

Homework 4 for Math 1530

Due day: Sunday October 15 Canvas.

Some problems will require results that haven't been covered in class yet. You can use what you know unless it is stated otherwise for example "prove directly from definition".

Problem 24. Find the limit $\lim_{n \rightarrow \infty} n \sin(2\pi en!)$.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 25. Find the limit

$$\lim_{n \rightarrow \infty} n^3 \left(\sqrt{n^2 + \sqrt{n^4 + 1}} - n\sqrt{2} \right).$$

Proof. WRITE YOUR SOLUTION HERE. □

Problem 26. Prove that there is a sequence such that the set of all possible limits of subsequences is the whole interval $[0, 1]$.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 27. Suppose that $\{a_n\}_n$ is a sequence such that for every integer $k \geq 2$ the sequence $\{a_{k \cdot n}\}_n$ is convergent. Does it follow that the sequence $\{a_n\}_n$ is convergent?

Proof. WRITE YOUR SOLUTION HERE. □

Problem 28. Prove that if $\{a_n\}_n$ is a sequence such that each of the sequences $\{a_{2n}\}_n$, $\{a_{2n+1}\}_n$ and $\{a_{3n}\}_n$ is convergent, then $\{a_n\}_n$ is convergent too.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 29. Provide an example of a convergent series $a_1 + a_2 + a_3 + \dots$, where $a_n > 0$, $n = 1, 2, 3, \dots$ such that the limit $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n}$ does not exist.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 30. Prove that the series

$$\sum_{n=3}^{\infty} \frac{1}{n(\log n)(\log \log n)^p}$$

diverges if $0 < p \leq 1$ and converges if $p > 1$.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 31. Prove that if the series $a_1 + a_2 + a_3 + \dots$ converges, where $a_n > 0$, $n = 1, 2, 3, \dots$, then the series

$$\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n} \quad \text{converges.}$$

Proof. WRITE YOUR SOLUTION HERE. □

DEFINITION. Let $a_1, a_2, a_3, \dots > 0$. We define the infinite product by

$$\prod_{n=1}^{\infty} a_n = \lim_{n \rightarrow \infty} a_1 a_2 \dots a_n.$$

We say that the infinite product *converges* if the limit is finite and *positive*. If the limit does not exist, equals 0 or ∞ then we say that the product *diverges*.

Problem 32. Prove that if $a_n > 0$, $n = 1, 2, \dots$, then the product $\prod_{n=1}^{\infty} (1 + a_n)$ converges if and only if the series $\sum_{n=1}^{\infty} a_n$ converges. **Hint:** You can use the inequality $e^x \geq 1 + x$ without proving it.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 33. Prove that if $0 < a_n < 1$, $n = 1, 2, \dots$, then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the series $\sum_{n=1}^{\infty} a_n / (1 - a_n)$ converges.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 34. Prove that if $0 < a_n < 1$, then the product $\prod_{n=1}^{\infty} (1 - a_n)$ converges if and only if the series $\sum_{n=1}^{\infty} a_n$ converges.

Proof. WRITE YOUR SOLUTION HERE. □