Homework 3 for Math 413

Due day: Thursday October 4 recitations.

Problem 19. Prove that $1 + 2 + \ldots + n = \dfrac{n(n+1)}{2}$ for all $n \in \mathbb{N}$.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 20. Prove that $1^3 + 2^3 + \ldots + n^3 = (1 + 2 + \ldots + n)^2$ for all $n \in \mathbb{N}$.
Hint: *You can use the result of Problem 19.*

Proof. WRITE YOUR SOLUTION HERE. □

Problem 21. Prove that $5^{2n} - 1$ is divisible by 8 for all $n \in \mathbb{N}$.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 22. Prove that $1 + \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}} + \cdots + \dfrac{1}{\sqrt{n}} > \sqrt{n}$ for all $n \in \mathbb{N}$, $n \geq 2$.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 23. Prove that $1 + \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}} + \cdots + \dfrac{1}{\sqrt{n}} > 2(\sqrt{n+1} - 1)$ for all $n \in \mathbb{N}$, $n \geq 2$.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 24. Prove that if $a, b > 0$ are real numbers, then $(a + b)^n < 2^n(a^n + b^n)$.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 25. Prove that for $q \neq 1$ and $n \in \mathbb{N}$ $1 + q + q^2 + \ldots + q^n = \dfrac{1 - q^{n+1}}{1 - q}$.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 26. Prove that if n is a natural number and $a, b \in \mathbb{R}$, then

$$a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \ldots + ab^{n-2} + b^{n-1}).$$

(Do it directly, you do not have to use a formal mathematical induction.)

Proof. WRITE YOUR SOLUTION HERE. □

Problem 27. Prove that if $0 \leq k \leq n$ are integers, then $\dbinom{n}{k-1} + \dbinom{n}{k} = \dbinom{n+1}{k}$.
(This problem does not involve mathematical induction.)

Proof. WRITE YOUR SOLUTION HERE. □

Problem 28. Let $a_1, \dots, a_n, b_1, \dots, b_n$ be positive numbers. Prove that

$$\prod_{i=1}^{n} (a_i + b_i)^{1/n} \geq \prod_{i=1}^{n} a_i^{1/n} + \prod_{i=1}^{n} b_i^{1/n} .$$

Hint: This problem does not involve mathematical induction. This is a problem for the application of the arithmetic-geometric mean inequality. Divide both sides by the expression on the left hand side and use the arithmetic-geometric mean inequality.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 29. Use mathematical induction to prove the Schwarz inequality

$$\left| \sum_{i=1}^{n} a_i \, b_i \right| \leq \left(\sum_{i=1}^{n} a_i^2 \right)^{1/2} \left(\sum_{i=1}^{n} b_i^2 \right)^{1/2} .$$

Proof. WRITE YOUR SOLUTION HERE. □

Problem 30. Use the Schwarz inequality to prove that if $a_1, \dots, a_n > 0$, then

$$\frac{n}{\frac{1}{a_1} + \ldots + \frac{1}{a_n}} \leq \frac{a_1 + \ldots + a_n}{n} .$$

Proof. WRITE YOUR SOLUTION HERE. □