Homework 5 for Math 1530

Due day: Tuesday October 8 recitations.

Problem 43. Prove that there is an increasing sequence of integers $a_1 < a_2 < a_3 < \dots$ such that for every $k \in \mathbb{N}$, the sequence $\{\sin(ka_n)\}_{n=1}^{\infty}$ converges.

Proof. WRITE YOUR SOLUTION HERE. ☐

Problem 44. Fix $k \in \mathbb{N}$ and define

$$z_n = \frac{1^k + 2^k + \dots + n^k}{n^{k+1}}, \quad n = 1, 2, 3, \dots$$

Prove that

$$\lim_{n\to\infty} n\left(z_n - \frac{1}{k+1}\right) = \frac{1}{2}.$$

Proof. WRITE YOUR SOLUTION HERE. ☐

Problem 45. Let $\beta > 0$ and $\{u_n\}$ be a sequence of positive real numbers such that $\dfrac{u_{n+1}}{u_n} \leq \beta$ for every $n \in \mathbb{N}$. Prove that

$$\limsup_{n\to\infty} \sqrt[n]{u_n} \leq \limsup_{n\to\infty} \left(\frac{u_{n+1}}{u_n}\right).$$

Proof. WRITE YOUR SOLUTION HERE. ☐

Problem 46. Prove that if a sequence (a_n) of real numbers is convergent to a finite limit, $\lim_{n\to\infty} a_n = g \in \mathbb{R}$, then

$$\lim_{x\to\infty} e^{-x} \sum_{n=0}^{\infty} a_n \frac{x^n}{n!} = g.$$

Proof. WRITE YOUR SOLUTION HERE. ☐

Problem 47. Suppose that a sequence of functions $f_1, f_2, \dots$ converges uniformly on $[0,1]$ to some function f. Suppose also that there is a constant M such that $|f_i(x)| \leq M$ for all i and x. Prove that the sequence of squares $f_1^2, f_2^2, \dots$ converges uniformly to f^2.

Proof. WRITE YOUR SOLUTION HERE. ☐

Problem 48. Prove that if $f : (0,1) \to \mathbb{R}$ is uniformly continuous, then there is a continuou function $F : [0,1] \to \mathbb{R}$ such that $F(x) = f(x)$ for all $x \in (0,1)$.

Proof. WRITE YOUR SOLUTION HERE. ☐

Problem 49. Prove that if $f : [0,\infty) \to \mathbb{R}$ is continuous and the limit $\lim_{x\to\infty} f(x)$ exists and is finite, then f is uniformly continuous.

Proof. WRITE YOUR SOLUTION HERE. ☐

Problem 50. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous, then there exist constants $a \geq 0$, $b \geq 0$ such that $|f(x)| \leq a|x| + b$ for all $x \in \mathbb{R}$.

Proof. WRITE YOUR SOLUTION HERE. ☐

Problem 51. Prove that there is no continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f(f(x)) = -x$ for all $x \in \mathbb{R}$.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 52. Suppose that $f : (0, 1) \to \mathbb{R}$ is continuous. Suppose also that there are two sequences $x_n, y_n \in (0, 1)$ both convergent to 0 such that $f(x_n) \to 0$, $f(y_n) \to 1$. Prove that there is a sequence $z_n \in (0, 1)$ convergent to 0 such that $f(z_n) \to 1/2$.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 53. Let $f : [a, b] \to \mathbb{R}$ be continuous. Prove that the function

$$g(x) = \sup_{t \in [a,x]} f(t)$$

is continuous.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 54. A function $f : [0, 1] \to \mathbb{R}$ is continuous and has the property that

$$\lim_{x \to 0^+} \frac{f(x + 1/3) + f(x + 2/3)}{x} = 1.$$

Prove that there is $x_0 \in [0, 1]$ such that $f(x_0) = 0$.

Proof. WRITE YOUR SOLUTION HERE. □