Homework 6 for Math 1530

Due day: Tuesday October 22 recitations.

Problem 55. Prove that the two series

$$\sum_{n=0}^{\infty} c_n x^n \quad \text{and} \quad \sum_{n=0}^{\infty} n(\log n) c_n x^{n+3}$$

have the same radius of convergence.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 56. Let $f : (-\infty, \infty) \to \mathbb{R}$ be continuous and $\lim\limits_{x\to\infty} f(f(x)) = \infty$. Prove that $\lim\limits_{x\to\infty} |f(x)| = \infty$.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 57. Let $f : [0, 1) \to \mathbb{R}$ be a function that is not necessarily continuous. Define

$$g(\delta) = \sup\{|f(y) - f(y')| : y, y' \in (1 - \delta, 1)\}.$$

Prove that $\lim\limits_{x\to 1^-} f(x)$ exists and is finite if and only if $\lim\limits_{\delta\to 0^+} g(\delta) = 0$.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 58. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is α-Hölder continuous with some $\alpha > 1$, then f is constant.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 59. Let $f : (1, \infty) \to \mathbb{R}$ be differentiable. Prove that if

$$\lim_{x\to\infty} f'(x) = g, \text{ then } \lim_{x\to\infty} \frac{f(x)}{x} = g.$$

Proof. WRITE YOUR SOLUTION HERE. □

Problem 60. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable and such that

$$\lim_{x\to\infty} f(x) = g_1 \in \mathbb{R}, \qquad \lim_{x\to\infty} f'(x) = g_2.$$

Prove that $g_2 = 0$.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 61. Suppose that a differentiable function $f : \mathbb{R} \to \mathbb{R}$ and its derivative f' have no common zeros. Prove that f has only finitely many zeros in $[0, 1]$.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 62. Suppose that $f : [0, \infty) \to \mathbb{R}$ is continuous on $[0, \infty)$ and differentiable on $(0, \infty)$, $f(0) = 0$, and $\lim\limits_{x\to\infty} f(x) = 0$. Prove that there exists $c \in \mathbb{R}$ such that $f'(c) = 0$.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 63. Let $f : [0,1] \to \mathbb{R}$ be continuous on $[0,1]$ and differentiable on $(0,1)$. Suppose that $f(0) < 0 < f(1)$ and $f'(x) \neq 0$ for all $x \in (0,1)$. Let $S_1 = \{x \in [0,1] : f(x) > 0\}$ and $S_2 = \{x \in [0,1] : f(x) < 0\}$. Prove that $\inf(S_1) = \sup(S_2)$.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 64. Let $f : [0,\infty) \to \mathbb{R}$ be a differentiable function on $[0,\infty)$ such that $f(0) > 0$ and

$$f'(x) = \frac{1}{x^2 + (f(x))^2} \quad \text{for all } x \in [0,\infty).$$

Prove that $\lim_{x\to\infty} f(x)$ exists and is finite.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 65. Prove that for $x \in \mathbb{R}$

$$\cos x \geq 1 - \frac{x^2}{2}.$$

Proof. WRITE YOUR SOLUTION HERE. □

Problem 66. Prove that for $x \in [0,1]$ and $p > 1$ the following inequality is satisfied

$$\frac{1}{2^{p-1}} \leq x^p + (1-x)^p \leq 1.$$

Proof. WRITE YOUR SOLUTION HERE. □

Problem 67. Let $W(x)$ be a polynomial such that $W(x) \geq 0$ for $x \in \mathbb{R}$. Prove that

$$u(x) = W(x) + W'(x) + W''(x) + \ldots \geq 0.$$

Proof. WRITE YOUR SOLUTION HERE. □

Problem 68. Prove that the polynomial

$$W_n(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}.$$

has no multiple roots.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 69. Suppose that $f \in C^\infty(\mathbb{R})$ and $f(a) = 0$. Prove that there is $g \in C^\infty(\mathbb{R})$ such that $f(x) = (x-a)g(x)$ for all $x \in \mathbb{R}$.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 70. Let $f(x) = e^{-1/x^2}$ for $x \neq 0$ and $f(0) = 0$. Prove that $f \in C^\infty(\mathbb{R})$ and $f^{(n)}(0) = 0$ for all $n = 0, 1, 2, \ldots$
Hint: *Use induction to prove that f is n-times differentiable, $f^{(n)}(0) = 0$ and $f(x) = W_n(1/x)e^{-1/x^2}$ for $x \neq 0$, where W_n is a polynomial.*
Remark. This is a very important example. Since all derivatives at 0 are equal zero, Maclaurin's series of f equals zero. However, $f(x) > 0$ for $x \neq 0$ so it is not equal to the Maclaurin series at any point except $x = 0$. Another reason why t is so important is that it allows to construct compactly supported smooth functions, see Problem 71.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 71. Use the function from Problem 63 to construct $f \in C^\infty(a,b)$ such that $f(x) = 0$ for $x \in \mathbb{R} \setminus (a,b)$.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 72. Let $n \geq 3$. Consider an n-times continuously differentiable function $f \in C^n(\mathbb{R})$ such that $f^{(k)}(0) = 0$, for $k = 2, 3, \ldots, n-1$ and $f^{(n)}(0) \neq 0$. Clearly, by the mean value theorem for any $h > 0$ there is $0 < \theta(h) < h$ such that

$$f(h) - f(0) = hf'(\theta(h)).$$

Prove that

$$\lim_{h \to 0} \frac{\theta(h)}{h} = \left(\frac{1}{n}\right)^{\frac{1}{n-1}}.$$

Hint: Expand f and f' using Taylor's formula.

Proof. WRITE YOUR SOLUTION HERE. □