Homework 5 for Math 413

Due day: Thursday November 8 recitations.

Problem 39. Write a formula for a bijection $f : (a, b) \to \mathbb{R}$, $a < b$, $a, b \in \mathbb{R}$.[1]
Hint: *Use* $\tan x$ *function and a linear change of variables.*

Proof. WRITE YOUR SOLUTION HERE. □

Problem 40. Write a formula for a bijection $f : (0, \infty) \to (0, 1)$.[2]

Proof. WRITE YOUR SOLUTION HERE. □

Problem 41. Prove that $(0, 1)$ and $[0, 1]$ have the same cardinality by constructing a bijection $f : [0, 1] \to (0, 1)$. Do not use the Cantor-Bernstein theorem.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 42. Prove that $(0, 1)$ and $[0, 1]$ have the same cardinality by applying the Cantor-Bernstein theorem.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 43. Prove that the power set $P(\mathbb{N})$ has the same cardinality as the set of all infinite sequences

$$\{(a_1, a_2, \ldots) : \ a_i \in \{0, 1\}, \ i = 1, 2, \ldots\}$$

Hint: *Associate with each such a sequence a subset of* $\mathbb{N}$. *Use* 0 *and* 1 *to denote which natural number you want to include in a set.*

Proof. WRITE YOUR SOLUTION HERE. □

Problem 44. Use Problem 43 and the Cantor theorem to show that the set

$$\{(a_1, a_2, \ldots) : \ a_i \in \{0, 1\}, \ i = 1, 2, \ldots\}$$

in uncountable.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 45. Prove that the set

$$\{(a_1, a_2, \ldots) : \ a_i \in \{0, 1\}, \ i = 1, 2, \ldots\}$$

is uncountable by mimicking the proof of uncountability of the set of real numbers.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 46. Prove directly using the definition, that $\lim_{n \to \infty} \frac{2n + 5}{3n - 7} = \frac{2}{3}$.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 47. Prove directly using the definition, that $\lim_{n \to \infty} \frac{2n + 5}{3n^2 - 7} = 0$.

Proof. WRITE YOUR SOLUTION HERE. □

[1] That proves that $\mathbb{R}$ and (a, b) have the same cardinality.
[2] That proves that $(0, \infty)$ and $(0, 1)$ have the same cardinality.

Problem 48. Prove that if $a_n > 0$, $a > 0$, and $\lim_{n\to\infty} a_n = a$, then $\lim\limits_{n\to\infty} \sqrt{a_n} = \sqrt{a}$.
Hint: *Use the formula* $a - b = (a^2 - b^2)/(a + b)$ *to estimate* $\sqrt{a_n} - \sqrt{a}$.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 49. Find the limit $\lim_{n\to\infty}(\sqrt{n^2 + 2n + 5} - n)$. You can use results proved in class, but clearly explain what results you use.
Hint: *Use the formula* $a - b = (a^2 - b^2)/(a + b)$. *You can use without proving it, that* $\lim_{n\to\infty} \sqrt{1 + \frac{2}{n} + \frac{5}{n^2}} = 1$.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 50. Find the limit $\lim_{n\to\infty} \sqrt[n]{n^2 + n}$. You can use results that have been proved in class.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 51. Prove that the sequence $\dfrac{n^3 + (-1)^n n^3}{n^2 + 1}$ is divergent by showing that it is unbounded.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 52. Although the sequence $\dfrac{n^3 + (-1)^n n^3}{n^2 + 1}$ is unbounded (Problem 51), show that it does not diverge to $+\infty$.

Proof. WRITE YOUR SOLUTION HERE. □