Homework 4 for Math 413

Due day: Thursday October 18 recitations.

Problem 31. Prove the property (Z14): for all $a, b, c \in \mathbb{Z}$ we have $a \le b \Rightarrow a + c \le b + c$.

Proof. WRITE YOUR SOLUTION HERE. $\square$

Problem 32. Let

$$F = \{(n, m) \in \mathbb{Z} \times \mathbb{Z} : m \neq 0\}.$$

Prove that the relation

$$(n_1, m_1) \sim (n_2, m_2) \quad \text{if and only if} \quad n_1 m_2 = n_2 m_1.$$

is an equivalence relation.

Proof. WRITE YOUR SOLUTION HERE. $\square$

Problem 33. Let[1] $F_* = \mathbb{Z} \times \mathbb{Z}$ and define the relation

$$(n_1, m_1) \sim (n_2, m_2) \quad \text{if and only if} \quad n_1 m_2 = n_2 m_1.$$

We can try to mimic the construction of rational numbers with the set F_* in place of F. Where does the theory fall apart?

Proof. WRITE YOUR SOLUTION HERE. $\square$

Problem 34. $\mathbb{Z}_{11} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ is a field. Find $7 + 8$, $7 \cdot 8$, $7/8$ in $\mathbb{Z}_{11}$.

Proof. WRITE YOUR SOLUTION HERE. $\square$

Problem 35. Let X be a non-empty set and let $P(X)$ be the power set. In the power set we have the operation of addition $A + B := A \cup B$. Check which of the axioms A1, A2, A3, A4 are satisfied.

Proof. WRITE YOUR SOLUTION HERE. $\square$

Problem 36. Give an example of a non-empty set X with a relation $\sim$ that is

(a) reflexive, but not symmetric or transitive;
(b) symmetric, but not reflexive or transitive;
(c) transitive, but not reflexive or symmetric;
(d) reflexive and symmetric, but not transitive;
(e) reflexive and transitive, but not symmetric;
(f) symmetric and transitive, but not reflexive.

Proof. WRITE YOUR SOLUTION HERE. $\square$

Problem 37. Let X be a nonempty set with a relation $\sim$. Consider the following argument. **Claim:** *If $\sim$ is symmetric and transitive, then it is also reflexive.*

Proof. $x \sim y \Rightarrow y \sim x$; also $x \sim y$ and $y \sim x \Rightarrow x \sim x$. Therefore $x \sim x$ for every $x \in X$. $\square$

This argument is at odds with part (f) of the previous problem. Where is the flaw?

Proof. WRITE YOUR SOLUTION HERE. $\square$

[1]It differs from F in Problem 32, because we include $(n, 0)$. Remember that $[(n, m)]$ represents the fraction $\frac{n}{m}$ so now we try to develop rational numbers, where we can divide by 0.

Problem 38. Let F be a set with operations of addition $+$ and multiplication $\cdot$ and assume that it satisfies axioms (A1)-(A9) and axiom (A10'): $1 = 0$. Prove that F consists of one point.

Proof. WRITE YOUR SOLUTION HERE. □