Homework 8 for Math 1530

Due day: December 13, 2019

Problem 84. Let (X, d) be a metric space. Prove that the set $A = \{x \in X : d(x, x_0) > 1\}$ is open, where $x_0 \in X$ is any fixed point.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 85. Show that the following sets are not compact, by exhibiting an open cover with no finite subcover
(a) $\{x \in \mathbb{R}^n : |x| < 1\}$,
(b) $\mathbb{Z} \subset \mathbb{R}$.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 86. Is it true that in a metric space the closed ball equals to the closure of the open ball, that is $\bar{B}(x, r) = \mathrm{cl}\,(B(x, r))$, where

$$B(x, r) = \{y : d(x, y) < r\} \quad \text{and} \quad \bar{B}(x, r) = \{y : d(x, y) \le r\}?$$

Proof. WRITE YOUR SOLUTION HERE. □

Problem 87. Let $(x_n)_{n=1}^{\infty}$ be a sequence of points in $\mathbb{R}^3$ such that $\|x_{n+1} - x_n\| \le 1/(n^2 + n)$, $n \ge 1$. Show that (x_n) converges.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 88. Prove that if K_1 and K_2 are nonempty compact and disjoint subsets of a metric space X, then the set $A = K_1 \cup K_2$ is disconnected.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 89. Prove that $(\mathbb{R}^n, \varrho)$, where

$$\varrho(x, y) = \frac{\|x - y\|}{1 + \|x - y\|}$$

is a metric space.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 90. Prove that every compact metric space is separable.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 91. Provide an example of a complete metric space that is not separable.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 92. Let X be a complete metric space and let V_n, $n = 1, 2, 3, \dots$ be open and dense sets. Prove that $\bigcap_{n=1}^{\infty} V_n$ is dense in X.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 93. Use previous problem to prove that the set of irrational numbers cannot be written as a union of countably many closed subsets of $\mathbb{R}$.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 94. Prove that ℓ^1 is a metric space, where

$$\ell^1 = \{x = (x_1, x_2, \dots) : \sum_{n=1}^{\infty} |x_i| < \infty\} \quad d(x,y) = \|x-y\|_1 = \sum_{n=1}^{\infty} |x_n - y_n|.$$

Proof. WRITE YOUR SOLUTION HERE. □

Problem 95. Prove that ℓ^1 is complete.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 96. Prove that ℓ^1 is separable.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 97. Prove that if $x \in \ell^1$ and $r > 0$, then the closed ball in ℓ^1

$$\bar{B}(x,1) = \{z \in \ell^1 : \|x-z\|_1 \leq 1\}$$

is not compact.[1]

Proof. WRITE YOUR SOLUTION HERE. □

Problem 98. Let

$$\ell^\infty = \{x = (x_1, x_2, \dots) : \sup_n |x_n| < \infty\} \quad d(x,y) = \|x-y\|_\infty = \sup_n |x_n - y_n|.$$

Prove that the metric space ℓ^∞ is not separable.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 99. Prove that for every separable metric space (X,d) there is an isometric embedding $\kappa : X \to \ell^\infty$.
Hint: *Let $x_0 \in X$ and let $\{x_i\}_{i=1}^\infty$ be a countable and a dense subset. For each $x \in X$ consider a sequence $(d(x,x_i) - d(x_i,x_0))_{i=1}^\infty$.*

Proof. WRITE YOUR SOLUTION HERE. □

Problem 100. Let $X \subset \mathbb{R}^n$ be a compact set. Prove that the set

$$Y = \big\{y \in \mathbb{R}^n : |x-y| = 2019 \text{ for some } x \in X\big\}$$

is compact.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 101. Construct an example of a decreasing family of connected sets

$$C_1 \supset C_2 \supset C_3 \supset \dots,$$

such that the intersection $\bigcap_{i=1}^\infty C_i$ is disconnected. (It is enough if you define C_i on a picture.)

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[1] This provides an example of a complete metric space where bounded and closed sets are not necessarily compact.

Proof. WRITE YOUR SOLUTION HERE. ☐

Problem 102. Let $(f_n)_{n=1}^{\infty}$, $f_n : [0,1] \to \mathbb{R}$ be sequence of continuous functions such that

(a) $f_n(x) \geq 0$ for all x and n,
(b) $f_{n+1} \leq f_n$ for all n,
(c) $\lim_{n\to\infty} f_n(x) = 0$ for all $x \in \mathbb{R}$.

Prove that $f_n \rightrightarrows 0$ converges uniformly to 0.

Proof. WRITE YOUR SOLUTION HERE. ☐

Problem 103. Let $F : \mathbb{R}^n \to \mathbb{R}$ be a norm, that is for all $x, y \in \mathbb{R}^n$ and $t \in \mathbb{R}$,

(a) $F(x) \geq 0$ and $F(x) = 0$ if and only if $x = 0$,
(b) $F(x + y) \leq F(x) + F(y)$,
(c) $F(tx) = |t|F(x)$.

