Homework 1 for Math 1530

Due day: Tuesday September 10 recitations.

Problem 1. Use the equivalence

$$p \wedge (q \vee r) \quad \equiv \quad (p \wedge q) \vee (p \wedge r) \tag{1}$$

to prove

$$p \vee (q \wedge r) \quad \equiv \quad (p \vee q) \wedge (p \vee r).$$

To this end apply (1) to $\neg p$, $\neg q$, $\neg r$ in place of p, q, r, and negate the statement using De Morgan's Laws.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 2. Negate the statement[1]

$$\forall \varepsilon > 0 \ \ \exists \delta > 0 \ \ \forall x \in \mathbb{R} \ \ \forall y \in \mathbb{R} \ \ (|x - y| < \delta \ \Rightarrow \ |\sin x - \sin y| < \varepsilon).$$

Proof. WRITE YOUR SOLUTION HERE. □

Problem 3. Negate the statement: *For all real numbers x, y satisfying $x < y$, there is a rational number q such that $x < q < y$.* Formulate the negation as a sentence and not as a formula involving quantifiers.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 4. Use an argument by contradiction prove that $\sqrt{3}$ is irrational.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 5. Prove the following statement[2]

$$\forall \varepsilon > 0 \ \ \exists n_0 \in \mathbb{N} \ \ \forall n \in \mathbb{N} \quad (n \geq n_0 \ \Rightarrow \ n^{-1} \leq \varepsilon).$$

Proof. WRITE YOUR SOLUTION HERE. □

Problem 6. Find a mistake in the solution to Problem 9 provided on page 19 in my notes and write a correct solution.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 7. Prove that for any set A and any family of sets $\{A_i\}_{i \in I}$

$$A \setminus \bigcup_{i \in I} A_i = \bigcap_{i \in I} (A \setminus A_i)\,,$$

$$A \setminus \bigcap_{i \in I} A_i = \bigcup_{i \in I} (A \setminus A_i)\,.$$

Proof. WRITE YOUR SOLUTION HERE. □

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[1] This is a true statement known as uniform continuity of the function $\sin x$. However, you are not asked to prove the statement only to negate it.

[2] Compare with Example 1.12.

Problem 8. Prove that if $f : X \to Y$ is a function and $A_1, A_2, A_3, \ldots$ are subsets of X, then

$$f\left(\bigcup_{i=1}^{\infty} A_i\right) = \bigcup_{i=1}^{\infty} f(A_i)\,,$$

and

$$f\left(\bigcap_{i=1}^{\infty} A_i\right) \subset \bigcap_{i=1}^{\infty} f(A_i). \tag{2}$$

Provide an example to show that we do not necessarily have equality in (2)

Proof. WRITE YOUR SOLUTION HERE. □

Problem 9. Prove that if $f : X \to Y$ is one-to-one and $A_1, A_2, A_3, \ldots$ are subsets of X, then

$$f\left(\bigcap_{i=1}^{\infty} A_i\right) = \bigcap_{i=1}^{\infty} f(A_i)\,.$$

Proof. WRITE YOUR SOLUTION HERE. □

Problem 10. Prove that $5^{2n} - 1$ is divisible by 8 for all $n \in \mathbb{N}$.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 11. Prove that $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} \geq \sqrt{n}$.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 12. Let $a_1, \ldots, a_n, b_1, \ldots, b_n$ be positive numbers. Prove that

$$\prod_{i=1}^{n} (a_i + b_i)^{1/n} \geq \prod_{i=1}^{n} a_i^{1/n} + \prod_{i=1}^{n} b_i^{1/n}\,.$$

Hint: Divithe both sides by the expression on the left hand side and use the arithmetic-geometric mean inequality.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 13. Prove that Schwartz inequality

$$\left|\sum_{i=1}^{n} a_i\, b_i\right| \leq \left(\sum_{i=1}^{n} a_i^2\right)^{1/2} \left(\sum_{i=1}^{n} b_i^2\right)^{1/2}\,.$$

Proof. WRITE YOUR SOLUTION HERE. □

Problem 14. Use the Schwarz inequality to prove that if $a_1, \ldots, a_n > 0$, then

$$\frac{n}{\frac{1}{a_1} + \ldots + \frac{1}{a_n}} \leq \frac{a_1 + \ldots + a_n}{n}\,.$$

Proof. WRITE YOUR SOLUTION HERE. □