

Homework for Math 1540

Due day: Tuesday, February 4, recitations

Problem 1. Prove that the trigonometric polynomials

$$T(x) = \sum_{k=0}^n a_k \cos kx + \sum_{k=0}^n b_k \sin kx, \quad a_k, b_k \in \mathbb{R}$$

form an algebra. *Hint:* $\cos x + i \sin x = e^{ix}$.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 2. Let $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle in the complex plane. Let $\mathcal{A}$ be the algebra of functions of the form

$$f(e^{i\theta}) = \sum_{n=0}^N c_n e^{in\theta}, \quad c_n \in \mathbb{C}, \theta \in \mathbb{R}.$$

It is easy to see that $f \equiv 1$ belongs of $\mathcal{A}$ and $\mathcal{A}$ separates points (do not prove it). Prove that there are complex valued functions on S^1 that cannot be uniformly approximated by functions in $\mathcal{A}$. *Hint:* For $f \in \mathcal{A}$

$$\int_0^{2\pi} f(e^{i\theta}) e^{i\theta} d\theta = 0.$$

Proof. WRITE YOUR SOLUTION HERE. □

Problem 3. Prove that complex polynomials

$$p(z) = \sum_{n=0}^N c_n z^n, \quad c_n \in \mathbb{C}$$

are not dense in $C(\overline{D}, \mathbb{C})$, where

$$\overline{D} = \{z \in \mathbb{C} : |z| \leq 1\}$$

is the unit disc in $\mathbb{C}$. *Hint:* Consider $f(z) = \bar{z}$. Is the previous exercise helpful?

Proof. WRITE YOUR SOLUTION HERE. □

Problem 4. We know that if $f : [a, b] \rightarrow \mathbb{R}$ is continuous and

$$(1) \quad \int_a^b f(x) x^n dx = 0$$

for $n = 0, 1, 2, 3, \dots$, then $f(x) = 0$ for all $a \leq x \leq b$. We proved it using the Weierstrass theorem. Suppose now that $f : [a, b] \rightarrow \mathbb{R}$ is continuous and (1) holds for all $n \geq 2011$. Does it follow that $f(x) = 0$ for all $a \leq x \leq b$?

Proof. WRITE YOUR SOLUTION HERE. □

Problem 5. Prove that if $f : [0, 1] \rightarrow \mathbb{R}$ is such that

$$\int_0^1 f(x)e^{nx} dx = 0 \quad \text{for all } n = 0, 1, 2, \dots,$$

then $f(x) = 0$ for all $0 \leq x \leq 1$. Provide two proofs following the methods:

- (a) Use the Stone-Weierstrass theorem.
- (b) Use the change of variables formula and apply the Weierstrass theorem.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 6. Prove that if X is a compact metric space and $f : X \rightarrow X$ is a continuous mapping such that

$$d(f(x), f(y)) < d(x, y) \quad \text{for all } x, y \in X,$$

then there is a fixed point of f , i.e. $x \in X$ such that $f(x) = x$.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 7. Find an example of a function $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$|f(x) - f(y)| < |x - y| \quad \text{for all } x, y \in \mathbb{R}$$

and f has no fixed point. You can find an explicit formula for f , but you do not have to. It is enough if you find a convincing argument that such a function exists. You do not have to be very precise, but your argument has to be convincing.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 8. Show that there is a unique continuous real valued function $f : [0, 1] \rightarrow \mathbb{R}$ such that

$$f(x) = \sin x + \int_0^1 \frac{f(y)}{e^{x+y+1}} dy.$$

Proof. WRITE YOUR SOLUTION HERE. □

Problem 9. Let (X, d) be a nonempty complete metric space. Let $S : X \rightarrow X$ be a given mapping and write S^2 for $S \circ S$ i.e. $S^2(x) = S(S(x))$. Suppose that S^2 is a contraction. Show that S has a unique fixed point.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 10. Let E be a compact set and let $\mathcal{F} \subset C(E, \mathbb{R})$ be an equicontinuous family of functions. Does it imply that the family $\mathcal{F}$ is bounded in $C(E, \mathbb{R})$?

Proof. WRITE YOUR SOLUTION HERE. □

Problem 11. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be bounded and uniformly continuous. Prove that the family of functions $\{g_z\}_{z \in \mathbb{R}^n}$, $g_z(x) = f(x)f(x - z)$ is equicontinuous.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 12. Suppose E is a compact metric space and $f_n : E \rightarrow \mathbb{R}$, $n = 1, 2, \dots$ is a bounded and equicontinuous sequence of functions. Suppose that f_n converges pointwise to a continuous function $f : E \rightarrow \mathbb{R}$ (i.e. $f_n(x) \rightarrow f(x)$ for every $x \in E$). Prove directly (i.e. without using Arzela-Ascoli theorem) that $f_n \Rightarrow f$ uniformly on E .

Proof. WRITE YOUR SOLUTION HERE. □

Problem 13. Suppose $f_n : \mathbb{R} \rightarrow \mathbb{R}$, $n = 1, 2, \dots$ is a bounded and equicontinuous sequence of functions. Suppose that f_n converges pointwise to a continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$. Does it imply that $f_n \Rightarrow f$ uniformly on $\mathbb{R}$?

Proof. WRITE YOUR SOLUTION HERE. □

Problem 14. Let $f_n : [a, b] \rightarrow \mathbb{R}$ be a sequence of increasing functions that is pointwise convergent to a continuous function $f : [a, b] \rightarrow \mathbb{R}$. Prove that $f_n \Rightarrow f$ uniformly on $[a, b]$.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 15. Let $\{f_n\}$ be a sequence of real valued C^1 functions on $[0, 1]$ such that, for all n ,

$$|f'_n(x)| \leq \frac{1}{\sqrt{x}} \quad (0 < x \leq 1),$$

$$\int_0^1 f_n(x) dx = 0.$$

Prove that the sequence has a subsequence that converges uniformly on $[0, 1]$.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 16. We know that every continuous function $f : [a, b] \rightarrow \mathbb{R}$ can be uniformly approximated by polynomials (Weierstrass' theorem). Prove that if a continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$ can be uniformly approximated by polynomials on all of $\mathbb{R}$, then f is a polynomial.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 17. Prove that if $f_n : \mathbb{R} \rightarrow \mathbb{R}$, $n = 1, 2, 3, \dots$ are differentiable functions such that

- (a) $f_n(0) = 0$ for all n ,
- (b) $|f'_n(x)| \leq e^x$ for all n and all x ,

then there is a subsequence of f_n that converges pointwise to a continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$.

Hint. Show that the family satisfies the assumptions of the Arzela-Ascoli theorem on every interval $[-n, n]$ and then apply the diagonal method.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 18. Let the functions $f_n : [0, 1] \rightarrow [0, 1]$, $n = 1, 2, \dots$, satisfy $|f_n(x) - f_n(y)| \leq |x - y|$ whenever $|x - y| \geq 1/n$. Prove that the sequence $\{f_n\}_{n=1}^\infty$ has a uniformly convergent subsequence.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 19. If $f = (f_1, \dots, f_n) : [a, b] \rightarrow \mathbb{R}^n$ is a continuous function, then we define

$$\int_a^b f(t) dt = \left\langle \int_a^b f_1(t) dt, \dots, \int_a^b f_n(t) dt \right\rangle.$$

Prove that

$$\left\| \int_a^b f(t) dt \right\| \leq \int_a^b \|f(t)\| dt.$$

Proof. WRITE YOUR SOLUTION HERE.

□