

Homework 3 for Math 1540

Due day: Tuesday, February 18, recitations

Problem 33. Prove that:

- (a) There is a unique continuous function $f : [0, 1] \rightarrow \mathbb{R}$ such that

$$f(x) = 1 + \int_0^x t^2 f(t) dt \quad \text{for all } x \in [0, 1].$$

- (b) The function from (a) is of class $f \in C^\infty(0, 1)$.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 34. Prove that if $f : \mathbb{R}^n \rightarrow \mathbb{R}$ satisfies $|f(x) - f(y)| \leq M\|x - y\|^{3/2}$, then f is constant.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 35. Prove that if the partial derivatives $\partial f/\partial x_1$ and $\partial f/\partial x_2$ of a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ exist at every point of $\mathbb{R}^2$, and the partial derivative $\partial f/\partial x_1$ is continuous on $\mathbb{R}^2$, then f is differentiable at every point of $\mathbb{R}^2$.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 36. Prove that if $f \in C^1(\mathbb{R}^n)$ and ∇f is L -Lipschitz, $\|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\|$, $x, y \in \mathbb{R}^n$, then

$$\frac{|f(y) - f(x) - \nabla f(x)(y - x)|}{\|y - x\|} \leq L\|y - x\| \quad \text{for all } x, y \in \mathbb{R}^n.$$

Proof. WRITE YOUR SOLUTION HERE. □

Problem 37. Let $f \in C^2(\mathbb{R}^2)$. Suppose that $\nabla f = 0$ on a compact set $E \subset \mathbb{R}^2$. Prove that there is a constant $M > 0$ such that $|f(x) - f(y)| \leq M|x - y|^2$ for all $x, y \in E$.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 38. Suppose that $f \in C^2(\mathbb{R}^n)$ has a local minimum at $x = 0$ and $f(0) = 0$. Prove that for any $x \in \mathbb{R}^n$

$$f(x) = \int_0^1 (1 - t)x^T D^2 f(tx)x dt,$$

where $x \in \mathbb{R}^n$ is a column vector and x^T is the horizontal vector.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 39. We know that $f \in C^2(\mathbb{R})$ is convex if and only if $f'' \geq 0$ on $\mathbb{R}$. Prove that $f \in C^2(\mathbb{R}^n)$ is convex if and only if $H_x(f) = D^2 f(x)$ is positive semidefinite for all $x \in \mathbb{R}^n$.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 40. Prove that $x_0 \in \mathbb{R}^n$ is a critical point of $f \in C^2(\mathbb{R}^n)$, and $H_{x_0}(f)$ is positive definite, then there are $M > 0$ and $\epsilon > 0$ such that

$$f(x) \geq f(x_0) + M\|x - x_0\|^2 \quad \text{whenever } \|x - x_0\| < \epsilon.$$

Proof. WRITE YOUR SOLUTION HERE. □

Problem 41. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a given function. Prove that if there are $M > 0$ and $\epsilon > 0$ such that

$$f(x) \geq f(x_0) + M\|x - x_0\| \quad \text{whenever } \|x - x_0\| < \epsilon,$$

then f is not differentiable at x_0 .

Proof. WRITE YOUR SOLUTION HERE. □

Problem 42. Consider an open ball $B = B(a, r) \subset \mathbb{R}^n$. Prove that the function

$$\varphi(x) = \begin{cases} \exp\left(\frac{1}{|x-a|^2-r^2}\right) & \text{if } x \in B, \\ 0 & \text{if } x \in \mathbb{R}^n \setminus B, \end{cases}$$

is infinitely differentiable on $\mathbb{R}^n$.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 43. A C^2 function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is called harmonic if

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

Prove that if a harmonic function has a local maximum at (x_0, y_0) , then *all* second order partial derivatives of f vanish at (x_0, y_0) .

Proof. WRITE YOUR SOLUTION HERE. □

Problem 44. Suppose that a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ of class C^2 satisfies the inequality $f_{xx} + f_{yy} \geq 0$ at every point of $\mathbb{R}^2$. Suppose also that all its critical points are non-degenerate, i.e. the matrix of second order derivatives at the critical point has non-zero determinant. Prove that f cannot have local maxima.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 45. Let $f \in C^2(\Omega) \cap C^0(\overline{\Omega})$, where $\Omega \subset \mathbb{R}^n$ is open and bounded. Let $\Delta f = \sum_{i=1}^n \partial^2 f / \partial x_i^2$ be the Laplace operator.

- (a) Show that if for some $\epsilon > 0$ and $x_0 \in \Omega$ we have $\Delta f(x_0) \geq \epsilon$, then f has no local maximum at x_0 .
- (b) Conclude that if $\Delta f(x) \geq \epsilon$ for some $\epsilon > 0$ and all $x \in \Omega$, then we have $\sup_{\Omega} f = \sup_{\partial\Omega} f$.
- (c) Conclude that if $\Delta f(x) \geq 0$ for all $x \in \Omega$, then we have $\sup_{\Omega} f = \sup_{\partial\Omega} f$.

Hint for part (c): Observe that $\Delta|x|^2 = 2n$. Use it to modify a function f in (c) so that you can apply part (b).

Proof. WRITE YOUR SOLUTION HERE. □

Problem 46. Let $A = [a_{ij}]$ be an $n \times n$ matrix. The *cofactor* A_{ij} is the product of $(-1)^{i+j}$ with the $(n-1) \times (n-1)$ determinant of A obtained by deleting the i th row and j th column in A . Prove that

$$\frac{\partial}{\partial a_{ij}}(\det A) = A_{ij}.$$

Proof. WRITE YOUR SOLUTION HERE. □

Problem 47. Let $A(t) = [x_{ij}(t)] : (a, b) \rightarrow \mathbb{R}^{n \times n}$ be a smooth matrix-valued curve. Prove that if $A(0) = I$, then

$$\left. \frac{d}{dt} \right|_{t=0} (\det A(t)) = \sum_{i=1}^n x'_{ii}(0) = \operatorname{tr} A'(0).$$

Proof. WRITE YOUR SOLUTION HERE. □

Problem 48. Let $M_{n \times n}$ denote the vector space of real $n \times n$ matrices. Define a map $f : M_{n \times n} \rightarrow M_{n \times n}$ by $f(X) = X^2$. Find the derivative of f .

Proof. WRITE YOUR SOLUTION HERE. □

Problem 49. The class of invertible matrices $GL(n, \mathbb{R})$ forms an open subset in the space of all $n \times n$ matrices $M_{n \times n} = \mathbb{R}^{n^2}$. Let $F : GL(n, \mathbb{R}) \rightarrow GL(n, \mathbb{R})$, $F(A) = A^{-1}$. Prove that the function F is C^∞ smooth (as a mapping from $\mathbb{R}^{n^2}$ to $\mathbb{R}^{n^2}$) and that for any $A \in \mathbb{R}^{n \times n}$ we have

$$DF(A)B = -A^{-1} \circ B \circ A^{-1} \quad \text{for all } B \in \mathbb{R}^{n \times n}$$

Proof. WRITE YOUR SOLUTION HERE. □