

Homework 1 for Math 1530
Due day: Tuesday September 17 Canvas.

Problem 1. Negate the statement¹

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall x \in \mathbb{R} \quad \forall y \in \mathbb{R} \quad (|x - y| < \delta \Rightarrow |\sin x - \sin y| < \varepsilon).$$

Proof. WRITE YOUR SOLUTION HERE. □

Problem 2. Negate the statement: *For all real numbers x, y satisfying $x < y$, there is a rational number q such that $x < q < y$.* Formulate the negation as a sentence and not as a formula involving quantifiers.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 3. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ that satisfy²

$$\forall \varepsilon > 0 \quad \forall x, y \in \mathbb{R} \quad \exists \delta > 0 \quad (|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon).$$

Proof. WRITE YOUR SOLUTION HERE. □

Problem 4. Prove that if $f : X \rightarrow Y$ is a function and $A_1, A_2, A_3, \dots$ are subsets of X , then

$$f \left(\bigcup_{i=1}^{\infty} A_i \right) = \bigcup_{i=1}^{\infty} f(A_i),$$

and

$$(1) \quad f \left(\bigcap_{i=1}^{\infty} A_i \right) \subset \bigcap_{i=1}^{\infty} f(A_i).$$

Provide an example to show that we do not necessarily have equality in (1)

Proof. WRITE YOUR SOLUTION HERE. □

Problem 5. Prove that if $f : X \rightarrow Y$ is one-to-one and $A_1, A_2, A_3, \dots$ are subsets of X , then

$$f \left(\bigcap_{i=1}^{\infty} A_i \right) = \bigcap_{i=1}^{\infty} f(A_i).$$

Proof. WRITE YOUR SOLUTION HERE. □

Problem 6. Prove that $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots + \frac{1}{\sqrt{n}} \geq \sqrt{n}$.

Proof. WRITE YOUR SOLUTION HERE. □

¹This is a true statement known as uniform continuity of the function $\sin x$. However, you are not asked to prove the statement only to negate it.

²The solution presented in class was intentionally incorrect. Did you find mistake?

Problem 7. Let $a_1, \dots, a_n, b_1, \dots, b_n$ be positive numbers. Prove that

$$\prod_{i=1}^n (a_i + b_i)^{1/n} \geq \prod_{i=1}^n a_i^{1/n} + \prod_{i=1}^n b_i^{1/n}.$$

Hint: Divide both sides by the expression on the left hand side and use the arithmetic-geometric mean inequality.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 8. Use the Schwarz inequality to prove that if $a_1, \dots, a_n > 0$, then

$$\frac{n}{\frac{1}{a_1} + \dots + \frac{1}{a_n}} \leq \frac{a_1 + \dots + a_n}{n}.$$

Proof. WRITE YOUR SOLUTION HERE. □