**Homework 3 for Math 1530**
Due day: Tuesday September 24 recitations.

**Problem 27.** Let $a_1, a_2, a_3, \ldots > 0$. Prove that if

$$\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) > 1\,,$$

then the series $a_1 + a_2 + a_3 + \ldots$ converges.

*Proof.* WRITE YOUR SOLUTION HERE. □

**Problem 28.** Provide an example of a convergent series $a_1 + a_2 + a_3 + \ldots$, where $a_n > 0$, $n = 1, 2, 3, \ldots$ such that the limit $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ does not exist.

*Proof.* WRITE YOUR SOLUTION HERE. □

**Problem 29.** Prove that there is a sequence of positive integers $n_1 < n_2 < n_3 < \ldots$ such that the sequence $a_k = \sin n_k$ converges.

*Proof.* WRITE YOUR SOLUTION HERE. □

**Problem 30.** Prove that the series

$$\sum_{n=3}^{\infty} \frac{1}{n(\log n)(\log\log n)^p}$$

diverges if $0 < p \leq 1$ and converges if $p > 1$.

*Proof.* WRITE YOUR SOLUTION HERE. □

**Problem 31.** Prove that if the series $a_1 + a_2 + a_3 + \ldots$ converges, where $a_n > 0$, $n = 1, 2, 3, \ldots$, then the series

$$\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n} \quad \text{converges.}$$

*Proof.* WRITE YOUR SOLUTION HERE. □

Definition. Let $a_1, a_2, a_3, \ldots > 0$. We define the infinite product by

$$\prod_{n=1}^{\infty} a_n = \lim_{n\to\infty} a_1 a_2 \ldots a_n\,.$$

We say that the infinite product *converges* if the limit is finite and *positive*. If the limit does not exist, equals 0 or $\infty$ then we say that the product *diverges*.

**Problem 32.** Prove that if $a_n > 0$, $n = 1, 2, \ldots$, then the product $\prod_{n=1}^{\infty}(1 + a_n)$ converges if and only if the series $\sum_{n=1}^{\infty} a_n$ converges. **Hint:** *You can use the inequality $e^x \geq 1 + x$ without proving it.*

*Proof.* WRITE YOUR SOLUTION HERE. □

**Problem 33.** Prove that if $0 < a_n < 1, n = 1, 2, \ldots$, then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the series $\sum_{n=1}^{\infty} a_n/(1 - a_n)$ converges.

*Proof.* WRITE YOUR SOLUTION HERE. □

**Problem 34.** Prove that if $0 < a_n < 1$, then the product $\prod_{n=1}^{\infty}(1 - a_n)$ converges if and only if the series $\sum_{n=1}^{\infty} a_n$ converges.

*Proof.* WRITE YOUR SOLUTION HERE. $\square$