic functions* and *Cheeger p-harmonic functions*, respectively [BBS03b].

The next result [GMP10, Lemma 4.2] is a local boundedness lemma for quasi-minimizers and is crucial to the proof of Theorem 3.1; see [KS01] for the original case of $a = b = 0$. Roughly speaking, it states that quasi-minimizers are locally bounded by their integral averages, up to an additive term that depends on the data of the functional. Below, we assume that Standing Hypotheses 2.7 and Structure Conditions 2.14 are in force.

Lemma 2.16 (Gong-Manfredi-Parviainen, 2010)**.** *Let Ω be a domain in X and fix $K \geq 1$. There exist $C = C(X, \mathrm{data}) > 0$ and $\beta = \beta(p, Q, s) \in (0,1)$ so that*

$$\sup_B |u| \;\leq\; C\Big\{ \Big(\fint_{2B} |u|^p \, d\mu \Big)^{1/p} \;+\; \big(\|a\|_{L^s(X)} + \|b\|_{L^s(X)} \big) r^{\beta} \Big\} \tag{2.11}$$

holds for all K-quasi-minimizers $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ and all balls $B = B(x,r)$ in X.

3. Differentiability Properties of Quasi-Minimizers

We now state our main result in terms of pointwise Lipschitz constants.

Theorem 3.1. *Let (X, d, μ) be a metric measure space satisfying Standard Hypotheses 2.7. If the functional in (2.7) satisfies Structure Conditions 2.14, then every quasi-minimizer $u \in N^{1,p}_{\mathrm{loc}}(X)$ satisfies*

$$\mathrm{Lip}[u](x) \;<\; \infty \tag{3.1}$$

for μ-a.e. $x \in X$.

Using the notion of Cheeger differentiability from Definition 2.10, the next result follows easily from Theorems 2.11 and 3.1.

Corollary 3.2. *Under the same hypotheses, every quasi-minimizer is Cheeger differentiable μ-a.e. on X. In particular, p-harmonic and Cheeger p-harmonic functions are Cheeger differentiable μ-a.e. on X.*

The proof of Theorem 3.1 follows a similar idea in [HS93]. Instead of rescaling the PDE, however, we show that generalized difference quotients of quasi-minimizers are themselves quasi-minimizers to appropriately *rescaled* energy functionals. Since arbitrary metric spaces do not admit Euclidean dilations of the form $x \mapsto \lambda x$, we will instead rescale the metric d.

In what follows, we will see that many crucial properties of metric measure spaces are preserved under rescalings of the metric. The rescaled energy functionals will also satisfy local bounds similar to Lemma 2.16. We will later see that the constants for these bounds will be independent of the scaling parameter, from which Theorem 3.1 follows.

3.1. **Rescalings of Metric Spaces.** Given a metric measure space (X, d, μ) and $\lambda \in (0,1)$, put $d_\lambda := \lambda^{-1} d$. We also write X_λ for the rescaled metric measure space (X, d_λ, μ). To fix notation, $B_\lambda = B_\lambda(x,r)$ denotes the ball(s)

$$\begin{aligned} B_\lambda(x,r) \;&:=\; \{ y \in X_\lambda \,:\, d_\lambda(x,y) < r \} \\ &=\; \{ y \in X \,:\, d(x,y) < \lambda r \} \;=\; B(x, \lambda r) \end{aligned}$$

and $\mathrm{diam}_\lambda(A)$ refers to the diameter of $A \subset X_\lambda$. Clearly the diameter functions of X and X_λ are related by the formula

$$\mathrm{diam}_\lambda(B_\lambda(x,r)) \;=\; \lambda^{-1} \, \mathrm{diam}(B(x, \lambda r)). \tag{3.2}$$

If $\gamma : [a, b] \to (X, d)$ is a curve then we write $\gamma_\lambda : [a, b] \to (X, d_\lambda)$ for the same curve in X_λ; the subscript λ only indicates the change in the metric. Clearly, the length $\ell_\lambda(\gamma_\lambda)$ of γ_λ satisfies

$$\ell_\lambda(\gamma_\lambda) \;\equiv\; \lambda^{-1}\ell(\gamma)$$

from which we conclude that, for all Borel functions $\rho : X \to [0, \infty]$,

$$\int_\gamma \rho\, ds \;=\; \lambda \int_{\gamma_\lambda} \rho\, ds_\lambda. \tag{3.3}$$

Here s_λ denotes the arc-length parameter of γ_λ.