Prove that there are constants $A, B > 0$ such that

$$A\|x\| \leq F(x) \leq B\|x\| \quad \text{for all } x \in \mathbb{R}^n.$$

Proof. WRITE YOUR SOLUTION HERE. ☐

Problem 104. Prove that if X is a metric space and $f : X \times [0,1] \to \mathbb{R}$ is continuous, then

$$g : X \to \mathbb{R}, \quad g(x) = \sup_{t\in[0,1]} f(x,t)$$

is continuous.

Proof. WRITE YOUR SOLUTION HERE. ☐

Problem 105. Prove that is $A \subset X$ is a dense subset of a metric pace X, and $f : A \to \mathbb{R}$ is continuous, then there is a unique function $F : X \to \mathbb{R}$ such that $F(x) = f(x)$ for all $x \in A$. Prove then that F is uniformly continuous.

Proof. WRITE YOUR SOLUTION HERE. ☐

Problem 106. Let $f : A \to X$ be a mapping between a dense subset $A \subset \mathbb{R}^n$ and a complete metric space (X, d). Assume that $d(f(x), f(y)) \leq |x - y|$ for all $x, y \in A$.

(a) Prove that there is a mapping $F : \mathbb{R}^n \to X$ such that $d(F(x), F(y)) \leq |x - y|$ for all $x, y \in \mathbb{R}^n$ and $F(x) = f(x)$ whenever $x \in A$.
(b) Provide an example showing that the claim in (a) is not true if we do not assume that the space (X, d) is complete.

Proof. WRITE YOUR SOLUTION HERE. ☐

Problem 107. Show that the Hilbert cube

$$\mathcal{H} = \{x = (x_1, x_2, \ldots) : 0 \leq x_n \leq 2^{-n} \text{ for each } n \in \mathbb{N}\}$$

is compact when equipped with the ℓ^1 metric $d(x, y) = \sum_{n=1}^{\infty} |x_n - y_n|$.

Proof. WRITE YOUR SOLUTION HERE. ☐

Problem 108. Let $f_n : \mathbb{R}^k \to \mathbb{R}^m$ be continuous maps $(n = 1, 2, \ldots)$ Let $K \subset \mathbb{R}^k$ be compact. Prove that if $f_n \rightrightarrows f$ uniformly on K, then the set

$$S = f(K) \cup \bigcup_{n=1}^{\infty} f_n(K) \quad \text{is compact.}$$

Proof. WRITE YOUR SOLUTION HERE. □

Problem 109. Let $f_n : X \to \mathbb{R}$, $n = 1, 2, \ldots$ be a sequence of continuous functions on a metric space X such that the series $\sum_{n=1}^{\infty} f_n(x)$ converges for all $x \in X$ and

$$\sup_{x \in X} \left(\sum_{n=1}^{\infty} f_n(x)^2 \right)^{1/2} < \infty.$$

Prove that if a series of real numbers c_n, $n = 1, 2, \ldots$ satisfies $\sum_{n=1}^{\infty} c_n^2 < \infty$, then the series

$$\sum_{n=1}^{\infty} c_n f_n(x)$$

converges uniformly to a continuous function.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 110. A graph of a mapping $f : X \to Y$ is defined as

$$G_f = \{(x, y) \in X \times Y : \, y = f(x)\}.$$

Prove that if X is a metric space and Y is a compact metric space, then the map $f : X \to Y$ is continuous if and only if G_f is a closed subset of $X \times Y$.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 111. Let (X, d) be a compact metric space and $z \in Z$. Let $T : X \to X$ be a maping that satisfies $d(x, y) \leq d(T(x), T(y)$ for all $x, y \in X$, that is the distances are non-decreasing under the mapping T. Define $\{x_n\}$ by

$$x_1 = T(z) \quad \text{and} \quad x_{n+1} = T(x_n) \text{ for } n \geq 1.$$

Prove that there is a subsequence of $\{x_n\}$ which converges to z.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 112. Let (X, d) be a compact metric space and $f : X \to \mathbb{R}$ be a continuous function. Prove that for any $\varepsilon > 0$, there is $C > 0$ such that

$$|f(x) - f(y)| \leq Cd(x, y) + \varepsilon \quad \text{for all } x, y \in X.$$

Proof. WRITE YOUR SOLUTION HERE. □

Problem 113. Let (X, d) be a metric space and $f : X \to X$ be a contraction mapping. Prove that if a non-empty and compact set $K \subset X$ satisfies $f(K) = K$, then K contains exactly one point.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 114. Let (X,d) be a compact metric space. Prove that if $f : X \to X$ satisfies $d(f(x), f(y)) < d(x,y)$ for all $x, y \in X$, $x \neq y$, then, there is a unique point $x \in X$ such that $f(x) = x$.

Proof. WRITE YOUR SOLUTION HERE. $\square$

Problem 115. Find an example of a function $f : \mathbb{R} \to \mathbb{R}$ such that

$$|f(x) - f(y)| < |x - y| \quad \text{for all } x, y \in \mathbb{R},\ x \neq y.$$

and f has no fixed point. You can find an explicit formula for f, but you do not have to. It is enough if you find a convincing argument that such a function exists. You do not have to be very precise, but your argument has to be convincing.

Proof. WRITE YOUR SOLUTION HERE. $\square$