

Homework 5 for Math 1530

Due day: Wednesday November 1 Canvas.

Some problems will require results that haven't been covered in class yet. You can use what you know unless it is stated otherwise for example "prove directly from definition".

Problem 35. Prove that there is an increasing sequence of integers $a_1 < a_2 < a_3 < \dots$ such that for every $k \in \mathbb{N}$, the sequence $\{\sin(ka_n)\}_{n=1}^\infty$ converges.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 36. Prove that if $\lim_{n \rightarrow \infty} a_n = a \in \mathbb{R}$, then for any sequence $\{b_n\}$

$$\limsup_{n \rightarrow \infty} (a_n + b_n) = a + \limsup_{n \rightarrow \infty} b_n.$$

Proof. WRITE YOUR SOLUTION HERE. □

Problem 37. Let $\beta > 0$ and $\{u_n\}$ be a sequence of positive real numbers such that $\frac{u_{n+1}}{u_n} \leq \beta$ for every $n \in \mathbb{N}$. Prove that

$$\limsup_{n \rightarrow \infty} \sqrt[n]{u_n} \leq \limsup_{n \rightarrow \infty} \left(\frac{u_{n+1}}{u_n} \right).$$

Proof. WRITE YOUR SOLUTION HERE. □

Problem 38. Let c_0 be the class of all sequences $\{x_n\}$ such that $\lim_{n \rightarrow \infty} x_n = 0$. Prove that if $\{a_n\}$ is a bounded sequence, then

$$\inf_{(x_n) \in c_0} \left(\sup_{n \in \mathbb{N}} \{a_n + x_n\} \right) = \limsup_{n \rightarrow \infty} a_n.$$

Proof. WRITE YOUR SOLUTION HERE. □

Problem 39. Prove that there is a continuous function $f : (0, 1] \rightarrow \mathbb{R}$ such that $\sup f$ and $\inf f$ are not attained.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 40. Prove that if $f : [0, 1] \rightarrow \mathbb{R}$ is α -Hölder continuous with exponent $\alpha > 1$, then f is constant. (Do not use derivatives.)

Proof. WRITE YOUR SOLUTION HERE. □

Problem 41. Prove that if $f : (0, 1) \rightarrow \mathbb{R}$ is uniformly continuous, then there is a continuous function $F : [0, 1] \rightarrow \mathbb{R}$ such that $F(x) = f(x)$ for all $x \in (0, 1)$.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 42. Prove that if $f : [0, \infty) \rightarrow \mathbb{R}$ is continuous and the limit $\lim_{x \rightarrow \infty} f(x)$ exists and is finite, then f is uniformly continuous.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 43. Prove that if $f : \mathbb{R} \rightarrow \mathbb{R}$ is uniformly continuous, then there exist constants $a \geq 0$, $b \geq 0$ such that $|f(x)| \leq a|x| + b$ for all $x \in \mathbb{R}$.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 44. Prove that there is no continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $f(f(x)) = -x$ for all $x \in \mathbb{R}$.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 45. Suppose that $f : (0, 1) \rightarrow \mathbb{R}$ is continuous. Suppose also that there are two sequences $x_n, y_n \in (0, 1)$ both convergent to 0 such that $f(x_n) \rightarrow 0$, $f(y_n) \rightarrow 1$. Prove that there is a sequence $z_n \in (0, 1)$ convergent to 0 such that $f(z_n) \rightarrow 1/2$.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 46. Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous. Prove that the function

$$g(x) = \sup_{t \in [a, x]} f(t)$$

is continuous.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 47. A function $f : [0, 1] \rightarrow \mathbb{R}$ is continuous and has the property that

$$\lim_{x \rightarrow 0^+} \frac{f(x + 1/3) + f(x + 2/3)}{x} = 1.$$

Prove that there is $x_0 \in [0, 1]$ such that $f(x_0) = 0$.

Proof. WRITE YOUR SOLUTION HERE. □