

Homework 6 for Math 1530

Due day: December 8 Canvas.

Some problems will require results that haven't been covered in class yet. You can use what you know unless it is stated otherwise for example "prove directly from definition".

Problem 48. Suppose that a differentiable function $f : \mathbb{R} \rightarrow \mathbb{R}$ and its derivative f' have no common zeros. Prove that f has only finitely many zeros in $[0, 1]$.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 49. Suppose that $f : [0, \infty) \rightarrow \mathbb{R}$ is continuous on $[0, \infty)$ and differentiable on $(0, \infty)$, $f(0) = 0$, and $\lim_{x \rightarrow \infty} f(x) = 0$. Prove that there exists $c \in \mathbb{R}$ such that $f'(c) = 0$.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 50. Prove that for $x \in [0, 1]$ and $p > 1$ the following inequality is satisfied

$$\frac{1}{2^{p-1}} \leq x^p + (1-x)^p \leq 1.$$

Proof. WRITE YOUR SOLUTION HERE. □

Problem 51. Let $W(x)$ be a polynomial such that $W(x) \geq 0$ for $x \in \mathbb{R}$. Prove that

$$u(x) = W(x) + W'(x) + W''(x) + \dots \geq 0.$$

Proof. WRITE YOUR SOLUTION HERE. □

Problem 52. Prove that the polynomial

$$W_n(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^n}{n!}.$$

has no multiple roots.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 53. Suppose that $f \in C^\infty(\mathbb{R})$ and $f(a) = 0$. Prove that there is $g \in C^\infty(\mathbb{R})$ such that $f(x) = (x - a)g(x)$ for all $x \in \mathbb{R}$.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 54. Prove that $\lim_{x \rightarrow \infty} e^{-x^2} \int_0^x e^{t^2} dt = 0$.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 55. For n a positive integer, put

$$t_n = \frac{1}{2n+1} - \frac{1}{2n+2} + \frac{1}{2n+3} - \frac{1}{2n+4} + \dots + \frac{1}{4n-1} - \frac{1}{4n}.$$

Find, with proof, the limit $\lim_{n \rightarrow \infty} nt_n$.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 56. Evaluate the limit $\lim_{n \rightarrow \infty} \int_1^2 \frac{nx^2}{1+n^2x^4} dx$.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 57. Prove that

$$\lim_{t \rightarrow 1^-} \int_0^t \left(\int_0^t \frac{dx}{1-xy} \right) dy = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Proof. WRITE YOUR SOLUTION HERE. □

Problem 58. Let $f : [0, 1] \rightarrow \mathbb{R}$ be a continuous function. Prove that

$$\lim_{n \rightarrow \infty} n \int_0^1 x^n f(x) dx = f(1).$$

Proof. WRITE YOUR SOLUTION HERE. □

Problem 59. Let $f : [0, 1] \rightarrow \mathbb{R}$ be continuously differentiable on $[0, 1]$ and satisfy $f(1) = 0$. Prove that

$$\int_0^1 |f(x)|^2 dx \leq 4 \int_0^1 x^2 |f'(x)|^2 dx.$$

Proof. WRITE YOUR SOLUTION HERE. □