Homework 7 for Math 1530

Due day: Tuesday November 5 recitations.

Problem 73. For $n \geq 2$ define $f_n : [0,1] \to [0,1]$ by

$$f_n(x) = \begin{cases} nx & \text{if } 0 \leq x \leq \frac{1}{n} \\ \frac{n}{n-1}(1-x) & \text{if } \frac{1}{n} \leq x \leq 1 \end{cases}$$

Show that $\sum_{n=2}^{\infty} [f_n(x)]^n$ converges pointwise on $[0,1]$ to a function $f(x)$ that is continuous on $(0,1]$, but that the improper integral $\int_0^1 f(x)\,dx$ diverges (the integral is improper at 0).

Proof. WRITE YOUR SOLUTION HERE. □

Problem 74. Prove that $\lim_{x\to\infty} e^{-x^2} \int_0^x e^{t^2}\,dt = 0$.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 75. For n a positive integer, put

$$t_n = \frac{1}{2n+1} - \frac{1}{2n+2} + \frac{1}{2n+3} - \frac{1}{2n+4} + \cdots + \frac{1}{4n-1} - \frac{1}{4n}.$$

Find, with proof, the limit $\lim_{n\to\infty} nt_n$.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 76. Evaluate the limit $\lim_{n\to\infty} \int_1^2 \frac{nx^2}{1+n^2x^4}\,dx$.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 77. Prove that

$$\lim_{t\to 1^-} \int_0^t \left(\int_0^t \frac{dx}{1-xy} \right) dy = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Proof. WRITE YOUR SOLUTION HERE. □

Problem 78. Let $f : [0,1] \to \mathbb{R}$ be a continuous function. Prove that

$$\lim_{n\to\infty} n \int_0^1 x^n f(x)\,dx = f(1).$$

Proof. WRITE YOUR SOLUTION HERE. □

Problem 79. Let $f : [0,1] \to \mathbb{R}$ be continuously differentiable on $[0,1]$ and satisfy $f(1) = 0$. Prove that

$$\int_0^1 |f(x)|^2\,dx \leq 4 \int_0^1 x^2 |f'(x)|^2\,dx.$$

Proof. WRITE YOUR SOLUTION HERE. □

Problem 80. Suppose $f : [1, \infty) \to \mathbb{R}$ is continuous and $\lim_{x\to\infty} xf(x) = 1$. Prove that

$$\lim_{t\to\infty} \frac{1}{t} \int_1^{e^t} f(x)\, dx = 1.$$

Proof. WRITE YOUR SOLUTION HERE. □

Problem 81. Prove that $\lim_{n\to\infty} \int_0^1 n \ln\left(1 + \frac{x}{n}\right) dx = \frac{1}{2}$.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 82. Evaluate the integral

$$I = \int_2^4 \frac{\sqrt{\ln(9-x)}}{\sqrt{\ln(9-x)} + \sqrt{\ln(3+x)}}\, dx.$$

Hint: *This is an easy problem if you look for symmetry.*

Proof. WRITE YOUR SOLUTION HERE. □

Problem 83. Show that for every positive integer n,

$$\left(\frac{2n-1}{e}\right)^{\frac{2n-1}{2}} < 1 \cdot 3 \cdot 5 \cdots (2n-1) < \left(\frac{2n+1}{e}\right)^{\frac{2n+1}{2}}$$

Hint: *Estimate the integral of* $\ln x$ *by Riemann sums both from above and from below.*

Proof. WRITE YOUR SOLUTION HERE. □