Regarding the Standard Hypotheses, it is easy to see that if X is κ-doubling, then so is X_λ. The next fact asserts that p-weak upper gradients of functions are rescaled whenever the metric is rescaled; in contrast, Poincaré inequalities are preserved under rescalings, with the same parameters as before.

To fix notation, the minimal p-weak upper gradient of u on X_λ is denoted by g_u^λ.

Lemma 3.3. *Let $\lambda \in (0, 1)$ and let $u : X \to \mathbb{R}$.*

(1) *A Borel function $g : X \to [0, \infty]$ is a p-weak upper gradient for u in X if and only if $\lambda g : X_\lambda \to [0, \infty]$ is a p-weak upper gradient for u in X_λ.*

(2) *Let μ be a doubling measure on X. Then X supports a p-Poincaré inequality if and only if X_λ supports a p-Poincaré inequality with the same parameters $C > 0$ and $\Lambda \geq 1$.*

Proof. Let $g : X \to [0, \infty]$ be a p-weak upper gradient of $u : X \to \mathbb{R}$. For a rectifiable curve $\gamma\colon [0, L] \to X$, Equation (3.3) implies

$$|u(y) - u(x)| \;\leq\; \int_\gamma g\, ds \;=\; \int_{\gamma_\lambda} g\lambda\, ds_\lambda$$

so λg is an upper gradient of $u : X_\lambda \to \mathbb{R}$. The other direction is symmetric, so this proves (1).

For (2), if $u \in N^{1,p}_{\mathrm{loc}}(X)$ then (1) implies that $u \in N^{1,p}_{\mathrm{loc}}(X_\lambda)$ and that

$$g_u \;=\; \lambda^{-1} g_u^\lambda.$$

Using this and Equation (3.2), we obtain the identities

$$\begin{aligned}
\mathrm{diam}_\lambda(B_\lambda)\Big(\fint_{\Lambda B_\lambda} (g_u^\lambda)^p\, d\mu\Big)^{\frac{1}{p}} \;&=\; \lambda^{-1}\,\mathrm{diam}(\lambda B)\Big(\fint_{\Lambda(\lambda B)} (g_u^\lambda)^p\, d\mu\Big)^{\frac{1}{p}} \\
&=\; \mathrm{diam}(\lambda B)\Big(\lambda^{-p}\fint_{\Lambda(\lambda B)} (g_u^\lambda)^p\, d\mu\Big)^{\frac{1}{p}} \\
&=\; \mathrm{diam}(\lambda B)\Big(\fint_{\Lambda(\lambda B)} g_u^p\, d\mu\Big)^{\frac{1}{p}}.
\end{aligned}$$

Moreover, it is clear that we have the identities $u_{B_\lambda} = u_{\lambda B}$ and

$$\fint_{B_\lambda} |u - u_{B_\lambda}|\, d\mu \;=\; \fint_{\lambda B} |u - u_{\lambda B}|\, d\mu.$$

So by replacing the balls λB with B, the lemma follows. $\square$

3.2. **Difference Quotients are Quasi-Minimizers.** To determine the differentiability properties of quasi-minimizers, we now study their generalized difference quotients. Given a ball $B = B(x_0, r)$ in X, a function $u : X \to \mathbb{R}$, and a fixed number $\tau \in \mathbb{R}$, we define

$$Q_B u(x) := \frac{u(x) - \tau}{r}.$$

Clearly $u \in N^{1,p}_{\mathrm{loc}}(X)$ implies $Q_B u \in N^{1,p}_{\mathrm{loc}}(X)$ and we have the identities

$$Q_{2B} u = 2(Q_B u) \tag{3.4}$$
$$u = r\,(Q_B u) + \tau \tag{3.5}$$
$$g_u = r\, g_{(Q_B u)}. \tag{3.6}$$

