

Homework for Math 1540

Due day: Tuesday, January 28, recitations

Problem 20. Let $A = [a_{ij}]$ be the matrix of a linear mapping $A \in L(\mathbb{R}^n, \mathbb{R}^m)$. Prove that the norm

$$\|A\| = \sup_{\|x\|=1} \|Ax\|$$

satisfies the inequality

$$\|A\| \leq \left(\sum_{i=1}^m \sum_{j=1}^n a_{ij}^2 \right)^{1/2}.$$

Hint: You may use the following argument: Write the components of the vector Ax as scalar products of rows on A and x . Then use the Schwarz inequality to estimate the length of the vector Ax .

Proof. WRITE YOUR SOLUTION HERE. □

Problem 21. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be differentiable and $F : \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined by $F(x, y) = f(xy)$. Prove that

$$x \frac{\partial F}{\partial x} = y \frac{\partial F}{\partial y}.$$

Proof. WRITE YOUR SOLUTION HERE. □

Problem 22. We say that a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is homogeneous of degree m if $f(tx) = t^m f(x)$ for all $x \in \mathbb{R}^n$ and all $t > 0$. Prove that if f is differentiable on $\mathbb{R}^n$ and homogeneous of degree m , then

$$\sum_{i=1}^n x_i \frac{\partial f}{\partial x_i}(x) = m f(x) \quad \text{for all } x \in \mathbb{R}^n.$$

Proof. WRITE YOUR SOLUTION HERE. □

Problem 23. We know that a function $f(x, y)$ is differentiable at $(0, 0)$. We also know the directional derivatives

$$\begin{aligned} D_u f(0, 0) &= 1 \quad \text{where } u = [1/\sqrt{5}, 2/\sqrt{5}], \\ D_v f(0, 0) &= 1 \quad \text{where } v = [1/\sqrt{2}, 1/\sqrt{2}]. \end{aligned}$$

Find the gradient $\nabla f(0, 0)$.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 24. Let $f \in C^1(\mathbb{R}^2)$ be such that $f(1, 1) = 1$ and $\nabla f(1, 1) = (a, b)$. Let $\varphi(x) = f(x, f(x, f(x, x)))$. Find $\varphi(1)$ and $\varphi'(1)$.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 25. A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is differentiable. Find the derivative of the function

$$F(t) = (f(t, t^2, \dots, t^n))^2, \quad t \in \mathbb{R}$$

of one variable.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 26. Verify by a direct computation that the vector field $F(x) = x|x|^{-n}$ defined on $\mathbb{R}^n \setminus \{0\}$ is divergence free, i.e.

$$\operatorname{div} F(x) = \sum_{i=1}^n \frac{\partial}{\partial x_i} \left(\frac{x_i}{|x|^n} \right) = 0 \quad \text{for all } x \neq 0.$$

Proof. WRITE YOUR SOLUTION HERE. □

Problem 27. Prove that for $\alpha > 0$ the function $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$,

$$\Phi(x) = x|x|^\alpha$$

is of class C^1 . Find $D\Phi(x)$.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 28. Find all the points $(x, y) \in \mathbb{R}^2$ where the function

$$f(x, y) = |e^x - e^y| \cdot (x + y - 2)$$

is differentiable.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 29. Consider the function $g : \mathbb{R}^2 \rightarrow \mathbb{R}$ given by

$$g(x, y) = x^{2/3}y^{2/3}, \quad \text{for all } (x, y) \in \mathbb{R}^2.$$

Prove that g is *differentiable* at $(0, 0)$.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 30. Find a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ that is differentiable at each point, but whose partial derivatives are not continuous at $(0, 0)$.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 31. Prove that if the partial derivatives (of first order) of a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ exist everywhere and they are bounded, then the function f is continuous.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 32. Prove that if $f, g \in C^k(\Omega)$, $\Omega \subset \mathbb{R}^n$, then for any multiindex α with $|\alpha| \leq k$ we have

$$D^\alpha(fg) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^\beta f D^{\alpha-\beta} g,$$

where $\beta \leq \alpha$ means that $\beta_i \leq \alpha_i$ for $i = 1, 2, \dots, n$, $\alpha - \beta = (\alpha_1 - \beta_1, \dots, \alpha_n - \beta_n)$ and

$$\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha - \beta)!}.$$

Proof. WRITE YOUR SOLUTION HERE. □