Math 413: Practice problems for the midterm exam

Problem 1. Prove that $\sqrt{3}$ is irrational.

Problem 2. Negate the statement: *For all real numbers x, y satisfying $x < y$, there is a rational number q such that $x < q < y$.* Formulate the negation as a sentence and not as a formula involving quantifiers.

Problem 3. Negate the statement[1]

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in A\ \forall y \in B\ \ \Big(|x-y| < \delta \ \Rightarrow\ (|f(x)-f(y)| < \varepsilon \vee |f(x)+f(y)| > \varepsilon) \Big).$$

Problem 4. Let $f : A \to B$ an $g : B \to C$ be functions. Prove that if $g \circ f : A \to C$ is surjective, then g is surjective.

Problem 5. Let $f : A \to B$ an $g : B \to C$ be functions. Prove that if $g \circ f : A \to C$ is injective, then f is injective.

Problem 6. Prove that if $f : X \to Y$ and $A \subset B \subset Y$, then $f^{-1}(A \cap B) = f^{-1}(A)$.

Problem 7. Prove that if $f : X \to Y$ and $A, B \subset X$, then $f(A \cup B) = f(A) \cup f(B)$.

Problem 8. Prove that if $f : X \to Y$ and $A, B \subset X$, then $f(A \cap B) \subset f(A) \cap f(B)$.

Problem 9. Prove that $1 + 2 + \ldots + n = \dfrac{n(n+1)}{2}$ for all $n \in \mathbb{N}$.

Problem 10. State the arithmetic-geometric mean inequality.

Problem 11. State the binomial formula.

Problem 12. Use mathematical induction to prove that $1 + 2 + \ldots + n = \dfrac{n(n+1)}{2}$ for all $n \in \mathbb{N}$.

Problem 13. Prove that $5^{2n} - 1$ is divisible by 8 for all $n \in \mathbb{N}$.

Problem 14. State what it means that two sets have the same cardinality.

Problem 15. Provide a definition of the equivalence relation.

Problem 16. Show an example of (a) an equivalence relation and (b) an example of a relation that is not an equivalence relation.

Problem 17. State the Cantor-Bernstein theorem.

Problem 18. Prove that $(0, 1)$ and $[0, 1]$ have the same cardinality by applying the Cantor-Bernstein theorem.

Problem 19. State and prove Bernoulli's inequality.

Problem 20. Let $a > 1$. Prove that for every $b > 0$ there is $n \in \mathbb{N}$ such that $a^n > b$. Use the following method. Write $a = 1 + c$, use Bernoulli's inequality to estimate $a^n = (1 + c)^n$ and use the the Archimedes Postulate.

Problem 21. State the result about density of rational numbers. Using this result prove that irrational numbers are dense in $\mathbb{R}$. **Hint:** *Consider numbers of the form $r\sqrt{2}$, $r \in \mathbb{Q}$.*

[1] Don't even think what the meaning of the statement is. Just negate it using formal rules of logic.

Problem 22. Let $\emptyset \neq A \subset \mathbb{R}$. State the definition of $\sup A$.

Problem 23. Fund $\inf A$ and $\sup A$, where $A = \{1 + 2^{-n} : n \in \mathbb{N}\}$.

Problem 24. Let $\emptyset \neq A \subset \mathbb{R}$ be bounded and let $-A = \{-x : x \in A\}$. Prove that $\sup A = -(\inf(-A))$.

Problem 25. Define what it means that $\lim_{n\to\infty} a_n = -\infty$.

Proof. Prove, directly from the definition that $\lim_{n\to\infty}(2018 \cdot (-1)^n + n) = \infty$. □

Problem 26. Prove directly using the definition of the limit that $\lim_{n\to\infty} 1 + \frac{2}{n} + \frac{3}{n^2} = 1$.

Problem 27. Prove that the sequence $\frac{n^3 + (-1)^n n^3}{n^2 + 1}$ is divergent by showing that it is unbounded.

Problem 28. Show that the sequence $\frac{n^3 + (-1)^n n^3}{n^2 + 1}$ does not diverge to $+\infty$.

Problem 29. Prove directly using the definition, that $\lim_{n\to\infty} \frac{2n+5}{3n-7} = \frac{2}{3}$.

Problem 30. Prove directly using the definition, that $\lim_{n\to\infty} \frac{2n+5}{3n^2-7} = 0$.