Lemma 3.4. *Let $K \geq 1$, fix a ball $B = B(x_0, r)$ in X, and put $\lambda = r$. Suppose that the functional (2.7) satisfies Structure Conditions 2.14 with data s, L, and $a, b \in L^s(X)$.*

(1) *If $u \in N^{1,p}_{\mathrm{loc}}(X)$ is a K-quasi-minimizer of the functional (2.7), then $Q_B u \in N^{1,p}_{\mathrm{loc}}(X_\lambda)$ is a K-quasi-minimizer of the functional*

$$\begin{aligned} \mathcal{F}_0(w; \Omega) &:= \int_\Omega F_0\big(x, w(x), g^\lambda_w(x)\big)\, d\mu(x) \\ \textit{where}\ \ F_0(x, y, z) &:= F\big(x,\, ry + \tau,\, z\big). \end{aligned}$$

(2) *F_0 satisfies the modified structure conditions*

$$|z|^p - \hat{b}(x)|y|^p + \hat{a}(x) \;\leq\; F_0(x, y, z) \;\leq\; L|z|^p + \hat{b}(x)|y|^p + \hat{a}(x) \tag{3.7}$$

with respect to the data

$$\begin{aligned} \hat{a}(x) &:= 2^{p-1}\big(|\tau|^p b(x) + a(x)\big) \\ \hat{b}(x) &:= 2^{p-1} r^p b(x) \end{aligned}$$

Proof. To prove (1), note first that Lemma 3.3 and Equations (3.5)-(3.6) imply that

$$g^\lambda_{(Q_B u)} \;=\; \lambda g_{(Q_B u)} \;=\; \lambda r^{-1} g_u \;=\; g_u$$

from which it follows that

$$F_0\big(x,\, Q_B u(x),\, g^\lambda_{Q_B u}(x)\big) \;=\; F\big(x,\, u(x),\, g_u(x)\big).$$

Let $V \in N^{1,p}_{\mathrm{loc}}(X)$ be such that $Q_B u - V \in N^{1,p}_0(\Omega)$. The function $v := rV + \tau$ satisfies $Q_B v = V$ and therefore $u - v \in N^{1,p}_0(\Omega)$. Using the above identities, we obtain

$$\begin{aligned} \mathcal{F}_0\big(Q_B u; \Omega \cap \{Q_B u \neq V\}\big) &= \int_{\Omega \cap \{Q_B u \neq V\}} F_0\big(x, Q_B u(x), g^\lambda_{(Q_B u)}(x)\big)\, d\mu(x) \\ &= \int_{\Omega \cap \{u \neq v\}} F\big(x,\, u(x),\, g_u(x)\big)\, d\mu(x) \end{aligned}$$

and using the quasi-minimizing property of u, we further obtain

$$\begin{aligned}
&\mathcal{F}_0\big(Q_B u; \Omega\cap\{Q_B u \neq V\}\big) \\
&\qquad \le\ K \int_{\Omega\cap\{u\neq v\}} F\big(x,\, v(x),\, g_v(x)\big)\, d\mu(x) \\
&\qquad =\ K \int_{\Omega\cap\{Q_B u\neq V\}} F_0\big(x,\, Q_B v(x),\, g^{\lambda}_{(Q_B v)}(x)\big)\, d\mu(x) \\
&\qquad =\ K\,\mathcal{F}_0(V; \Omega\cap\{Q_B u\neq V\}).
\end{aligned}$$

Hence $Q_B u$ is a K-quasi-minimizer of $\mathcal{F}_0$.

Since F from the functional (2.7) satisfies Structure Conditions 2.14, the convexity of the function $t \mapsto |t|^p$, for $p > 1$, gives

$$\begin{aligned}
F_0(x,y,z) &= F\big(x,\, ry+\tau,\, z\big) \\
&\le L|z|^p + b(x)\Big|ry+\tau\Big|^p + a(x) \\
&\le L|z|^p + b(x)2^{p-1}\Big(r^p|y|^p + |\tau|^p\Big) + a(x).
\end{aligned}$$

The other inequality is similar, which proves (3.7). □

We are now ready to prove our main result. As before, g_u^λ denotes the minimal p-weak upper gradient of u in X_λ.

