Homework 2 for Math 1530

Due day: Tuesday September 17 recitations.

Some of the problems might require material that will be covered in class later, like for example properties of sin and cos functions. You can use it and other similar results, even if it was not covered in class yet. For Problem 26, see Theorem 9.8.

Problem 15. Find $\sup A$ and $\inf A$, where

$$A = \left\{ \frac{n^2 + 2n - 3}{n+1} : \ n = 1, 2, 3, \dots \right\}.$$

Proof. WRITE YOUR SOLUTION HERE. □

Problem 16. Prove that is $A, B \subset \mathbb{R}$ are bounded and non-empty, then

$$\sup(A + B) = \sup A + \sup B, \quad \text{where} \quad A + B = \{x + y : \ x \in A, \ y \in B\}.$$

Proof. WRITE YOUR SOLUTION HERE. □

Problem 17. Prove that is $A, B \subset (0, \infty)$ are bounded and non-empty, then

$$\sup(A \cdot B) = \sup A \cdot \sup B, \quad \text{where} \quad A \cdot B = \{xy : \ x \in A, \ y \in B\}.$$

Proof. WRITE YOUR SOLUTION HERE. □

Problem 18. Prove that if $\lim_{n\to\infty} a_n = \infty$, then

$$\lim_{n\to\infty} \frac{a_1 + a_2 + \ldots + a_n}{n} = \infty.$$

Proof. WRITE YOUR SOLUTION HERE. □

Problem 19. Prove that if $\lim_{n\to\infty} a_n = g \in \mathbb{R}$ and $a_n > 0$ for all n, then

$$\lim_{n\to\infty} \sqrt[n]{a_1 a_2 \cdots a_n} = g.$$

Proof. WRITE YOUR SOLUTION HERE. □

Problem 20. Prove that the sequence defined by

$$a_1 = 0, \qquad a_{n+1} = \sqrt{6 + a_n} \text{ for } n \geq 1$$

is convergent and find its limit.

Remark. It is natural to denote the limit of this sequence by

$$\sqrt{6 + \sqrt{6 + \sqrt{6 + \ldots}}}$$

Proof. WRITE YOUR SOLUTION HERE. □

Problem 21. Prove that

$$2\cos\left(\frac{\pi}{2^{n+1}}\right) = \underbrace{\sqrt{2 + \sqrt{2 + \sqrt{2 + \ldots + \sqrt{2}}}}}_{n \text{ square roots}}.$$

Proof. WRITE YOUR SOLUTION HERE. □

Problem 22. Find the limit

$$\lim_{n\to\infty} 2^n \underbrace{\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}}}}_{n \text{ square roots}}.$$

That is not a typo. We have one "$-$" and the rest are "$+$" signs.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 23. Find the limit $\lim\limits_{n\to\infty} \dfrac{n}{e^{1+\frac{1}{2}+\cdots+\frac{1}{n}}}$.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 24. Find the limit $\lim\limits_{n\to\infty} \sin\left(2\pi\sqrt{n^2+1}\right)$.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 25. Prove that the sequence $\sin n$ is divergent.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 26. Prove that the sequence

$$\left(1+\frac{1}{n}\right)^{n+1}$$

is decreasing.

Proof. WRITE YOUR SOLUTION HERE. □