Proof of Theorem 3.1. Let $u \in N^{1,p}_{\mathrm{loc}}(X)$ be a K-quasi-minimizer. In particular, we have $|u(x_0)| < \infty$ and $|Du(x_0)| < \infty$ for μ-a.e. $x_0 \in X$. For such points $x_0 \in X$, let $r \in (0, \frac{1}{2})$ and put $\lambda = r$. It follows that

$$\mathbb{B}_\lambda \ :=\ B_\lambda(x_0, 1)\ =\ B(x_0, r)\ =:\ B.$$

With $\tau = u(x_0)$, the difference quotient

$$Q_B u(x)\ =\ \frac{u(x) - u(x_0)}{r}$$

is a K-quasi-minimizer of the modified functional $\mathcal{F}_0$ from Lemma 3.4.

By hypothesis, (X, d, μ) satisfies Standard Hypotheses 2.7, so by Lemma 3.3 the rescaled space X_λ satisfies the same hypotheses and with the same parameters κ, C, and Λ.

For the functions $\hat{a}$ and $\hat{b}$ from Lemma 3.4 and using the ball $2\mathbb{B}_\lambda$ of radius 2, observe that the expression

$$\big(\|\hat{a}\|_{L^s(X)} + \|\hat{b}\|_{L^s(X)}\big)2^\beta$$

in inequality (2.11) is bounded by a constant $c_0 > 0$ that is independent of r. Hence Lemma 2.16 and equation (3.4) imply that

$$\begin{aligned}
\sup_B |Q_B u|\ =\ \sup_{\mathbb{B}_\lambda} |Q_B u|\ &\le\ C\Big(\fint_{2\mathbb{B}_\lambda} |Q_B u|^p\, d\mu\Big)^{\frac{1}{p}} + c_0 \\
&\le\ C\Big(\fint_{2B} 2^{-p}|Q_{2B} u|^p\, d\mu\Big)^{\frac{1}{p}} + c_0
\end{aligned} \tag{3.8}$$

for constants $C, c_0 > 0$ independent of r.

Let $\{X_k\}_{k=1}^\infty$ and $\{\xi_k\}_{k=1}^\infty$ be as in Theorem 2.8. Theorem 2.12 implies that for μ-a.e. $x_0 \in X_k$, we have

$$\Big(\fint_{2B} \Big| Q_{2B}u - D_k u(x_0)\, \frac{\xi_k(y) - \xi_k(x_0)}{r} \Big|^p \, d\mu \Big)^{\frac{1}{p}} \;\le\; 1.$$

for sufficiently small $r > 0$. Since each ξ_k is 1-Lipschitz (Remark 2.9) we further obtain

$$\begin{aligned} \Big(\fint_{2B} \big| Q_{2B}u \big|^p \, d\mu \Big)^{\frac{1}{p}} \;&\le\; \Big(\fint_{2B} \Big| Q_{2B}u - D_k u(x_0)\, \frac{\xi_k(y) - \xi_k(x_0)}{r} \Big|^p \, d\mu \Big)^{\frac{1}{p}} \\ &\quad + \Big(\fint_{2B} \Big| D_k u(x_0)\, \frac{\xi_k(y) - \xi_k(x_0)}{r} \Big|^p \, d\mu \Big)^{\frac{1}{p}} \\ &\le\; 1 + |D_k u(x_0)| \end{aligned}$$

which, combined with Equation (3.8), implies that

$$\sup_B \big| Q_B u \big| \;\le\; C \Big(\fint_{2B} \big| Q_{2B}u \big|^p \, d\mu \Big)^{\frac{1}{p}} + c_0 \;\le\; C\big(1 + |D_k u(x_0)|\big) + c_0.$$

In particular, the upper bound is independent of $r = \lambda$. Taking limits as $r \to 0$, the LHS converges to $\mathrm{Lip}[u](x_0)$, so the theorem follows. $\square$

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Jasun Gong, Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15260, United States
E-mail address: jasun@pitt.edu

Piotr Hajłasz, Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15260, United States
E-mail address: hajlasz@pitt.edu